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A new perspective on Fibonacci and Lucas numbers

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Abstract. In the present work, we introduce a new version of Fibonacci and Lucas numbers which we will call non-Newtonian Fibonacci and non-Newtonian Lucas numbers. Also, we discuss a variety of some properties of them. Furthermore, we give some formulas and identities such as Binet's formula, the d'Ocagne's identity, Cassini's identity and Gelin-Cesàro identity involving these new types and we find the generating functions for these numbers.

1. Introduction and Preliminaries

Since Newton and Leibnitz introduced modern calculus, many calculi have been created with different aspects. A known and favored technique of introducing a novel mathematical system is to modify the axioms of a known one. Grossman and Katz [19] established a new family of calculi, named non-Newtonian calculus which culminated in their little book Non-Newtonian Calculus, and gave definitions of contemporary types of integrals and derivatives transforming the acts of addition and subtraction into multiplication and division in the period between 1967 and 1970. They defined an infinite family of calculus which involves some special calculi such as geometric calculus, harmonic calculus, bigeometric calculus, anageometric calculus (see [17, 18]). Furthermore, a mathematical problem, which is difficult or impossible to solve in one calculus, can be effortlessly exposed through another calculus.

Since the leading-edge work of Grossman and Katz, non-Newtonian calculi have become a hot issue in recent times due to prevalence of excellent applications for some problems, e.g., in economy, quantum calculus, biomathematics, calculus in variantions, actuarial science, finance, economics, demography, signal processing and thermostatistics [3, 6, 12, 13, 15, 16, 20, 27, 30–32, 34].

An arithmetic is a complete ordered field whose realm is a subset of $\mathbb{R}$. Non-Newtonian calculi utilize different types of arithmetic and their generators. Let α be a bijection whose domain $\mathbb{R}$ and whose range is a subset A of $\mathbb{R}$. Then, it is called a generator with range A and defines an arithmetic. The range of generator α is denoted by $\mathbb{R}_\alpha$. Also, every element of $\mathbb{R}_\alpha$ is called a non-Newtonian real number. Choosing $\alpha = I$ and $\alpha = \exp$, the classical arithmetic and the geometric arithmetic are obtained, respectively, and also, $\mathbb{R}_I = \mathbb{R}$

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and $\mathbb{R}_{\exp} = \mathbb{R}^{+}$.

	$\alpha-$arithmetic
Realm	$A\,(=\mathbb{R}_{\alpha})$
Addition	$x \dot{+} y = \alpha\left\{\alpha^{-1}(x) + \alpha^{-1}(y)\right\}$
Subtraction	$x \dot{-} y = \alpha\left\{\alpha^{-1}(x) - \alpha^{-1}(y)\right\}$
Multiplication	$x \dot{\times} y = \alpha\left\{\alpha^{-1}(x) \times \alpha^{-1}(y)\right\}$
Division	$x \dot{/} y = \frac{x}{y}\alpha = \alpha\left\{\frac{\alpha^{-1}(x)}{\alpha^{-1}(y)}\right\}\ \left(y \neq \dot{0}\right)$
Ordering	$x \dot{\leq} y \Longleftrightarrow \alpha^{-1}(x) \leq \alpha^{-1}(y)$

If $x \in \mathbb{R}_{\alpha}$ and $\dot{0} \dot{<} x$ (or $x \dot{<} \dot{0}$), then we say that it is a $\alpha-$positive number (or $\alpha-$negative number). Additionally, $\mathbb{R}_{\alpha}^{+}$ denotes the set of $\alpha-$positive numbers. Also, $\alpha(-x) = \alpha\left\{-\alpha^{-1}\left(\dot{x}\right)\right\} = \dot{-}\dot{x}$ for all $x \in \mathbb{R}$. On the other hand, the number $x \dot{\times} x$ is called the $\alpha-$ square of x, denoted by $x^{\dot{2}}$. If $x \in \mathbb{R}_{\alpha}^{+} \cup \left\{\dot{0}\right\}$, then we say that $\alpha\left[\sqrt{\alpha^{-1}(x)}\right]$ is the $\alpha-$ square root of x, denoted by $\dot{\sqrt{x}}$ [7, 19].

Italian mathematician Leonardo Fibonacci created a new number sequence called Fibonacci numbers. These numbers worked as a model for studying the growth of rabbit populations (see [9]). Also, the rate of two consecutive Fibonacci numbers reaches the golden ratio 1, 61803399....Fibonacci numbers are connected with Lucas numbers.

Subsequently, the study of numerical sequences of such numbers is a great topic of research and since the second half of 20th century it starts to become more popular for researchers. Also, Fibonacci numbers, Lucas numbers and the golden mean arise in the investigation of numerous areas of art and science, and they have many original generalizations in different ways with various aspects. In addition, [25] is a good resource for the rich applications and usefulness of these numbers. Some pivotal attempts at generalizing these numbers are [1, 2, 5, 14, 21, 22, 24, 28, 29] which each approached them with a different perspective.

For self consistency, we give a brief introductory of Fibonacci and Lucas numbers which focuses on the nomenclature used in this paper.

Fibonacci numbers are the terms of the integer sequence

$$\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ..., F_n, ...\}$$

defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2} \text{ for each } n \in \{2, 3, 4, ...\}$$

with $F_0 = 0, F_1 = 1$, it is well known as the $n-$th term of the Fibonacci sequence (F_n) which is a numerical sequence.

Lucas numbers are the terms of the integer sequence

$$\{2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, ..., L_n, ...\}$$

defined by the recurrence relation

$$L_n = L_{n-1} + L_{n-2} \text{ for each } n \in \{2, 3, 4, ...\}$$

with $L_0 = 2, L_1 = 1$, it is well known as the $n-$th term of the Lucas sequence (L_n) which is a numerical sequence.

For $n, m \geq 0$, the following relations hold (see [4, 8, 10, 23, 25, 26, 33]):

$$F_n + F_{n+1} = F_{n+2}. \quad (1)$$

$$L_n + L_{n+1} = L_{n+2}. \quad (2)$$

$$L_n = F_{n-1} + F_{n+1}. \tag{3}$$

$$L_n = F_{n+2} - F_{n-2}. \tag{4}$$

$$F_n^2 + F_{n+1}^2 = F_{2n+1}. \tag{5}$$

$$F_{n+1}^2 - F_{n-1}^2 = F_{2n}. \tag{6}$$

$$F_n F_m + F_{n+1} F_{m+1} = F_{n+m+1}. \tag{7}$$

$$F_m F_{n+1} - F_n F_{m+1} = (-1)^n F_{m-n}. \tag{8}$$

$$F_{n+m} = F_{n-1} F_m + F_n F_{m+1}. \tag{9}$$

$$F_n = \frac{\gamma^n - \beta^n}{\gamma - \beta} \tag{10}$$

and

$$L_n = \gamma^n + \beta^n \tag{11}$$

where $\gamma = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

$$F_n^4 - F_{n-1} F_{n-2} F_{n+1} F_{n+2} = 1. \tag{12}$$

$$F_{n+1} F_{n+2} F_{n+6} - F_{n+3}^3 = (-1)^n F_n. \tag{13}$$

$$\sum_{k=0}^{n} F_k = F_{n+2} - 1, \ \sum_{k=0}^{n} F_{2k} = F_{2n+1} - 1, \ \sum_{k=0}^{n} F_{2k+1} = F_{2n+2}. \tag{14}$$

