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A note on the algebraic representation of coframes via the Scott closed set monad over the category of S_0-convex spaces

Changchun Xia[a]

[a]*School of Mathematics and Statistics, Northwestern Polytechnical University, Xi'an 710129, China*

Abstract. In this note, we shall give a complete answer to the question that what kind of lattice structures corresponds to the Φ-algebras with respect to the Scott closed set monad over the category of S_0-convex spaces and show that the Eilenberg-Moore algebras with respect to the Scott closed set monad are precisely coframes endowed with the Scott convex structure. Meanwhile, we shall also prove that the category of coframes is strictly monadic over the category of S_0-convex spaces.

1. Introduction

A *monad* over a category $\mathbf{X}$ is a triple (T, η, μ) consisting of a functor $T \colon \mathbf{X} \longrightarrow \mathbf{X}$ and natural transformations $\mu \colon T \circ T \longrightarrow T$ and $\eta \colon id_{\mathbf{X}} \longrightarrow T$ such that

$$\mu \circ T\mu = \mu \circ \mu T, \ \mu \circ T\eta = \mu \circ \eta T = id_{\mathbf{X}}.$$

Given a monad (T, η, μ) over $\mathbf{X}$, a *T-algebra* (or an *Eilenberg-Moore algebra*) is a pair (X, α_X), where X is an $\mathbf{X}$-object and the structured morphism $\alpha_X \colon T(X) \longrightarrow X$ satisfies

$$\alpha_X \circ T\alpha_X = \alpha_X \circ \mu_X, \ \alpha_X \circ \eta_X = id_X.$$

In [2], Day investigated the Eilenberg-Moore algebras of the open filter monad over **Set** and $\mathbf{Top}_0$ respectively, and characterized them both exactly as continuous lattices endowed with the Scott topology. From another perspective, Scott [5] also in his paper on the mathematical models for the Church-Curry λ-calculus characterized continuous lattices endowed with the Scott topology precisely as T_0-spaces injective over all the subspace embeddings. Furthermore, Wyler [8] studied the categorical algebraic theories of filters, ultrafilter monad, powerset monad, Vietoris monad, open filter monad. One uses the fact that the filter monad is of Kock-Zöberlein type, and in that poset-enriched category with such a monad structure, the injective objects over a certain class of embedding defined in terms of monad structures are precisely the algebras. From the above results, a bridge among domain theory, topology and categorical algebras is built, which will further promote the mutual applications among them. In [3], Jankowski characterized

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Email address: xiachangchun@xjtu.edu.cn (Changchun Xia)

frames endowed with the filter-convex structure precisely as the S_0-convex spaces injective over all the convex-subspace embeddings, where a convex space and an injective object are called a closure space which satisfies the compact theorem and an absolute extensor respectively in [3]. As a matter of fact, one should notice that the injective S_0-convex spaces over all the convex-subspace embeddings can also be characterized by coframes endowed with the Scott convex structure.

Based on the above facts, a natural question was posed by Yue, Yao and Ho [10] that what kind of lattice structures corresponds to the Eilenberg-Moore algebras of Scott closed set monad, coframes or C-continuous lattices? Also, in their paper, based on the remotehood system, the convex structure by Kleisli monoid over the Scott-closed set monad was characterized and the convex convergence spaces were proved precisely to be the reflexive and transitive lax algebras. Recently, in [4], Liu, Yue and Wei showed that the Eilenberg-Moore algebras with respect to the Scott open set monad over **Set** are precisely frames. However, the question that what kind of lattice structures corresponds to the Φ-algebras with respect to the Scott closed set monad over the category of S_0-convex spaces is still unknown. In this note, we shall mainly devote to giving a complete answer to this question and further show that the category of coframes is strictly monadic over the category of S_0-convex spaces.

2. Preliminaries

In this section, we shall recall some basic facts about order theory and convex spaces. The readers can refer to [1] for category theory.

