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Certain properties of η-Ricci soliton on η-Einstein para-Kenmotsu manifolds

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Abstract. The objective of present research paper is to be investigate the geometric properties of η-Ricci solitons on η-Einstein para-Kenmotsu manifolds. In this manner, we consider η-Ricci solitons on η-Einstein para-Kenmotsu manifolds satistfying $R.S = 0$. Further, we obtain results for η-Ricci solitons on η-Einstein para-Kenmotsu manifolds with quasi-conformal flat property. Moreover, we get result for η-Ricci solitins in η-Einstein para-Kenmotsu manifolds admitting Codazzi type of Ricci tensor and cyclic parallel Ricci tensor, η-quasi-conformally semi-symmetric, η-Ricci symmetric and quasi-conformally Ricci semi-symmetric. At last, we construct an example of a such manifold which justify the existence of proper η-Ricci solitons.

1. Introduction

In 1982, Hamilton [19] introduced the notion of the Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold

$$\frac{\partial}{\partial t} g_{it}(t) = -2R_{ij} \tag{1}$$

A Ricci soliton is a natural generalization of an Einstein metric and is defined on a Riemannian manifold (M, g) [6] . A Ricci soliton is a triple (g, V, λ) with g a Riemannian metric, V a vector field (called the potential vector fields), and λ a real scalar such that

$$\pounds_V g + 2S + 2\lambda g = 0 \tag{2}$$

where S is a Ricci tensor of M and $\pounds_V g$ denotes the Lie derivative operator along the vector field V. The Ricci soliton is said to be shrinking, steady and expanding accordingly as λ is negative, zero and positive, respectively [20]. A Ricci soliton with V zero is reduced to Einstein equation. Metrices satisfying (2) is interesting and useful in Physics and is often referred as quasi-Einstein ([27], [28]). Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t} g = -2S$, projected from the space of metrices onto its quotient modulo

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diffeomorphisms and scalings, and often arise blow-up limits for the Ricci flow on compact manifolds. Theoretical physicists have also been looking into the equation of Ricci soliton in relation with string theory. The initial contributions in this direction is due to Friedmannn [7], who discusses some aspects of it. Ricci solitons were introduced in Riemannian geometry [19], as the self-similar solutions of the Ricci flow, and play an important role in understanding its singularities. Ricci solitons have been studied in many contexts by several authors such as ([22], [26], [31], [32], [33]) and many others. As a generalization of Ricci-soliton, the notion of η-Ricci soliton introduced by J. T. Cho and M. Kimura [10], which was treated by C. Calin and M.Crasmareanu on Hopf hypersurfaces in complex space forms [6]. An η-Ricci soliton is a tuple (g, V, λ, μ), where V is a vector field on M, and λ, μ are constants and g is a Riemannian (or pseudo-Riemannian) metric satisfying the equation

$$\pounds_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0 \tag{3}$$

where S is the Ricci tensor associated to g. In particular, if μ=0 then the notion of η-Ricci solitons (g, V, λ, μ) reduces to the notion of Ricci soliton (g, V, λ). If $\mu \neq 0$, then the η-Ricci solitons are called proper η- Ricci solitons. We refer to ([11], [16], [24], [25], [35]) and references there in for a survey and further references on the geometry of Ricci solitons on pseudo-Riemannian manifolds. Recently an η- Ricci soliton has been studied by several authors such as ([3], [4], [5], [8], [15], [18], [22] [29], [30]) and they found many interesting geometric properties. These above results motivated me to study η-Ricci solitons on η-Einstein para-Kenmotsu manifolds satisfying certain curvature conditions. The paper is organized in the following way. In section 2, we give a brief introduction of an η-Einstein para- Kenmotsu manifold. Section 3 deals with the study of Ricci solitons and η-Ricci solitons in η-Einstein para-Kenmotsu manifolds. In section 4, we study η-Ricci solitons on η-Einstein para-Kenmotsu manifolds satisfying $R.S$=0. η-Ricci solitons on quasi-conformally flat η-Einstein para-Kenmotsu manifolds have been studied in section 5. In section 6, we study η-Ricci solitons in η- Einstein para-Kenmotsu manifolds admitting Codazzi type of Ricci tensor and cyclic parallel Ricci tensor. Section 7 is devoted to the study of ϕ-quasi conformally semi-symmetric η-Ricci solitons on η-Einstein para-Kenmotsu manifolds. Beside these we study η- Ricci solitons on ϕ-Ricci symmetric η-Einstein para-Kenmotsu manifolds. Also η-Ricci solitons on conformally Ricci semi-symmetric η-Einstein para-Kenmotsu manifolds has been considered. Finally we construct a 3-dimensional example of a η-Einstein para-Kenmotsu manifold which admits an η-Ricci soliton.