Besides, for $n, r \geq 1$, the following relations hold (see [4, 10, 23, 25, 33]):

$$F_n^2 - F_{n-1} F_{n+1} = (-1)^{n+1}. \tag{15}$$

$$L_n^2 - L_{n-1} L_{n+1} = 5 (-1)^n. \tag{16}$$

$$F_n^2 - F_{n-r} F_{n+r} = (-1)^{n-r} F_r^2. \tag{17}$$

$$L_n^2 - L_{n-r} L_{n+r} = 5 (-1)^{n-r} F_r^2. \tag{18}$$

Motivated by rich applications of both non-Newtonian calculus and numerical sequences, in this article, we present and study non-Newtonian Fibonacci and non-Newtonian Lucas numbers as a new addition to the existing literature. Hereupon, we connect such numbers with the classical Fibonacci and Lucas numbers. Also, we investigate non-Newtonian versions of some important identities and remarkable formulas given for classical Fibonacci and Lucas numbers in a new and direct way. Furthermore, such numbers generalize the known corresponding numbers, so our results are stronger than counterparts in the literature and they gain importance as a starting point for new applications to many interesting problems in various aspects e.g. the applications to problems in encryption theory.

2. Non-Newtonian Fibonacci and Non-Newtonian Lucas Numbers with Some Properties

In this part, we define the concepts of a non-Newtonian Fibonacci number and a non-Newtonian Lucas number with a new perspective on the concepts of a Fibonacci number and a Lucas number. We also deal with the non-Newtonian versions of some formulas and identities in analogy with some well-known identities and formulas for classical counterparts and evince their relationships with each other.

Definition 2.1. *The non-Newtonian Fibonacci and non-Newtonian Lucas numbers are defined*

$$\aleph\aleph F_n = \dot{F_n} = \alpha(F_n)$$

and

$$\aleph\aleph L_n = \dot{L_n} = \alpha(L_n),$$

respectively where F_n and L_n are the $n-th$ Fibonacci and Lucas numbers, respectively. The set of non-Newtonian Fibonacci and non-Newtonian Lucas numbers are denoted by $\aleph\aleph F$ and $\aleph\aleph L$, respectively. That is,

$$\begin{aligned} \aleph\aleph F &= \{\aleph\aleph F_n : n \in \mathbb{N}\} \\ &= \left\{\dot{0}, \dot{1}, \dot{1}, \dot{2}, \dot{3}, \dot{5}, \dot{8}, \dot{13}, \dot{21}, \dot{34}, \dot{55}, \dot{89}, \dot{144}, ..., \dot{F_n}, ...\right\} \end{aligned}$$

and

$$\begin{aligned} \aleph\aleph L &= \{\aleph\aleph L_n : n \in \mathbb{N}\} \\ &= \left\{\dot{2}, \dot{1}, \dot{3}, \dot{4}, \dot{7}, \dot{11}, \dot{18}, \dot{29}, \dot{47}, \dot{76}, \dot{123}, \dot{199}, ..., \dot{L_n}, ...\right\}. \end{aligned}$$

If we use the generator I defined by $\alpha(x) = x$ for all $x \in \mathbb{R}$, we obtain Fibonacci and Lucas numbers with respect to classical arithmetic, respectively.

Also, by choosing the generator exp defined by $\alpha(x) = e^x$ for all $x \in \mathbb{R}$, we obtain Fibonacci and Lucas numbers with respect to geometric arithmetic, respectively, as follows:

$$\begin{aligned} \aleph\aleph GF &= \{\alpha(F_n) : n \in \mathbb{N}\} \\ &= \left\{e^{F_n} : n \in \mathbb{N}\right\} \\ &= \left\{e^0, e^1, e^1, e^2, e^3, e^5, e^8, e^{13}, ..., e^{F_n}, ...\right\} \end{aligned}$$

and

$$\begin{aligned} \aleph\aleph GL &= \{\alpha(L_n) : n \in \mathbb{N}\} \\ &= \left\{e^{L_n} : n \in \mathbb{N}\right\} \\ &= \left\{e^2, e^1, e^3, e^4, e^7, e^{11}, e^{18}, ..., e^{L_n}, ...\right\}. \end{aligned}$$

In what follows, we focus on some relations for non-Newtonian Fibonacci and non-Newtonian Lucas numbers and relationship between each other.

Theorem 2.2. *Let $\aleph\aleph F_n$ and $\aleph\aleph L_n$ be a non-Newtonian Fibonacci number and a non-Newtonian Lucas number, respectively. For $n, m \geq 0$, the following equalities hold:*

1) $\aleph\aleph F_n \dot{+} \aleph\aleph F_{n+1} = \aleph\aleph F_{n+2}$.

2) $\aleph\aleph L_n \dot{+} \aleph\aleph L_{n+1} = \aleph\aleph L_{n+2}$.

3) $\aleph\aleph F_{n-1} \dot{+} \aleph\aleph F_{n+1} = \aleph\aleph L_n$.

4) $\aleph\aleph F_{n+2} \dot{-} \aleph\aleph F_{n-2} = \aleph\aleph L_n$.

5) $\aleph\aleph F_n^{\dot{2}} \dot{+} \aleph\aleph F_{n+1}^{\dot{2}} = \aleph\aleph F_{2n+1}$.

6) $\aleph\aleph F_{n+1}^{\dot{2}} \dot{-} \aleph\aleph F_{n-1}^{\dot{2}} = \aleph\aleph F_{2n}$.

7) $\aleph\aleph F_n \dot{\times} \aleph\aleph F_m \dot{+} \aleph\aleph F_{n+1} \dot{\times} \aleph\aleph F_{m+1} = \aleph\aleph F_{n+m+1}$.

Proof. Based on addition and subtraction property of non-Newtonian real numbers, from (1), (2), (3) and (4), the proofs of 1), 2), 3) and 4) are clear.

5) From (5) and addition and multiplication property of non-Newtonian real numbers, we have

$$\aleph\aleph F_n^{\dot{2}} \dot{+} \aleph\aleph F_{n+1}^{\dot{2}}$$

$$
\begin{aligned}
&= \aleph\aleph F_n \dot{\times} \aleph\aleph F_n \dot{+} \aleph\aleph F_{n+1} \dot{\times} \aleph\aleph F_{n+1} \\
&= \alpha(F_n) \dot{\times} \alpha(F_n) \dot{+} \alpha(F_{n+1}) \dot{\times} \alpha(F_{n+1}) \\
&= \alpha\left\{\alpha^{-1}\alpha(F_n) \times \alpha^{-1}\alpha(F_n)\right\} \dot{+} \alpha\left\{\alpha^{-1}\alpha(F_{n+1}) \times \alpha^{-1}\alpha(F_{n+1})\right\} \\
&= \alpha\left\{\alpha^{-1}\alpha\left\{\alpha^{-1}\alpha(F_n) \times \alpha^{-1}\alpha(F_n)\right\} + \alpha^{-1}\alpha\left\{\alpha^{-1}\alpha(F_{n+1}) \times \alpha^{-1}\alpha(F_{n+1})\right\}\right\} \\
&= \alpha\left(F_n^2 + F_{n+1}^2\right) \\
&= \alpha(F_{2n+1}) \\
&= \aleph\aleph F_{2n+1}
\end{aligned}
$$

which is the desired result.