For a subset Y of a poset $(X, \leq)$, we write $\downarrow Y = \{x \in X\colon \exists y \in Y\ s.t.\ x \leq y\}$. We always write $F \subseteq_\omega X$ to denote F is a finite subset of X. A subset Y is a *lower set* of X if $Y = \downarrow Y$. When $Y = \{y\}$, we write $\downarrow \{y\}$ simply as $\downarrow y$ and call it a *principal ideal* of X. Dually, the concepts of upper sets and principle filters can be defined. A lower (An upper) subset S of a join-semilattice (semilattice) L is called an *ideal* (a *filter*) of L, if $\vee F \in S$ ($\wedge F \in S$) for any $F \subseteq_\omega S$. Let $Idl(L)$ ($Fil(L)$) denote the set of all ideals (filters) of a join-semilattice (semilattice) L. A subset Y of a complete lattice X is *Scott closed* if it satisfies (1) $Y = \downarrow Y$ and (2) for any directed subset D, $D \subseteq Y$ implies $\bigvee D \in Y$. The complements of Scott closed sets are called *Scott open sets*, such subsets of a poset X form a topology on X, which is the well-known Scott topology. A *frame* is a complete lattice satisfying the distributive law of binary meets over arbitrary joins. Dually, a *coframe* is a complete lattice satisfying the distributive law of binary joins over arbitrary meets. A map $f\colon L \longrightarrow M$ between frames (coframes) is called a frame (coframe) homomorphism if f preserves finite meets (joins) and arbitrary joins (meets). Let **Frm** (**CoFrm**) denote the category of frames (coframes) and frame (coframe) homomorphisms and let $\mathbf{Frm}_\wedge$ ($\mathbf{CoFrm}_\vee$) denote the category of frames (coframes) and finite-meets-preserving maps (finite-joins-preserving maps). Let $f\colon P \longrightarrow Q$ and $g\colon Q \longrightarrow P$ be two monotone maps between posets. Then f is called a *left adjoint* of g (g is a *right adjoint* of f) if $f(x) \leq y \Leftrightarrow x \leq g(y)$ holds for all $x \in P$ and $y \in Q$. In particular, if P and Q are complete lattices, then f is a left adjoint of g iff f preserves arbitrary joins iff g preserves arbitrary meets.

Definition 2.1. ([7]) Let X be a set. A subfamily $\mathfrak{C}$ of 2^X is called a *convex structure* on X, if it satisfies the following conditions:

(1) $\emptyset, X \in \mathfrak{C}$;

(2) For any $\{A_i\}_{i\in I} \subseteq \mathfrak{C}$, $\bigcap\limits_{i\in I} A_i \in \mathfrak{C}$;

(3) For any directed family $\{D_i\}_{i\in I} \subseteq \mathfrak{C}$, $\bigcup\limits_{i\in I} D_i \in \mathfrak{C}$.

We call the pair $(X, \mathfrak{C})$, or simply X, a *convex space*, and every element in $\mathfrak{C}$ a *convex set*. We shall always denote by $\mathfrak{C}_X$ the set of all convex sets of X.

Let $(X, \mathfrak{C}_X)$ be a convex space. For any subset A of X, the *hull* $co_X(A)$ of A is defined as

$$co_X(A) = \bigcap \{B \in \mathfrak{C}_X\colon A \subseteq B\}$$

The operator co_X is called the hull on X. A convex set C is called a *polytope* if it is the hull of a finite set of X. For convenience, we always write $co_X(x)$ for $co_X(\{x\})$ for any $x \in X$. One can easily check that

$co_X(A) = \bigcup_{F \subseteq_\omega A} co_X(F)$ for any $A \subseteq X$. A convex space X is called S_0, if $co_X(x) = co_X(y)$ implies $x = y$ for any $x, y \in X$. A convex space $(X, \mathfrak{C}_X)$ is called *sober*, if every non-empty polytope is the hull of a unique singleton (see [6]). The specialization preorder $\leq_{\mathfrak{C}_X}$ on X is defined by $x \leq_{\mathfrak{C}_X} y$ iff $x \in co_X(y)$, or alternatively $x \leq y$ iff $y \in C$ implies $x \in C$ for all $C \in \mathfrak{C}_X$. The specialization preorder is a partial order iff X is an S_0-convex space. For an S_0-convex space X, one can easily verify that every convex set in $\mathfrak{C}_X$ is always a lower set of X with respect to the specialization order. For a join-semilattice $(L, \leq)$, one can check that the collection $Idl(L)$ of all the ideals of L is a convex structure, which is called the *Scott convex structure* on L. It should be noted that an S_0-convex space X is *sober* iff $(X, \leq_{\mathfrak{C}_X})$ is a join-semilattice and the convex structure $\mathfrak{C}_X$ is coarser than the Scott convex structure on $(X, \leq_{\mathfrak{C}_X})$.