2. Preliminaries

Let (M^n, g) be n-dimensional smooth manifold equipped with an almost paracontact metric structure (ϕ, ξ, η, g) that is ϕ is a tensor field of type $(1, 1)$, ξ is a vector field, η is a 1-form and g is a pseudo-Riemannnian metric such that

$$\phi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = 1, \tag{4}$$

which implies

$$\phi\xi = 0, \quad \eta(\phi X) = 0 \tag{5}$$

Then M^n admits a pseudo-Riemannian metric g, such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{6}$$

$$g(X, \xi) = \eta(X), \tag{7}$$

$$g(\phi X, Y) = -g(X, \phi Y), \tag{8}$$

for all vector fields X, $Y \in TM^n$ [23]. Then (M^n, g) is called almost paracontact metric manifold.

Definition 2.1. An almost paracontact metric manifold M^n is called almost para-Kenmotsu manifold if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)\phi X, \tag{9}$$

for any vector field X, Y on TM^n [23]. A normal almost para-Kenmotsu manifold is called a para-Kenmotsu manifold. From the above equation it follows that

$$\nabla_X \xi = X - \eta(X)\xi, \tag{10}$$

$$(\nabla_X \eta)(Y) = g(X,Y) - \eta(X)\eta(Y), \tag{11}$$

$$\eta(\nabla_X \xi) = 0, \quad \nabla_\xi \xi = 0, \tag{12}$$

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = -g(Y,Z)\eta(X) + g(X,Z)\eta(Y), \tag{13}$$

$$R(X,Y)\xi = -\eta(Y)X + \eta(X)Y, \tag{14}$$

$$R(\xi,X)Y = -g(X,Y)\xi + \eta(Y)X, \tag{15}$$

$$R(\xi,X)\xi = -\eta(X)\xi + X, \tag{16}$$

$$S(X,\xi) = -(n-1)\eta(X) \tag{17}$$

$$Q\xi = -(n-1)\xi, \tag{18}$$

for any X, Y, Z on TM^n.

Definition 2.2. A manifold M^n is called η-Einstein, if the Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{19}$$

where a, b are smooth functions on M [9].

Definition 2.3. A para-Kenmotsu manifold is called η-Einstein para-Kenmotsu manifold M if,

$$S(X,Y) = \left(1 + \frac{r}{2n}\right)(X,Y) - \left\{(2n+1) + \frac{r}{2n}\right\}\eta(X)\eta(Y). \tag{20}$$

for all vector fields X, Y on M holds [9].

Definition 2.4. The quasi-conformal curvature tensor W in a η-Einstein para-Kenmotsu manifold M is defined by

$$W(X,Y)Z = aR(X,Y)Z + b(S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY) - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)(g(Y,Z)X - g(X,Z)Y). \tag{21}$$

Where a and b are constants such that $ab \neq 0$ and R, S, Q and r are the Riemannian curvature tensor of type $(1,3)$, the Ricci tensor of type $(0,2)$, the Ricci operator defined by $g(QX,Y) = S(X,Y)$ and the scalar curvature of the manifold respectively [13].

If $a = 1$ and $b = -\frac{1}{(n-2)}$ then (21) takes the form

$$W(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)}\Big(S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\Big) + \frac{r}{(n-1)(n-2)}\Big(g(Y,Z)X - g(X,Z)Y\Big) = C(X,Y)Z \tag{22}$$

Where C is the conformal curvature tensor ([12], [14]). Thus the conformal curvature tensor C is a particular case of quasi-conformal curvature tensor W. The manifold is said to be quasi-conformally flat if W vanishes identically on M.