6) Similarly, by (6) we get

$$
\begin{aligned}
&\aleph\aleph F_{n+1}^{\dot{2}} \dot{-} \aleph\aleph F_{n-1}^{\dot{2}} \\
&= \aleph\aleph F_{n+1} \dot{\times} \aleph\aleph F_{n+1} \dot{-} \aleph\aleph F_{n-1} \dot{\times} \aleph\aleph F_{n-1} \\
&= \alpha(F_{n+1}) \dot{\times} \alpha(F_{n+1}) \dot{-} \alpha(F_{n-1}) \dot{\times} \alpha(F_{n-1}) \\
&= \alpha\left\{\alpha^{-1}\alpha(F_{n+1}) \times \alpha^{-1}\alpha(F_{n+1})\right\} \dot{-} \alpha\left\{\alpha^{-1}\alpha(F_{n-1}) \times \alpha^{-1}\alpha(F_{n-1})\right\} \\
&= \alpha\left\{\alpha^{-1}\alpha\left\{\alpha^{-1}\alpha(F_{n+1}) \times \alpha^{-1}\alpha(F_{n+1})\right\} + \alpha^{-1}\alpha\left\{\alpha^{-1}\alpha(F_{n-1}) \times \alpha^{-1}\alpha(F_{n-1})\right\}\right\} \\
&= \alpha\left(F_{n+1}^2 + F_{n-1}^2\right) \\
&= \alpha(F_{2n}) \\
&= \aleph\aleph F_{2n}.
\end{aligned}
$$

It results that $\aleph\aleph F_{n+1}^{\dot{2}} \dot{-} \aleph\aleph F_{n-1}^{\dot{2}} = \aleph\aleph F_{2n}$.

7) The equation (7) implies that

$$
\begin{aligned}
&\aleph\aleph F_n \dot{\times} \aleph\aleph F_m \dot{+} \aleph\aleph F_{n+1} \dot{\times} \aleph\aleph F_{m+1} \\
&= \alpha(F_n) \dot{\times} \alpha(F_m) \dot{+} \alpha(F_{n+1}) \dot{\times} \alpha(F_{m+1}) \\
&= \alpha\left\{\alpha^{-1}\alpha(F_n) \times \alpha^{-1}\alpha(F_m)\right\} \dot{+} \alpha\left\{\alpha^{-1}\alpha(F_{n+1}) \times \alpha^{-1}\alpha(F_{m+1})\right\} \\
&= \alpha\left\{\alpha^{-1}\alpha\left\{\alpha^{-1}\alpha(F_n) \times \alpha^{-1}\alpha(F_m)\right\} + \alpha^{-1}\alpha\left\{\alpha^{-1}\alpha(F_{n+1}) \times \alpha^{-1}\alpha(F_{m+1})\right\}\right\} \\
&= \alpha(F_nF_m + F_{n+1}F_{m+1}) \\
&= \alpha(F_{n+m+1}) \\
&= \aleph\aleph F_{n+m+1}
\end{aligned}
$$

as desired. □

Remark 2.3. *1) If we choose the identity function I instead of α in the definition of non-Newtonian Fibonacci and non-Newtonian Lucas numbers, then we obtain classical Fibonacci and Lucas numbers. Therefore, Theorem 2.2 generalizes related relations in the literature.*

2) Taking $\alpha = \exp$, we obtain some geometric relations as follows:

$$
e^{F_n+F_{n+1}} = e^{F_{n+2}}, e^{L_n+L_{n+1}} = e^{L_{n+2}}, e^{F_{n-1}+F_{n+1}} = e^{L_n}, e^{F_{n+2}-F_{n-2}} = e^{L_n},
$$

$$
e^{F_n^2+F_{n+1}^2} = e^{F_{2n+1}}, e^{F_{n+1}^2-F_{n-1}^2} = e^{F_{2n}}, e^{F_nF_m+F_{n+1}F_{m+1}} = e^{F_{n+m+1}},
$$

for $n, m \geq 0$.

In the next theorem, we derive the D'Ocagne identity including non-Newtonian Fibonacci numbers.

Theorem 2.4. *The D'Ocagne identity of the non-Newtonian Fibonacci numbers* $\aleph\aleph F_n$ *and* $\aleph\aleph F_m$ *is given as*

$$\aleph\aleph F_m \dot{\times} \aleph\aleph F_{n+1} \dot{-} \aleph\aleph F_{m+1} \dot{\times} \aleph\aleph F_n = \left(\dot{-}\dot{1}\right)^{\dot{n}} \dot{\times} \aleph\aleph F_{m-n}$$

for $n, m \geq 0$.

Proof. From the D'Ocagne identity (8) of Fibonacci numbers, we obtain that

$$\begin{aligned}
& \aleph\aleph F_m \dot{\times} \aleph\aleph F_{n+1} \dot{-} \aleph\aleph F_{m+1} \dot{\times} \aleph\aleph F_n \\
= \ & \alpha(F_m) \dot{\times} \alpha(F_{n+1}) \dot{-} \alpha(F_{m+1}) \dot{\times} \alpha(F_n) \\
= \ & \alpha\left\{\alpha^{-1}\alpha(F_m) \times \alpha^{-1}\alpha(F_{n+1})\right\} \dot{-} \alpha\left\{\alpha^{-1}\alpha(F_{m+1}) \times \alpha^{-1}\alpha(F_n)\right\} \\
= \ & \alpha\left\{\alpha^{-1}\alpha\left\{\alpha^{-1}\alpha(F_m) \times \alpha^{-1}\alpha(F_{n+1})\right\} - \alpha^{-1}\alpha\left\{\alpha^{-1}\alpha(F_{m+1}) \times \alpha^{-1}\alpha(F_n)\right\}\right\} \\
= \ & \alpha(F_m F_{n+1} - F_{m+1} F_n) \\
= \ & \alpha\left((-1)^n F_{m-n}\right) \\
= \ & \left(\dot{-}\dot{1}\right)^{\dot{n}} \dot{\times} \aleph\aleph F_{m-n}
\end{aligned}$$

which is what we wanted to see. □

Remark 2.5. *1) If we use the generator* $\alpha = I$ *in the definition of non-Newtonian Fibonacci and non-Newtonian Lucas numbers, Theorem 2.4 turns into the D'Ocagne identity for Fibonacci numbers.*