Let $f\colon (X, \mathfrak{C}_X) \longrightarrow (Y, \mathfrak{C}_Y)$ be a map between convex spaces. Then f is called *convexity-preserving* (*CP* for short), if for any $C \in \mathfrak{C}_Y$, $f^{-1}(C) \in \mathfrak{C}_X$. One can easily check that f is *CP* iff $f(co_X(A)) \subseteq co_Y(f(A))$ for any $A \subseteq X$ iff $f(co_X(F)) \subseteq co_Y(f(F))$ for any $F \subseteq_\omega X$. A collection $\mathcal{S} \subseteq \mathfrak{C}_X$ is called a *subbase* of a convex space $(X, \mathfrak{C}_X)$, if $\mathfrak{C}_X$ is the coarsest among all convex structures that include $\mathcal{S}$. Let $\mathbf{ConvexS_0}$ denote the category of S_0-convex spaces with *CP*-maps.

A *morphism* of T-algebras $f\colon (X, \alpha_X) \longrightarrow (Y, \beta_Y)$ is an $\mathbf{X}$-morphism $f\colon X \longrightarrow Y$ such that $f \circ \alpha_X = \beta_Y \circ Tf$. Let $\mathbf{X}^T$ denote the category of all T-algebras and T-morphisms. A monad (T, η, μ) is always associated with an adjoint pair $F \dashv G\colon \mathbf{A} \longrightarrow \mathbf{X}$ such that $T = GF$ and $\mu = G\varepsilon F$. The related comparison functor $K\colon \mathbf{A} \longrightarrow \mathbf{X}^T$ is given by $A \mapsto (GA, G\varepsilon_A)$ and $(f\colon A \longrightarrow B) \mapsto (G(f)\colon G(A) \longrightarrow G(B))$. The category $\mathbf{A}$ is called monadic (resp., strict monadic) over $\mathbf{X}$ via the monad T if the comparison functor K is an equivalence (resp., isomorphism).

3. Main results

In this section, we shall aim at showing that the coframes endowed with the Scott convex structure are precisely the Φ-algebras of the monad (Φ, μ, η) over $\mathbf{ConvexS_0}$, and $\mathbf{CoFrm}$ is strictly monadic over $\mathbf{ConvexS_0}$. In the following, a complete lattice or a coframe L is always assumed to be equipped with the Scott convex structure.

For an S_0-convex space $(X, \mathfrak{C}_X)$, we let ΦX be the set of all Scott closed sets on the poset $(\mathfrak{C}_X, \subseteq)$ and $\Phi\mathfrak{C}_X$ be the convex structure on ΦX generated by the subbase $\mathfrak{C}_X^* = \{C^*\colon C \in \mathfrak{C}_X\}$, where $C^* = \{\mathbb{I} \in \Phi X\colon C \notin \mathbb{I}\}$. Obviously, $(\Phi X, \Phi\mathfrak{C}_X)$ is an S_0 convex space and the specialization order on ΦX is just the inclusion order.

Define $\Phi\colon \mathbf{ConvexS_0} \longrightarrow \mathbf{ConvexS_0}$ by $\Phi(X) = \Phi X$ for any S_0-convex space X and $\Phi f(\mathbb{I}) = \{B \in \mathfrak{C}_Y\colon f^{-1}(B) \in \mathbb{I}\}$ for any *CP* map $f\colon (X, \mathfrak{C}_X) \longrightarrow (Y, \mathfrak{C}_Y)$. Then it can be checked that Φ is a functor. For an S_0-convex space $(X, \mathfrak{C}_X)$, the natural transformations $\eta\colon id_{\mathbf{ConvexS_0}} \longrightarrow \Phi$ and $\mu\colon \Phi\Phi \longrightarrow \Phi$ is given by $\eta_X(x) = R_x$ and $\mu_X(\mathcal{A}) = \{A \in \mathfrak{C}_X\colon A^* \in \mathcal{A}\}$, where $R_x = \{C \in \mathfrak{C}_X\colon x \notin C\}$. In [10], it was proved that η_X, μ_X are *CP* and (Φ, η, μ) is a Kock-Zöberlein type monad on $\mathbf{ConvexS_0}$. Then it follows that $\eta_X \dashv \alpha$ for any Φ-algebra (X, α_X) over (Φ, η, μ).

Lemma 3.1. *Let $f\colon (X, \mathfrak{C}_X) \longrightarrow (Y, \mathfrak{C}_Y)$ be a CP map between S_0-convex spaces. Then the maps Φf and μ_X preserve arbitrary meets.*

Proof. Proof is easy. $\square$

The following Theorem can be found in Theorem 4.9, Theorem 4.11 and Theorem 4.12 of [9] for the case $L = \mathbf{2}$.