Definition 2.5. If (M, V, λ, μ) is an η-Ricci soliton, then the 1-form ξ is said to be potential vector field.

3. η-Ricci solitons on η-Einstein para-Kenmotsu manifolds

Let (M, ϕ, ξ, η, g) be an almost paracontact metric manifold. Consider the equation

$$£_\xi g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0 \tag{23}$$

where $£_\xi$ is the Lie derivative operator along the vector field ξ, S is the Ricci curvature tensor of the metric g, and λ and μ are real constants. Writing $£_\xi g$ in terms of the Levi-civita connection ∇, we have

$$2S(X, Y) = -g(\nabla_Y \xi, X) - g(Y, \nabla_X \xi) - 2\lambda g(X, Y) - 2\mu\eta(X)\eta(Y), \tag{24}$$

for any $X, Y \in \chi(M)$, or equivalent

$$S(X, Y) = -(1 + \lambda)g(X, Y) - (\mu - 1)\eta(X)\eta(Y), \tag{25}$$

for any $X, Y \in \chi(M)$. The above equation yields

$$S(X, \xi) = S(\xi, X) = -(\mu + \lambda)\eta(X), \tag{26}$$

$$QX = -(1 + \lambda)X + (1 - \mu)\eta(X) \tag{27}$$

comparing (17) with (26), we get

$$\mu + \lambda = n - 1. \tag{28}$$

The data (g, ξ, η, μ) which satisfy the equation (23) is said to be an η-Ricci soliton on M [23] .

Thus, we can state the following theorem:

Theorem 3.1. Let M be an n-dimensional η-Einstein para-Kenmotsu manifold. If the manifold admits an η-Ricci soliton (g, ξ, λ, μ), then M an η-Einstein manifold of the form (25), and the scalars λ and μ are related by $\mu+\lambda=n - 1$.

In particular, if we take $\mu=0$ in (25) and (28), then we obtain $S(X, Y)=-(1 + \lambda)g(X, Y) + \eta(X)\eta(Y)$, and $\lambda=n - 1$, respectively. Thus, we have

Corollary 3.1. Let M be an n-dimensional η-Einstein para-Kenmotsu manifold. If the manifold admits a Ricci soliton (g, ξ, λ), then M is an η-Einstein manifold and the manifold is expanding or shrinking according to the vector field ξ being spacelike or timelike.

4. η-Ricci solitons on η-Einstein para-Kenmotsu manifolds satisfying $R.S$=0

In this section we are going to study, an n-dimensional η-Einstein para-Kenmotsu manifold admitting an η-Ricci soliton satisfies $R.S$=0, which implies

$$(R(X, Y).S)(Z, W) = 0 \tag{29}$$

From (29), we have

$$S(R(X, Y)Z, W) + S(Z, R(X, Y)W) = 0 \tag{30}$$

Putting $X=\xi$ in (30), we obtain

$$S(R(\xi, Y)Z, W) + S(Z, R(\xi, Y)W) = 0 \tag{31}$$

Replacing the expression of S from (25) and from the symmetries of R, we find

$$(\mu - 1)[\eta(Y)g(X, Z) + \eta(Z)g(X, Y) - 2\eta(X)\eta(Y)\eta(Z)] = 0, \tag{32}$$

taking $Z=\xi$ in (32), we have

$$(\mu - 1)[g(X, Y) - \eta(X)\eta(Y)] = 0, \tag{33}$$

from which it follows that μ=1. From the relation (28), we get $\lambda=(n - 2)$.

Thus, we can state the following theorem:

Theorem 4.1. Let (g, ξ, λ, μ) be an n-dimensional η-Einstein para-Kenmotsu manifold admitting a proper η-Ricci soliton satisfies $R.S$=0, then μ=1 and $\lambda=(n - 2)$.