2) The generator $\alpha = \exp$ *yields the identity* $e^{F_m F_{n+1} - F_{m+1} F_n} = e^{(-1)^n F_{m-n}}$ *for* $n, m \geq 0$ *which we call the geometric D'Ocagne identity. In fact, the equalities*

$$\begin{aligned}
\aleph\aleph F_m \dot{\times} \aleph\aleph F_{n+1} \dot{-} \aleph\aleph F_{m+1} \dot{\times} \aleph\aleph F_n &= e^{F_m} \dot{\times} e^{F_{n+1}} \dot{-} e^{F_{m+1}} \dot{\times} e^{F_n} \\
&= e^{\left(\ln e^{F_m} \ln e^{F_{n+1}}\right) - \left(\ln e^{F_{m+1}} \ln e^{F_n}\right)} \\
&= e^{F_m F_{n+1} - F_{m+1} F_n}
\end{aligned}$$

and

$$\begin{aligned}
\left(\dot{-}\dot{1}\right)^{\dot{n}} \dot{\times} \aleph\aleph F_{m-n} &= \overbrace{e^{-1} \dot{\times} \ldots \dot{\times} e^{-1}}^{n \text{ times}} \dot{\times} e^{F_{m-n}} \\
&= e^{\overbrace{\ln e^{-1} \ldots \ln e^{-1}}^{n \text{ times}} \ln e^{F_{m-n}}} \\
&= e^{(-1)^n F_{m-n}}
\end{aligned}$$

explain it.

We are ready to give the Honsberger's identity of the non-Newtonian Fibonacci numbers.

Theorem 2.6. *The Honsberger's identity of the non-Newtonian Fibonacci numbers for* $n, m \geq 0$ *is given as*

$$\aleph\aleph F_{n+m} = \aleph\aleph F_{n-1} \dot{\times} \aleph\aleph F_m \dot{+} \aleph\aleph F_n \dot{\times} \aleph\aleph F_{m+1}.$$

Proof. Taking into account the Honsberger's identity (9) of Fibonacci numbers, one can easily reach that

$$\begin{aligned}
& \aleph\aleph F_{n-1} \dot{\times} \aleph\aleph F_m \dot{+} \aleph\aleph F_n \dot{\times} \aleph\aleph F_{m+1} \\
= \ & \alpha(F_{n-1}) \dot{\times} \alpha(F_m) \dot{-} \alpha(F_n) \dot{\times} \alpha(F_{m+1}) \\
= \ & \alpha\left\{\alpha^{-1}\alpha(F_{n-1}) \times \alpha^{-1}\alpha(F_m)\right\} \dot{+} \alpha\left\{\alpha^{-1}\alpha(F_n) \times \alpha^{-1}\alpha(F_{m+1})\right\} \\
= \ & \alpha\left\{\alpha^{-1}\alpha\left\{\alpha^{-1}\alpha(F_{n-1}) \times \alpha^{-1}\alpha(F_m)\right\} + \alpha^{-1}\alpha\left\{\alpha^{-1}\alpha(F_n) \times \alpha^{-1}\alpha(F_{m+1})\right\}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \alpha\left(F_{n-1}F_m + F_nF_{m+1}\right)\\
&= \alpha\left(F_{n+m}\right)\\
&= \aleph\aleph F_{n+m}.
\end{aligned}
$$

□

Remark 2.7. *1) Note that for the generator $\alpha = I$, we obtain the well-known Honsberger's identity.*
2) By putting $\alpha = \exp$, the geometric Honsberger's identity is derived as follows:

$$e^{F_{n+m}} = e^{F_{n-1}F_m + F_nF_{m+1}},$$

for $n, m \geq 0$.

The following theorem reveals the Binet formulas for a Fibonacci number and a Lucas number with respect to the non-Newtonian calculus.

Theorem 2.8. *Assume that $\aleph\aleph F_n$ and $\aleph\aleph L_n$ be a non-Newtonian Fibonacci and a non-Newtonian Lucas number, respectively. For $n \geq 0$, the Binet formulas for them are given by*

$$\aleph\aleph F_n = \frac{\dot{\gamma}^{\dot{n}} \dot{-} \dot{\beta}^{\dot{n}}}{\dot{\gamma} \dot{-} \dot{\beta}}\alpha$$

and

$$\aleph\aleph L_n = \dot{\gamma}^{\dot{n}} \dot{+} \dot{\beta}^{\dot{n}}$$

where $\dot{\gamma} = \frac{\dot{1} \dot{+} \dot{\sqrt{5}}}{\dot{2}}\alpha$ and $\dot{\beta} = \frac{\dot{1} \dot{-} \dot{\sqrt{5}}}{\dot{2}}\alpha$.

Proof. Considering subtraction and division operations in the set of non-Newtonian real numbers and by virtue of Binet formula (10) for Fibonacci numbers, we get

$$
\begin{aligned}
\frac{\dot{\gamma}^{\dot{n}} \dot{-} \dot{\beta}^{\dot{n}}}{\dot{\gamma} \dot{-} \dot{\beta}}\alpha
&= \alpha\left\{\frac{\alpha^{-1}\left(\dot{\gamma}^{\dot{n}} \dot{-} \dot{\beta}^{\dot{n}}\right)}{\alpha^{-1}\left(\dot{\gamma} \dot{-} \dot{\beta}\right)}\right\}\\
&= \alpha\left\{\frac{\alpha^{-1}\left(\overbrace{\left(\dot{\gamma} \dot{\times} \dot{\gamma} \dot{\times} \ldots \dot{\times} \dot{\gamma}\right)}^{n \text{ times}} \dot{-} \overbrace{\left(\dot{\beta} \dot{\times} \dot{\beta} \dot{\times} \ldots \dot{\times} \dot{\beta}\right)}^{n \text{ times}}\right)}{\alpha^{-1}\left(\dot{\gamma} \dot{-} \dot{\beta}\right)}\right\}\\
&= \alpha\left\{\frac{\alpha^{-1}\left(\alpha\left[\left(\alpha^{-1}\left(\dot{\gamma}\right)\right)^n - \left(\alpha^{-1}\left(\dot{\beta}\right)\right)^n\right]\right)}{\alpha^{-1}\left(\alpha\left[\alpha^{-1}\left(\dot{\gamma}\right) - \alpha^{-1}\left(\dot{\beta}\right)\right]\right)}\right\}\\
&= \alpha\left\{\frac{\gamma^n - \beta^n}{\gamma - \beta}\right\}
\end{aligned}
$$

$$= \alpha(F_n)$$
$$= \aleph\aleph F_n.$$

On the other hand, by doing the necessary calculations and using addition operation in the set of non-Newtonian real numbers and Binet formula (11) for Lucas numbers, one can uncomplicatedly see that

$$\begin{aligned}
\dot{\gamma}^{\dot{n}} \dot{+} \dot{\beta}^{\dot{n}} &= \alpha\left[\alpha^{-1}\left(\overbrace{\left(\dot{\gamma} \dot{\times} \dot{\gamma} \dot{\times} \ldots \dot{\times} \dot{\gamma}\right)}^{n \text{ times}} \dot{+} \overbrace{\left(\dot{\beta} \dot{\times} \dot{\beta} \dot{\times} \ldots \dot{\times} \dot{\beta}\right)}^{n \text{ times}}\right)\right] \\
&= \alpha\left[\alpha^{-1}\left(\alpha\left[\left(\alpha^{-1}\left(\dot{\gamma}\right)\right)^{n}\right] \dot{+} \alpha\left[\left(\alpha^{-1}\left(\dot{\beta}\right)\right)^{n}\right]\right)\right] \\
&= \alpha\left[\alpha^{-1}\left(\alpha\left[\alpha^{-1}\alpha\left[\left(\alpha^{-1}\left(\dot{\gamma}\right)\right)^{n}\right] + \alpha^{-1}\alpha\left[\left(\alpha^{-1}\left(\dot{\beta}\right)\right)^{n}\right]\right]\right)\right] \\
&= \alpha\left(\gamma^{n} + \beta^{n}\right) \\
&= \alpha\left(L_n\right) \\
&= \aleph\aleph L_n.
\end{aligned}$$