Theorem 3.2. *Every frame equipped with the filter-convex structure is an injective S_0-convex space and conversely, the specialization ordered set of an injective S_0-convex space is a frame. Furthermore, $\leq_{\mathfrak{C}_X}^{op} \circ \mathbf{Fil} = id_{\mathbf{Frm}_\wedge}$ and $\mathbf{Fil} \circ \leq_{\mathfrak{C}_X}^{op} = id_{\mathbf{ConvexS_0}}$, that is, the categories of injective S_0-convex spaces and $\mathbf{Frm}_\wedge$ are isomorphic.*

One should note that the specialization order on an S_0-convex space $(X, \mathfrak{C}_X)$ in Theorem 3.2 is just the opposite order of the specialization order $\leq_{\mathfrak{C}_X}$ defined in the preliminaries. Then we immediately get the following Corollary.

Corollary 3.3. *Every coframe equipped with the Scott convex structure is an injective S_0-convex space and conversely, the specialization ordered set $(X, \leq_{\mathfrak{C}_X})$ of an injective S_0-convex space $(X, \mathfrak{C}_X)$ is a coframe. Furthermore, $\leq_{\mathfrak{C}_X} \circ \mathbf{Idl} = id_{\mathbf{CoFrm}_\wedge}$ and $\mathbf{Idl} \circ \leq_{\mathfrak{C}_X} = id_{\mathbf{ConvexS_0}}$, that is, the category of injective S_0-convex spaces and $\mathbf{CoFrm}_\wedge$ are isomorphic.*

Lemma 3.4. *Let $(X, \mathfrak{C}_X)$ be an S_0-convex space. Then $(\Phi X, \Phi\mathfrak{C}_X)$ is injective in $\mathbf{ConvexS_0}$.*

Proof. Obviously, the specialization order on $(\Phi X, \Phi\mathfrak{C}_X)$ is the inclusion order and $(\Phi X, \subseteq)$ is a coframe. Then it is routine to check that $\Phi\mathfrak{C}_X = Idl(\Phi X)$, as every C^* is an ideal of ΦX and $L(\mathbb{I}) = \bigcap\limits_{C \notin \mathbb{I}} C^*$ for any $\mathbb{I} \in \Phi X$, where $L(\mathbb{I}) = \{\mathbb{J} \in Idl(\mathfrak{C}_X) \colon \mathbb{J} \subseteq \mathbb{I}\}$. Thus, Corollary 3.3 gives that $(\Phi X, \Phi\mathfrak{C}_X)$ is injective in $\mathbf{ConvexS_0}$. □

Lemma 3.5. *Let $f \colon (X, \mathfrak{C}_X) \longrightarrow (Y, \mathfrak{C}_Y)$ be a map between sober convex spaces. Then f is CP iff $f \colon (X, \leq_{\mathfrak{C}}) \longrightarrow (Y, \leq_{\mathfrak{C}})$ preserves finite joins.*

Proof. Proof is easy. □

Let L be a complete lattice. We define a map $\alpha_L \colon (\Phi L, \Phi Idl(L)) \longrightarrow (L, Idl(L))$ by $\alpha_L(\mathbb{I}) = \bigwedge\{x \in L \colon \downarrow x \notin \mathbb{I}\}$ for any $\mathbb{I} \in \Phi L$. It is obvious that α_L preserves arbitrary meets.

Lemma 3.6. *Let L be a complete lattice. Then α_L is the right adjoint to η_L and $\alpha_L \circ \eta_L = id_L$.*

Proof. Let $\eta_L(x) \subseteq \mathbb{I}$. For any $\downarrow y \notin \mathbb{I}$, we have $x \in \downarrow y$ and then $x \leq \alpha_L(\mathbb{I})$. Conversely, let $x \leq \alpha_L(\mathbb{I})$, we can assert that $\eta_L(x) \subseteq \mathbb{I}$. If not, then there exists $C \in Idl(L)$ with $x \notin C$ but $C \notin \mathbb{I}$. So there exists $x_1 \in C$ such that $\downarrow x_1 \notin \mathbb{I}$. This implies that $x \leq x_1$ and hence $x \in C$, a contradiction. Thus, $\eta_L(x) \subseteq \mathbb{I}$. Furthermore, for any $x \in L$, we have $\alpha_L(\eta_L(x)) = \bigwedge\{y \in L \colon \downarrow y \notin R_x\} = \bigwedge\{y \in L \colon x \leq y\} = x$. □

Proposition 3.7. *Let L be a complete lattice. Then the following statements are equivalent:*
(1) α_L is CP;
(2) α_L preserves finite joins;
(3) L is a coframe.