Corollary 4.1. On a η-Einstein para-Kenmotsu manifold M satisfying $R.S$=0, there is no Ricci soliton with the potential vector field ξ.

5. η-Ricci solitons on quasi-conformally flat η-Einstein para-Kenmotsu manifolds

In this constituent we review an η-Ricci solitons on quasi-conformally flat η-Einstein para-Kenmotsu manifolds. Let us assume that the manifold M admitting η-Ricci solitons is quasi-conformally flat, that is W=0. Then, from (21) it follows that

$$R(X,Y)Z = -\frac{b}{a}\Big(S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\Big) + \frac{r}{an}\Big(\frac{a}{n-1} + 2b\Big)(g(Y,Z)X - g(X,Z)Y) \tag{34}$$

Taking the inner product of (34) with ξ and using (7), (25) and (26), we get

$$\eta(R(X,Y)Z) = \Big[\frac{\lambda b}{a} + \frac{nb}{a} + \frac{r}{an}\Big(\frac{a}{n-1} + 2b\Big)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \tag{35}$$

By virtue of (13) and (35), we obtain

$$\Big[\frac{\lambda b}{a} + \frac{nb}{a} + \frac{r}{an}\Big(\frac{a}{n-1} + 2b\Big) + 1\Big][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] = 0 \tag{36}$$

Now putting X=ξ in the last equation, we find

$$\Big[\frac{\lambda b}{a} + \frac{nb}{a} + \frac{r}{an}\Big(\frac{a}{n-1} + 2b\Big) + 1\Big][g(Y,Z) - \eta(Y)\eta(Z)] = 0, \tag{37}$$

from which it follows that

$$\lambda = -n - \frac{a}{b} - \frac{r}{bn}\Big(\frac{a}{n-1} + 2b\Big)$$

From the relation (28), we obtain

$$\mu = \Big[2n + \frac{a}{b} + \frac{r}{bn}\Big(\frac{a}{n-1} + 2b\Big) - 1\Big]$$

Thus, we can state the following theorem:

Theorem 5.1. A quasi-conformally flat η-Einstein para-Kenmotsu manifold admits a proper η-Ricci solitons with λ=$\frac{a}{b}\Big(-1 - \frac{r}{n(n-1)}\Big) - \Big(n + \frac{2r}{n}\Big)$ and μ=$\frac{a}{b}\Big(1 + \frac{r}{n(n-1)}\Big) + \Big(2n + \frac{2r}{n} - 1\Big)$.

Corollary 5.1. On a η-Einstein para-Kenmotsu manifold satisfying W=0, there is no Ricci soliton with the potential vector field ξ.

6. η-Ricci solitons on η-Einstein para-Kenmotsu manifolds admitting Codazzi type of Ricci tensor and cyclic parallel Ricci tensor

In this section we consider η-Ricci solitons in η-Einstein para-Kenmotsu manifolds admitting codazzi type of Ricci tensor and cyclic parallel Ricci tensor. Gray [2] introduced the notion of cyclic parallel Ricci tensor and codazzzi type of Ricci tensor.

Definition 6.1. An η-Einstein para-Kenmotsu manifold is said to have codazzi type of Ricci tensor if its Ricci tensor S of the type $(0,2)$ is non-zero and satisfies the following conditions

$$(\nabla_Z S)(X,Y) = (\nabla_X S)(Y,Z) \tag{38}$$

for all X, Y, Z on M.

Taking covariant derivative of (25) along Z, we get

$$(\nabla_Z S)(X,Y) = -(\mu - 1)[(\nabla_Z \eta)(Y)\eta(X) + \eta(Y)(\nabla_Z \eta)(X)] \tag{39}$$

by virtue of (11) and (39), we obtain

$$(\nabla_Z S)(X,Y) = (1-\mu)[g(Z,X)\eta(Y) + g(Z,Y)\eta(X) - 2\eta(X)\eta(Y)\eta(Z)] \tag{40}$$

By hypothesis the Ricci tensor S is of codazzi type. Then

$$(\nabla_Z S)(X,Y) = (\nabla_X S)(Y,Z) \tag{41}$$

Making use of (40), (41) takes the form

$$(1-\mu)[g(Z,Y)\eta(X) - g(X,Y)\eta(Z)] = 0 \tag{42}$$

substituting Z=ξ in (42), we find

$$(1-\mu)[\eta(Y)\eta(X) - g(X,Y)] = 0 \tag{43}$$

from which it follows that μ=1. From the relation (28), we obtain λ=$n-2$.