□

Remark 2.9. *1) We draw attention that the Binet formulas for non-Newtonian Fibonacci and Lucas numbers are generalizations of Binet formulas for Fibonacci and Lucas numbers, respectively which are obtained by putting the generator $\alpha = I$.*

2) According to geometric arithmetic we obtain the following formulas and call them the geometric Binet formula for Fibonacci numbers and the geometric Binet formula for Lucas numbers, respectively:

$$e^{F_n} = e^{\frac{\gamma^n - \beta^n}{\gamma - \beta}}, \quad e^{L_n} = e^{\gamma^n + \beta^n}$$

for $n \geq 0$.

The next theorem is non-Newtonian versions of Cassini's identities for Fibonacci and Lucas numbers.

Theorem 2.10. *The followings are the Cassini's identities for $\aleph\aleph F_n$ and $\aleph\aleph L_n$ for $n \geq 1$:*

1) $\aleph\aleph F_n^{\dot{2}} \dot{-} \aleph\aleph F_{n-1} \dot{\times} \aleph\aleph F_{n+1} = \left(\dot{-}\dot{1}\right)^{\dot{n}\dot{+}\dot{1}}$.

2) $\aleph\aleph L_n^{\dot{2}} \dot{-} \aleph\aleph L_{n-1} \dot{\times} \aleph\aleph L_{n+1} = \dot{5} \dot{\times} \left(\dot{-}\dot{1}\right)^{\dot{n}}$.

Proof. 1) Using Cassini's identity (15) and making the necessary calculations we get the subsequent result.

$$\begin{aligned}
& \aleph\aleph F_n^{\dot{2}} \dot{-} \aleph\aleph F_{n-1} \dot{\times} \aleph\aleph F_{n+1} \\
= \; & \aleph\aleph F_n \dot{\times} \aleph\aleph F_n \dot{-} \aleph\aleph F_{n-1} \dot{\times} \aleph\aleph F_{n+1} \\
= \; & \alpha\left(F_n\right) \dot{\times} \alpha\left(F_n\right) \dot{-} \alpha\left(F_{n-1}\right) \dot{\times} \alpha\left(F_{n+1}\right) \\
= \; & \alpha\left\{\alpha^{-1}\alpha\left(F_n\right) \times \alpha^{-1}\alpha\left(F_n\right)\right\} \dot{-} \alpha\left\{\alpha^{-1}\alpha\left(F_{n-1}\right) \times \alpha^{-1}\alpha\left(F_{n+1}\right)\right\} \\
= \; & \alpha\left\{\alpha^{-1}\alpha\left\{\alpha^{-1}\alpha\left(F_n\right) \times \alpha^{-1}\alpha\left(F_n\right)\right\} - \alpha^{-1}\alpha\left\{\alpha^{-1}\alpha\left(F_{n-1}\right) \times \alpha^{-1}\alpha\left(F_{n+1}\right)\right\}\right\} \\
= \; & \alpha\left(F_n^2 - F_{n-1}F_{n+1}\right) \\
= \; & \alpha\left((-1)^{n+1}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \alpha\left(\overbrace{(-1)\ldots(-1)}^{n+1 \text{ times}}\right) \\
&= \alpha\left(\overbrace{\alpha^{-1}(\alpha(-1))\ldots\alpha^{-1}(\alpha(-1))}^{n+1 \text{ times}}\right) \\
&= \alpha(-1)\,\dot{\times}\ldots\dot{\times}\,\alpha(-1) \\
&= \overbrace{\left(\dot{-}\dot{1}\right)\dot{\times}\ldots\dot{\times}\left(\dot{-}\dot{1}\right)}^{n+1 \text{ times}} \\
&= \left(\dot{-}\dot{1}\right)^{n\dot{+}\dot{1}}
\end{aligned}
$$

2) Using the identity (16) for the proof of 2), we get the results. □

Let's continue with the Catalan identities for F_n and L_n in non-Newtonian sense.

Theorem 2.11. *For $n, r \geq 1$, the Catalan identities for $\aleph\aleph F_n$ and $\aleph\aleph L_n$ are as follows:*

1) $\aleph\aleph F_n^{\dot{2}} \dot{-} \aleph\aleph F_{n-r}\,\dot{\times}\aleph\aleph F_{n+r} = \left(\dot{-}\dot{1}\right)^{n\dot{-}\dot{r}}\dot{\times}\,\aleph\aleph F_r^{\dot{2}}$.

2) $\aleph\aleph L_n^{\dot{2}} \dot{-} \aleph\aleph L_{n-r}\,\dot{\times}\aleph\aleph L_{n+r} = \dot{5}\,\dot{\times}\left(\dot{-}\dot{1}\right)^{n\dot{-}\dot{r}}\dot{\times}\,\aleph\aleph F_r^{\dot{2}}$.

Proof. 1) After some elementary calculations, it can be computed similar to the property 1) in Theorem 2.10 taking it into account (17) and the proof is straightforward.

2) Using Catalan identity (18) and making simple computations we compute the following expression:

$$
\begin{aligned}
& \aleph\aleph L_n^{\dot{2}} \dot{-} \aleph\aleph L_{n-r}\,\dot{\times}\,\aleph\aleph L_{n+r} \\
&= \aleph\aleph L_n\,\dot{\times}\,\aleph\aleph L_n \dot{-} \aleph\aleph L_{n-r}\,\dot{\times}\,\aleph\aleph L_{n+r} \\
&= \alpha(L_n)\,\dot{\times}\,\alpha(L_n) \dot{-} \alpha(L_{n-r})\,\dot{\times}\,\alpha(L_{n+r}) \\
&= \alpha\left\{\alpha^{-1}\alpha(L_n)\times\alpha^{-1}\alpha(L_n)\right\} \dot{-} \alpha\left\{\alpha^{-1}\alpha(L_{n-r})\times\alpha^{-1}\alpha(L_{n+r})\right\} \\
&= \alpha\left\{\alpha^{-1}\alpha\left\{\alpha^{-1}\alpha(L_n)\times\alpha^{-1}\alpha(L_n)\right\} - \alpha^{-1}\alpha\left\{\alpha^{-1}\alpha(L_{n-r})\times\alpha^{-1}\alpha(L_{n+r})\right\}\right\} \\
&= \alpha\left(L_n^2 - L_{n-r}L_{n+r}\right) \\
&= \alpha\left(5(-1)^{n-r}F_r^2\right) \\
&= \dot{5}\,\dot{\times}\left(\dot{-}\dot{1}\right)^{n\dot{-}\dot{r}}\dot{\times}\,\aleph\aleph F_r^{\dot{2}}
\end{aligned}
$$

which ends the proof. □

Remark 2.12. *1) Notice that Theorem 2.10 and Theorem 2.11 extend the known Cassini's identity and Catalan identity, respectively.*

2) We say that Theorem 2.10 is a special case of Theorem 2.11 choosing $r = 1$.