Thus, we can state the following theorem.

Theorem 6.1. Let (g, ξ, λ, μ) be a proper η-Ricci soliton in an n-dimensional η-Einstein para-Kenmotsu manifold. If the manifold has Ricci tensor of codazzi type, then μ=1 and λ=$(n-2)$.

Corollary 6.1. An η-Einstein para-Kenmotsu manifold Ricci tensor is of codazzi type does not admit Ricci solitons with potential field ξ.

Definition 6.2. An η-Einstein para-Kenmotsu manifold is said to have cyclic parallel Ricci tensor if its Ricci tensor S of type $(0,2)$ is non-zero and satisfies the following condition [19]

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0, \tag{44}$$

for all X, Y, Z on M.

Let (g, ξ, λ, μ) be an η-Ricci soliton in an n-dimensional η-Einstein para-Kenmotsu manifold and the manifold has cyclic parallel Ricci tensor, then (44) holds.

Taking covariant derivative of (25) and using (11), we obtain

$$(\nabla_Z S)(X,Y) = (1-\mu)[g(Z,X)\eta(Y) + g(Z,Y)\eta(X) - 2\eta(X)\eta(Y)\eta(Z)], \tag{45}$$

Similarly, we have

$$(\nabla_X S)(Y,Z) = (1-\mu)[g(X,Y)\eta(Z) + g(X,Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)], \tag{46}$$

and

$$(\nabla_Y S)(Z,X) = (1-\mu)[g(Y,Z)\eta(X) + g(Y,X)\eta(Z) - 2\eta(X)\eta(Y)\eta(Z)], \tag{47}$$

By using (45)-(47) in (44), we find

$$2(1-\mu)[g(X,Y)\eta(Z) + g(X,Z)\eta(Y) + g(Y,Z)\eta(X) - 3\eta(X)\eta(Y)\eta(Z)] = 0. \tag{48}$$

Now putting Z=ξ in (48), we have

$$2(1-\mu)[g(X,Y) - \eta(X)\eta(Y)] = 0 \tag{49}$$

from which it follws that μ=1. From the relation (28), we obtain λ=$(n-2)$.

Thus, we can state the following theorem.

Theorem 6.2. Let (g, ξ, λ, μ) be a proper η-Ricci soliton in an n-dimensional η-Einstein para-Kenmotsu manifold. If the manifold has cyclic parallel Ricci tensor, then μ=1 and λ=$(n-2)$.

Corollary 6.2. An η-Einstein para-Kenmotsu manifold has cyclic parallel Ricci tensor does not admit Ricci solitons with potential vector field ξ.

7. η-Ricci solitons on ϕ-quasi-conformally semi symmetric η-Einstein para-Kenmotsu manifolds

This section is devoted to the study of ϕ-quasi-conformally semi-symmetric η-Einstein para-Kenmotsu manifolds.Then, we have

$$W \cdot \phi = 0, \tag{50}$$

from which it follows that

$$W(X,Y)\phi Z - \phi(W(X,Y)Z) = 0. \tag{51}$$

Taking Z=ξ in (51), we have

$$\phi(W(X,Y)\xi) = 0. \tag{52}$$

Now replacing Z=ξ in (21) and using (5), (14), (17) and (18), we obtain

$$W(X,Y)\xi = \Big(a + nb - b\lambda + \frac{ra}{n(n-1)} + \frac{2rb}{n}\Big)[-\eta(Y)X + \eta(X)Y]. \tag{53}$$

In view of (52) and (53), we get

$$\phi(W(X,Y)\xi) = \Big(a + nb - b\lambda + \frac{ra}{n(n-1)} + \frac{2rb}{n}\Big)[-\eta(Y)\phi X + \eta(X)\phi Y] = 0 \tag{54}$$

Now substituting X by ϕX in (54), we find

$$\Big(a + nb - b\lambda + \frac{ra}{n(n-1)} + \frac{2rb}{n}\Big)\eta(Y)\phi^2 X = 0. \tag{55}$$

By virtue of (4) and (55), we have

$$\Big(a + nb - b\lambda + \frac{ra}{n(n-1)} + \frac{2rb}{n}\Big)[X - \eta(X)\xi] = 0 \tag{56}$$

Taking inner product of (56) with respect to U, we find

$$\Big(a + nb - b\lambda + \frac{ra}{n(n-1)} + \frac{2rb}{n}\Big)[g(X,U) - \eta(X)\eta(Y)] = 0, \tag{57}$$

from which it follows that λ=$\frac{a}{b}\Big(1 + \frac{r}{n(n-1)}\Big) + \Big(n + \frac{2r}{n}\Big)$.

From the relation (28), we obtain μ=$-\frac{a}{b}\Big(1 + \frac{r}{n(n-1)}\Big) - \Big(\frac{2r}{n} + 1\Big)$.

Thus, we can state the following theorem:

Theorem 7.1. A ϕ-conformally semi-symmetric η-Einstein para-Kenmotsu manifold admits a proper η-Ricci soliton with λ=$\frac{a}{b}\Big(1 + \frac{r}{n(n-1)}\Big) + \Big(n + \frac{2r}{n}\Big)$ and μ=$-\frac{a}{b}\Big(1 + \frac{r}{n(n-1)}\Big) - \Big(\frac{2r}{n} + 1\Big)$.

Corollary 7.1. On a η-Einstein para-Kenmotsu manifold satisfying $W.\phi$=0, there is no Ricci soliton with the potential vector field ξ.

8. η-Ricci solitons on ϕ-Ricci symmetric η-Einstein para-Kenmotsu manifolds

In this section, we study ϕ- Ricci symmetric η-Ricci soliton on η-Einstein para-Kenmotsu manifolds.

Definition 8.1. An η-Einstein para-Kenmotsu manifold is said to be ϕ-Ricci symmetric if

$$\phi^2(\nabla_X Q)Y = 0 \tag{58}$$

holds for all smooth vector fields X, Y.

If X, Y are orthogonal to ξ, then the manifold is said to be ϕ-Ricci symmetric. It is well known that ϕ-symmetric implies ϕ-Ricci symmetric, but the converse is not true, in general true. ϕ-Ricci symmetric Sasakian manifolds have been studied by De and Sarkar [34].

We know that, the Ricci tensor for an η-Ricci soliton on η-Einstein para-Kenmotsu manifold is given by

$$S(X, Y) = -(1 + \lambda)g(X, Y) - (\mu - 1)\eta(X)\eta(Y). \tag{59}$$

Then it follows that

$$QY = -(1 + \lambda)Y - (\mu - 1)\eta(Y)\xi, \tag{60}$$

for smooth vector fields Y.Taking covariant derivative of above equation with respect to X, we have

$$(\nabla_X Q)Y = \nabla_X QY - Q(\nabla_X Y) = -(\mu - 1)[(\nabla_X \eta)(Y)\xi + \eta(Y)\nabla_X \xi]. \tag{61}$$

Using (10) and (11), we get

$$(\nabla_X Q)Y = -(\mu - 1)[g(X, Y)\xi + \eta(Y)X - 2\eta(X)\eta(Y)\xi]. \tag{62}$$

Applying ϕ^2 on both sides of the above equation, we find

$$\phi^2(\nabla_X Q)Y = -(\mu - 1)\eta(Y)\phi^2 X. \tag{63}$$

From (58) and (63), we obtain

$$(\mu - 1)\eta(Y)\phi^2 X = 0, \tag{64}$$

by virtue of (4) and (64), we find

$$(\mu - 1)\eta(Y)[X - \eta(X)\xi] = 0 \tag{65}$$

Taking inner product of (65) with respect to U, we get

$$(\mu - 1)[g(X, U) - \eta(X)\eta(U)] = 0 \tag{66}$$

from which it follows that μ=1. From the relation (28), we obtain λ=$(n - 2)$.