3) Substituting exp *for the generator α, the following identities are added to the literature:*

$$
\begin{aligned}
e^{F_n^2-F_{n-1}F_{n+1}} &= e^{(-1)^n} \quad \textit{(Geometric Cassini's identity for Fibonacci numbers),} \\
e^{L_n^2-L_{n-1}L_{n+1}} &= e^{5(-1)^n} \quad \textit{(Geometric Cassini's identity for Lucas numbers),} \\
e^{F_n^2-F_{n-r}F_{n+r}} &= e^{(-1)^{n-r}F_r^2} \quad \textit{(Geometric Catalan identity for Fibonacci numbers),}
\end{aligned}
$$

$$e^{L_n^2-L_{n-r}L_{n+r}} = e^{5(-1)^{n-r}F_r^2} \text{ (Geometric Catalan identity for Lucas numbers)},$$

for $n, r \geq 1$.

In the following theorem, we explore the Gelin-Cesàro identity in non-Newtonian sense.

Theorem 2.13. *For* $n \geq 0$*, the Gelin-Cesàro identity of the non-Newtonian Fibonacci numbers* $\aleph\aleph F_n$ *is given as*

$$\aleph\aleph F_n^{\dot{4}} \dot{-} \aleph\aleph F_{n-1} \dot{\times} \aleph\aleph F_{n-2} \dot{\times} \aleph\aleph F_{n+1} \dot{\times} \aleph\aleph F_{n+2} = \dot{1}.$$

Proof. In view of (12), we get

$$\begin{aligned}
& \aleph\aleph F_n^{\dot{4}} \dot{-} \aleph\aleph F_{n-1} \dot{\times} \aleph\aleph F_{n-2} \dot{\times} \aleph\aleph F_{n+1} \dot{\times} \aleph\aleph F_{n+2} \\
= & \aleph\aleph F_n \dot{\times} \aleph\aleph F_n \dot{\times} \aleph\aleph F_n \dot{\times} \aleph\aleph F_n \\
& \dot{-} \aleph\aleph F_{n-1} \dot{\times} \aleph\aleph F_{n-2} \dot{\times} \aleph\aleph F_{n+1} \dot{\times} \aleph\aleph F_{n+2} \\
= & \alpha(F_n) \dot{\times} \alpha(F_n) \dot{\times} \alpha(F_n) \dot{\times} \alpha(F_n) \dot{-} \alpha(F_{n-1}) \dot{\times} \alpha(F_{n-2}) \dot{\times} \alpha(F_{n+1}) \dot{\times} \alpha(F_{n+2}) \\
= & \alpha\left\{\alpha^{-1}\alpha(F_n) \times \alpha^{-1}\alpha(F_n) \times \alpha^{-1}\alpha(F_n) \times \alpha^{-1}\alpha(F_n)\right\} \\
& \dot{-} \alpha\left\{\alpha^{-1}\alpha(F_{n-1}) \times \alpha^{-1}\alpha(F_{n-2}) \times \alpha^{-1}\alpha(F_{n+1}) \times \alpha^{-1}\alpha(F_{n+2})\right\} \\
= & \alpha\left\{\begin{array}{c} \alpha^{-1}\alpha\left\{\alpha^{-1}\alpha(F_n) \times \alpha^{-1}\alpha(F_n) \times \alpha^{-1}\alpha(F_n) \times \alpha^{-1}\alpha(F_n)\right\} \\ -\alpha^{-1}\alpha\left\{\alpha^{-1}\alpha(F_{n-1}) \times \alpha^{-1}\alpha(F_{n-2}) \times \alpha^{-1}\alpha(F_{n+1}) \times \alpha^{-1}\alpha(F_{n+2})\right\} \end{array}\right\} \\
= & \alpha\left(F_n^4 - F_{n-1}F_{n-2}F_{n+1}F_{n+2}\right) \\
= & \alpha(1) \\
= & \dot{1}.
\end{aligned}$$

The proof is completed. □

Remark 2.14. *1) We note that Theorem 2.13 turns into the known Gelin-Cesàro identity when the generator is chosen as* $\alpha = I$.

2) According to geometric arithmetic, the Gelin-Cesàro identity of the non-Newtonian Fibonacci numbers turns into

$$e^{F_n^4 - F_{n-1}F_{n-2}F_{n+1}F_{n+2}} = e$$

where $n \geq 0$.

Now, we derive the Melham's identity for $\aleph\aleph F_n$'s.

Theorem 2.15. *For* $n \geq 0$*, the Melham's identity of the non-Newtonian Fibonacci numbers* $\aleph\aleph F_n$ *is given by*

$$\aleph\aleph F_{n+1} \dot{\times} \aleph\aleph F_{n+2} \dot{\times} \aleph\aleph F_{n+6} \dot{-} \aleph\aleph F_{n+3}^{\dot{3}} = \left(\dot{-}\dot{1}\right)^{\dot{n}} \dot{\times} \aleph\aleph F_n.$$

Proof. From (13), it can be seen that

$$\begin{aligned}
& \aleph\aleph F_{n+1} \dot{\times} \aleph\aleph F_{n+2} \dot{\times} \aleph\aleph F_{n+6} \dot{-} \aleph\aleph F_{n+3}^{\dot{3}} \\
= & \aleph\aleph F_{n+1} \dot{\times} \aleph\aleph F_{n+2} \dot{\times} \aleph\aleph F_{n+6} \dot{-} \aleph\aleph F_{n+3} \dot{\times} \aleph\aleph F_{n+3} \dot{\times} \aleph\aleph F_{n+3} \\
= & \alpha(F_{n+1}) \dot{\times} \alpha(F_{n+2}) \dot{\times} \alpha(F_{n+6}) \dot{-} \alpha(F_{n+3}) \dot{\times} \alpha(F_{n+3}) \dot{\times} \alpha(F_{n+3}) \\
= & \alpha\left\{\alpha^{-1}\alpha(F_{n+1}) \times \alpha^{-1}\alpha(F_{n+2}) \times \alpha^{-1}\alpha(F_{n+6})\right\} \\
& \dot{-} \alpha\left\{\alpha^{-1}\alpha(F_{n+3}) \times \alpha^{-1}\alpha(F_{n+3}) \times \alpha^{-1}\alpha(F_{n+3})\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \alpha \left\{ \begin{array}{c} \alpha^{-1}\alpha\left\{\alpha^{-1}\alpha\left(F_{n+1}\right)\times\alpha^{-1}\alpha\left(F_{n+2}\right)\times\alpha^{-1}\alpha\left(F_{n+6}\right)\right\} \\ -\alpha^{-1}\alpha\left\{\alpha^{-1}\alpha\left(F_{n+3}\right)\times\alpha^{-1}\alpha\left(F_{n+3}\right)\times\alpha^{-1}\alpha\left(F_{n+3}\right)\right\} \end{array} \right\} \\
&= \alpha\left(F_{n+1}F_{n+2}F_{n+6}-F_{n+3}^{3}\right) \\
&= \alpha\left(\left(-1\right)^{n}F_{n}\right) \\
&= \alpha\left(\alpha^{-1}\alpha\left[\left(-1\right)^{n}\right]\times\alpha^{-1}\alpha\left(F_{n}\right)\right) \\
&= \alpha\left[\left(-1\right)^{n}\right]\dot{\times}\,\alpha\left(F_{n}\right) \\
&= \left(\dot{-}\dot{1}\right)^{\dot{n}}\dot{\times}\,\aleph\aleph F_{n},
\end{aligned}
$$

as desired. □

Remark 2.16. *1) Notice that if we choose the generator as $\alpha = I$, the last theorem produces the Melhan's identity provided for the F_n's.*