Thus, we can state the following theorem:

Theorem 8.1. On ϕ-Ricci symmetric η-Einstein para-Kenmotsu manifold admits a proper η-Ricci soliton with μ=1 and λ=$(n - 2)$.

Corollary 8.1. On a ϕ-Ricci symmetric η-Einstein para-Kenmotsu manifold, there is no Ricci solitons with potential vector field ξ.

9. η-Ricci solitons on quasi-conformally Ricci semi-symmetric η-Einstein para-Kenmotsu manifolds

In this section, we study η-Ricci solitons on quasi-conformally Ricci semi-symmetric η-Einstein para-Kenmotsu manifolds, that is,

$$W \cdot S = 0 \tag{67}$$

which implies

$$(W(X, Y) \cdot S)(Z, U) = 0 \tag{68}$$

From (68), we get

$$S(W(X, Y)Z, U) + S(Z, W(X, Y)U) = 0. \tag{69}$$

Using (25) in (69), we have

$$-(1+\lambda)g(W(X,Y)Z,U)-(\mu-1)\eta(W(X,Y)Z)\eta(U)-(1+\lambda)g(Z,W(X,Y)U)-(\mu-1)\eta(Z)\eta(W(X,Y)U)=0 \quad (70)$$

Taking X=U=ξ in (70), we get

$$-(1+\lambda)g(W(\xi,Y)Z,\xi)-(\mu-1)\eta(W(\xi,Y)Z)-(1+\lambda)g(Z,W(\xi,Y)\xi)-(\mu-1)\eta(Z)\eta(W(\xi,Y)\xi)=0 \quad (71)$$

From (53) we obtain

$$W(\xi,Y)\xi=\Big[a+nb-b\lambda+\frac{r}{n}\Big(\frac{a}{n-1}+2b\Big)\Big][-\eta(Y)\xi+Y]. \quad (72)$$

By virtue of (72), we get

$$\eta(W(\xi,Y)Z)=g(W(\xi,Y)Z,\xi)\ =-g(W(\xi,Y)\xi,Z)$$

$$=-\Big[a+nb-b\lambda+\frac{r}{n}\Big(\frac{a}{n-1}+2b\Big)\Big][g(Y,Z)-\eta(Y)\eta(Z)]. \quad (73)$$

Also from (73), we find

$$\eta(W(\xi,Y)\xi)=0. \quad (74)$$

Making use of (72), (73) and (74) in (71), we have

$$(\mu-1)\Big[a+nb-b\lambda+\frac{r}{n}\Big(\frac{a}{n-1}+2b\Big)\Big][g(Y,Z)-\eta(Y)\eta(Z)]=0, \quad (75)$$

$$\textit{either}\quad \mu-1=0,\quad \textit{or}\quad \Big[a+nb-b\lambda+\frac{r}{n}\Big(\frac{a}{n-1}+2b\Big)\Big]=0, \quad (76)$$

$$\textit{either}\quad \mu=1,\quad \textit{or}\quad \Big[a+nb-b\lambda+\frac{r}{n}\Big(\frac{a}{n-1}+2b\Big)\Big]=0,$$

Case I: if μ=1 then from (28), we find λ=$2-n$.
Case II: if $\Big[a+nb-b\lambda+\frac{r}{n}\Big(\frac{a}{n-1}+2b\Big)\Big]=0$,
implies that λ=$\frac{a}{b}\Big(1+\frac{r}{n(n-1)}\Big)+\Big(n+\frac{2r}{n}\Big)$.
From the relation (28), we obtain μ=-$\frac{a}{b}\Big(1+\frac{r}{n(n-1)}\Big)+\Big(\frac{2r}{n}-1\Big)$.

Theorem 9.1. If a conformally Ricci semi-symmetric η-Einstein para-Kenmotsu manifold admits a proper η-Ricci soliton, then μ=1 and λ=$n-2$ or λ=$\frac{a}{b}\Big(1+\frac{r}{n(n-1)}\Big)+\Big(n+\frac{2r}{n}\Big)$ and μ=-$\frac{a}{b}\Big(1+\frac{r}{n(n-1)}\Big)+\Big(\frac{2r}{n}-1\Big)$.

Corollary 9.1. On conformally Ricci semi-symmetric η-Einstein para-Kenmotsu manifold, there is no Ricci solitons with potential vector field ξ.

10. Example

Now, we consider the 3-dimensional manifold

$$M=\{(x,y,z)\in R^3 : z\neq 0,\} \quad (77)$$

where x, y, z are the standard coordinates in R^3.

The vector fields

$$e_1=\frac{\partial}{\partial x},\quad e_2=\frac{\partial}{\partial y}+2x\frac{\partial}{\partial z},\quad e_3=\frac{\partial}{\partial z}=\xi \quad (78)$$

are lineraly independent at each point of M. Let g be the Riemannian metric defined by
$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1$,
$g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0$
Let η be the 1-form defined by

$$\eta(Z) = g(Z, e_3) = g(Z, \xi), \tag{79}$$

for any vector field Z on M.
Let ϕ be the (1,1)-tensor field defined by

$$\phi(e_1) = -e_2, \ \ \phi(e_2) = e_1, \ \ \phi(e_3) = 0. \tag{80}$$

Then, using the linearity of ϕ and g, we have

$$\eta(e_3) = 1, \quad \phi^2 Z = Z - \eta(Z)\xi, \tag{81}$$

$$g(\phi Z, \phi W) = -g(Z, W) + \eta(Z)\eta(W), \tag{82}$$

for any vector field Z, W on M.

It is easy to see that

$$\eta(e_1) = 0, \ \ \eta(e_2) = 0, \ \ \eta(e_3) = 1. \tag{83}$$

Thus for e_3=ξ, the structure (ϕ, ξ, η, g) defines a η-Einstein almost paracontact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to the Riemannian metric g.

Then we have

$$[e_1, e_2] = 2e_3, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 0., \tag{84}$$

Using Koszul's formula for Levi-Civita connection ∇ with respect to g, i.e.,

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y])$$

One can easily calculate

$$\nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_2 = e_3, \quad \nabla_{e_1} e_3 = -e_2, \tag{85}$$

$$\nabla_{e_2} e_1 = -e_3, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_3 = e_1, \tag{86}$$

$$\nabla_{e_3} e_1 = -e_2, \quad \nabla_{e_3} e_2 = e_1, \quad \nabla_{e_3} e_3 = 0. \tag{87}$$

From the above calculations, we see that the manifold under consideration satisfies ∇, i.e.,

$$\nabla_Z \xi = -Z - \eta(Z)\xi, \quad \textit{and} \quad (\nabla_Z \phi)W = -g(Z, W)\xi - \eta(W)\phi Z. \tag{88}$$

Also, the Riemannian curvature tensor R is given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{89}$$

Then

$$R(e_1, e_2)e_1 = 3e_2, \ \ R(e_1, e_2)e_2 = -3e_1, \ \ R(e_1, e_2)e_3 = 0, \tag{90}$$

$$R(e_2, e_3)e_1 = 0, \ \ R(e_2, e_3)e_2 = -e_3, \ \ R(e_2, e_3)e_3 = e_2, \tag{91}$$

$$R(e_1, e_3)e_1 = -e_3, \ \ R(e_1, e_3)e_1 = -e_3, \ \ R(e_1, e_3)e_3 = e_1, \tag{92}$$

Then, the Ricci tensor S is given by

$$S(e_1, e_1) = S(e_2, e_2) = -2, \ \ S(e_3, e_3) = 2. \tag{93}$$

From (23), we obtain $S(e_1, e_1)$=$S(e_2, e_2)$= -$(1 - \lambda)$ and $S(e_3, e_3)$=$\lambda - \mu$, therefore λ=-1 and μ=-3. The data (g, ξ, λ, μ) for λ=-1 and μ=-3 defines an η-Ricci soliton on η-Einstein para-Kenmotsu manifold M.

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