2) The Melham's identity of the non-Newtonian Fibonacci numbers is

$$e^{F_{n+1}F_{n+2}F_{n+6}-F_{n+3}^{3}} = e^{(-1)^{n}F_{n}}$$

for $n \geq 0$ in the geometric arithmetic.

The subsequent theorem introduces some summing formulas of non-Newtonian Fibonacci numbers.

Theorem 2.17. *We have the following formulas where the symbol ${}_{\alpha}\sum\limits_{k=0}^{n}$ denotes the finite sum according to $\alpha-$arithmetic and $n \geq 0$:*

1) ${}_{\alpha}\sum\limits_{k=0}^{n}\aleph\aleph F_{k} = \aleph\aleph F_{n+2}\,\dot{-}\,\dot{1}.$
2) ${}_{\alpha}\sum\limits_{k=0}^{n}\aleph\aleph F_{2k} = \aleph\aleph F_{2n+1}\,\dot{-}\,\dot{1}.$
3) ${}_{\alpha}\sum\limits_{k=0}^{n}\aleph\aleph F_{2k+1} = \aleph\aleph F_{2n+2}.$

Proof. If we use (14), it is obtained that

$$
\begin{aligned}
{}_{\alpha}\sum_{k=0}^{n}\aleph\aleph F_{k} &= \alpha\left(\sum_{k=0}^{n}\alpha^{-1}\left(\aleph\aleph F_{k}\right)\right) = \alpha\left(\sum_{k=0}^{n}F_{k}\right) \\
&= \alpha\left(F_{n+2}-1\right) = \alpha\left(\alpha^{-1}\alpha\left(F_{n+2}\right)-\alpha^{-1}\alpha\left(1\right)\right) \\
&= \alpha\left(F_{n+2}\right)\dot{-}\,\alpha\left(1\right) = \aleph\aleph F_{n+2}\,\dot{-}\,\dot{1},
\end{aligned}
$$

$$
\begin{aligned}
{}_{\alpha}\sum_{k=0}^{n}\aleph\aleph F_{2k} &= \alpha\left(\sum_{k=0}^{n}\alpha^{-1}\left(\aleph\aleph F_{2k}\right)\right) = \alpha\left(\sum_{k=0}^{n}F_{2k}\right) \\
&= \alpha\left(F_{2n+1}-1\right) = \alpha\left(\alpha^{-1}\alpha\left(F_{2n+1}\right)-\alpha^{-1}\alpha\left(1\right)\right) \\
&= \alpha\left(F_{2n+1}\right)\dot{-}\,\alpha\left(1\right) = \aleph\aleph F_{2n+1}\,\dot{-}\,\dot{1},
\end{aligned}
$$

$$
\begin{aligned}
{}_{\alpha}\sum_{k=0}^{n}\aleph\aleph F_{2k+1} &= \alpha\left(\sum_{k=0}^{n}\alpha^{-1}\left(\aleph\aleph F_{2k+1}\right)\right) = \alpha\left(\sum_{k=0}^{n}F_{2k+1}\right) \\
&= \alpha\left(F_{2n+2}\right) = \aleph\aleph F_{2n+2}.
\end{aligned}
$$

This finalizes the proof of 1), 2) and 3). □

Remark 2.18. *1) Theorem 2.17 generalizes the formulas of sum of the first n terms, sum of the first n even terms, sum of the first n odd terms of Fibonacci numbers.*

2) The generator $\alpha = \exp$ produces some new formulas as follows:

$$e^{\sum_{k=0}^{n} F_k} = e^{F_{n+2}-1}, \quad e^{\sum_{k=0}^{n} F_{2k}} = e^{F_{2n+1}-1}, \quad e^{\sum_{k=0}^{n} F_{2k+1}} = e^{F_{2n+2}}$$

for $n \geq 0$.

After obtaining these famous identities and formulas, in closing of this article, we present generating functions of the non-Newtonian Fibonacci numbers and non-Newtonian Lucas numbers.

Theorem 2.19. *Generating function of the non-Newtonian Fibonacci numbers is*

$$g_{\aleph\aleph F} : \mathbb{R}_\alpha \to \mathbb{R}_\alpha, \quad g_{\aleph\aleph F}(y) = \frac{y}{\dot{1} \dot{-} y \dot{-} y^{\dot{2}}}\alpha.$$

Proof. Assume that the generating function of the non-Newtonian Fibonacci number $\aleph\aleph F_n$ has the form

$$g_{\aleph\aleph F}(y) =_\alpha \sum_{n=0}^{\infty} \left(\aleph\aleph F_n \dot{\times} y^{\dot{n}}\right),$$

where the symbol ${}_\alpha\sum_{n=0}^{\infty}$ denotes the non-Newtonian real number series which can be found in [11].

Then, after the needed calculations, we get the following equations:

$$\begin{aligned}
g_{\aleph\aleph F}(y) &= {}_\alpha\sum_{n=0}^{\infty} \left(\aleph\aleph F_n \dot{\times} y^{\dot{n}}\right) \\
&= \dot{0} \dot{+} y \dot{+} {}_\alpha\sum_{n=2}^{\infty} \left(\aleph\aleph F_n \dot{\times} y^{\dot{n}}\right) \\
&= y \dot{+} {}_\alpha\sum_{n=2}^{\infty} \left(\aleph\aleph F_{n-1} \dot{\times} y^{\dot{n}}\right) \dot{+} {}_\alpha\sum_{n=2}^{\infty} \left(\aleph\aleph F_{n-2} \dot{\times} y^{\dot{n}}\right), \\
y \dot{\times} g_{\aleph\aleph F}(y) &= {}_\alpha\sum_{n=1}^{\infty} \left(\aleph\aleph F_n \dot{\times} y^{\dot{n}\dot{+}\dot{1}}\right) =_\alpha \sum_{n=2}^{\infty} \left(\aleph\aleph F_{n-1} \dot{\times} y^{\dot{n}}\right), \\
y^{\dot{2}} \dot{\times} g_{\aleph\aleph F}(y) &= {}_\alpha\sum_{n=0}^{\infty} \left(\aleph\aleph F_n \dot{\times} y^{\dot{n}\dot{+}\dot{2}}\right) =_\alpha \sum_{n=2}^{\infty} \left(\aleph\aleph F_{n-2} \dot{\times} y^{\dot{n}}\right).
\end{aligned}$$

So, it follows that

$$\begin{aligned}
&\left(\dot{1} \dot{-} y \dot{-} y^{\dot{2}}\right) \dot{\times} g_{\aleph\aleph F}(y) \\
= \; & g_{\aleph\aleph F}(y) \dot{-} \left(y \dot{\times} g_{\aleph\aleph F}(y)\right) \dot{-} \left(y^{\dot{2}} \dot{\times} g_{\aleph\aleph F}(y)\right) \\
= \; & \left[y \dot{+} {}_\alpha\sum_{n=2}^{\infty} \left(\aleph\aleph F_{n-1} \dot{\times} y^{\dot{n}}\right) \dot{+} {}_\alpha\sum_{n=2}^{\infty} \left(\aleph\aleph F_{n-2} \dot{\times} y^{\dot{n}}\right)\right] \\
& \dot{-} \left[{}_\alpha\sum_{n=2}^{\infty} \left(\aleph\aleph F_{n-1} \dot{\times} y^{\dot{n}}\right)\right] \dot{-} \left[{}_\alpha\sum_{n=2}^{\infty} \left(\aleph\aleph F_{n-2} \dot{\times} y^{\dot{n}}\right)\right] \\
= \; & y.
\end{aligned}$$

Hereupon, we derive that the function $g_{\aleph\aleph F}(y) = \frac{y}{\dot{1} \dot{-} y \dot{-} y^{\dot{2}}}\alpha$ as the desired result. □

Theorem 2.20. *Generating function of the non-Newtonian Lucas numbers is*

$$g_{\aleph\aleph L} : \mathbb{R}_\alpha \to \mathbb{R}_\alpha, \;\; g_{\aleph\aleph L}(y) = \frac{\dot{2} \dot{-} y}{\dot{1} \dot{-} y \dot{-} y^{\dot{2}}} \alpha.$$

Proof. Suppose that the generating function of the non-Newtonian Lucas number $\aleph\aleph L_n$ has the form

$$g_{\aleph\aleph L}(y) =_\alpha \sum_{n=0}^{\infty} \left(\aleph\aleph L_n \dot{\times} y^{\dot{n}} \right).$$

Thus, with some computations, we get $g_{\aleph\aleph L}(y)$, $y \dot{\times} g_{\aleph\aleph L}(y)$ and $y^{\dot{2}} \dot{\times} g_{\aleph\aleph L}(y)$, as follows:

$$\begin{aligned}
g_{\aleph\aleph L}(y) &= \dot{2} \dot{+} y \dot{+} \;_\alpha\sum_{n=2}^{\infty} \left(\aleph\aleph L_n \dot{\times} y^{\dot{n}} \right) \\
&= \dot{2} \dot{+} y \dot{+} \;_\alpha\sum_{n=2}^{\infty} \left(\aleph\aleph L_{n-1} \dot{\times} y^{\dot{n}} \right) \dot{+} \;_\alpha\sum_{n=2}^{\infty} \left(\aleph\aleph L_{n-2} \dot{\times} y^{\dot{n}} \right), \\
y \dot{\times} g_{\aleph\aleph L}(y) &= \;_\alpha\sum_{n=0}^{\infty} \left(\aleph\aleph L_n \dot{\times} y^{\dot{n} \dot{+} \dot{1}} \right) \\
&= \dot{2} \dot{\times} y \dot{+} \;_\alpha\sum_{n=1}^{\infty} \left(\aleph\aleph L_n \dot{\times} y^{\dot{n} \dot{+} \dot{1}} \right) \\
&= \dot{2} \dot{\times} y \dot{+} \;_\alpha\sum_{n=2}^{\infty} \left(\aleph\aleph L_{n-1} \dot{\times} y^{\dot{n}} \right), \\
y^{\dot{2}} \dot{\times} g_{\aleph\aleph L}(y) &= \;_\alpha\sum_{n=0}^{\infty} \left(\aleph\aleph L_n \dot{\times} y^{\dot{n} \dot{+} \dot{2}} \right) =_\alpha \sum_{n=2}^{\infty} \left(\aleph\aleph L_{n-2} \dot{\times} y^{\dot{n}} \right).
\end{aligned}$$

So, one can easily see that

$$\begin{aligned}
& \left(\dot{1} \dot{-} y \dot{-} y^{\dot{2}} \right) \dot{\times} g_{\aleph\aleph L}(y) \\
= \;& g_{\aleph\aleph L}(y) \dot{-} \left(y \dot{\times} g_{\aleph\aleph L}(y) \right) \dot{-} \left(y^{\dot{2}} \dot{\times} g_{\aleph\aleph L}(y) \right) \\
= \;& \dot{2} \dot{+} y \dot{+} \;_\alpha\sum_{n=2}^{\infty} \left(\aleph\aleph L_{n-1} \dot{\times} y^{\dot{n}} \right) \dot{+} \;_\alpha\sum_{n=2}^{\infty} \left(\aleph\aleph L_{n-2} \dot{\times} y^{\dot{n}} \right) \\
& \dot{-} \left[\dot{2} \dot{\times} y \dot{+} \;_\alpha\sum_{n=2}^{\infty} \left(\aleph\aleph L_{n-1} \dot{\times} y^{\dot{n}} \right) \right] \dot{-} \left[_\alpha\sum_{n=2}^{\infty} \left(\aleph\aleph L_{n-2} \dot{\times} y^{\dot{n}} \right) \right] \\
= \;& \dot{2} \dot{-} y.
\end{aligned}$$

It results that $g_{\aleph\aleph L}(y) = \frac{\dot{2} \dot{-} y}{\dot{1} \dot{-} y \dot{-} y^{\dot{2}}} \alpha$. The proof is completed. □

Remark 2.21. *1) The generating functions $g_{\aleph\aleph F}$ and $g_{\aleph\aleph L}$ are analogues of the generating functions of F_n's and L_n's defined by $g_F(y) = \frac{y}{1-y-y^2}$ and $g_L(y) = \frac{2-y}{1-y-y^2}$, respectively. In fact, it is enough to write $\alpha = I$.*

2) Geometric generating functions of Fibonacci and Lucas numbers are

$$g_{\aleph\aleph GF}(y) = e^{\frac{\ln y}{1 - \ln y - (\ln y)^2}}$$

and

$$g_{\aleph\aleph GL}(y) = e^{\frac{2-\ln y}{1-\ln y-(\ln y)^2}},$$

respectively.

3. Conclusion and Future Works

The aim of this article is to derive Fibonacci and Lucas numbers in the non-Newtonian sense and to construct some important properties and identities of them such as Binet's formula, summing formulas, ngenerating functions, Cassini's identity and Catalan's identity. The study fills the gap here by introducing non-Newtonian Fibonacci and non-Newtonian Lucas numbers to the most existing literature by combining the definitions of Fibonacci numbers, Lucas numbers and non-Newtonian real numbers. Due to the fact that Fibonacci numbers are used in encryption theory, we believe that our findings contribute to researchers for future works as a new perspective.

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