Proof. (1) ⇔ (2) By Lemma 3.5.

(2) ⇒ (3) It is clear that $(\Phi L, \subseteq)$, as the lattice of Scott closed sets of $Idl(L)$, is a coframe. By the assumption and Lemma 3.6, L is a retract of ΦL. This implies that L is a coframe with $\bigwedge\limits_{L} A = \alpha_L(\bigwedge\limits_{\Phi L} \eta_L(A))$ for any $A \subseteq L$.

(3) ⇒ (2) For any $\mathbb{I}, \mathbb{J} \in \Phi L$, we have that

$$\begin{aligned} \alpha_L(\mathbb{I}) \vee \alpha_L(\mathbb{J}) &= \bigwedge\{y \in L \colon \downarrow y \notin \mathbb{I}\} \vee \bigwedge\{z \in L \colon \downarrow z \notin \mathbb{J}\} \\ &= \bigwedge\{y \vee z \colon \downarrow y \notin \mathbb{I}, \downarrow z \notin \mathbb{J}, y, z \in L\} \\ &= \bigwedge\{t \in L \colon \downarrow t \notin \mathbb{I} \cup \mathbb{J}\} \\ &= \alpha_L(\mathbb{I} \cup \mathbb{J}), \end{aligned}$$

where the third equality is valid because $A \triangleq \{y \vee z \colon \downarrow y \notin \mathbb{I}, \downarrow z \notin \mathbb{J}, y, z \in L\} = \{t \in L \colon \downarrow t \notin \mathbb{I} \cup \mathbb{J}\} \triangleq B$. Indeed, let $y \vee z \in A$, then $\downarrow y \notin \mathbb{I}$ and $\downarrow z \notin \mathbb{J}$ and so $\downarrow (y \vee z) \notin \mathbb{I} \cup \mathbb{J}$, as $\mathbb{I}$ and $\mathbb{J}$ are lower sets, which implies that $y \vee z \in B$. Conversely, let $t \in B$, then $\downarrow t \notin \mathbb{I}$ and $\downarrow t \notin \mathbb{J}$. Hence the fact $t = t \vee t$ gives that $t \in A$. □

Lemma 3.8. *Let L be a coframe. Then $\mathbb{I} = \bigcap\limits_{x \notin \bigcup \mathbb{I}} R_x$ for any $\mathbb{I} \in \Phi L$.*

Proof. Let $I \in \mathbb{I}$. It is obvious that $x \notin I$ for any $x \notin \bigcup \mathbb{I}$ and hence $\mathbb{I} \subseteq \bigcap\limits_{x \notin \bigcup \mathbb{I}} R_x$. Conversely, if $I \in R_x$ for any $x \notin \bigcup \mathbb{I}$, then $I \subseteq \bigcup \mathbb{I}$. Since I is an ideal of L and $\mathbb{I}$ is a Scott closed subset of ΦL, it follows that $I \in \mathbb{I}$ and so $\bigcap\limits_{x \notin \bigcup \mathbb{I}} R_x \subseteq \mathbb{I}$. □

Proposition 3.9. *Let L be a coframe. Then $\alpha_L \circ \Phi\alpha_L = \alpha_L \circ \mu_L$.*

Proof. For any $\mathbb{I} \in \Phi L$, we have that

$$\begin{aligned}
\alpha_L(\Phi\alpha_L(\mathbb{I}^*)) &= \alpha_L(\{J \in Idl(L) \colon \alpha_L^{-1}(J) \in \mathbb{I}^*\}) \\
&= \alpha_L(\{J \in Idl(L) \colon \alpha_L(\mathbb{I}) \notin J\}) \\
&= \alpha_L(\{J \in Idl(L) \colon J \in R_{\alpha_L(\mathbb{I})}\}) \\
&= \alpha_L(R_{\alpha_L(\mathbb{I})}) \\
&= \bigwedge\{y \in L \colon \downarrow y \notin R_{\alpha_L(\mathbb{I})}\} \\
&= \bigwedge\{y \in L \colon \alpha_L(\mathbb{I}) \le y\} \\
&= \alpha_L(\mathbb{I}) \\
&= \alpha_L(\{J \in Idl(L) \colon J \in \mathbb{I}\}) \\
&= \alpha_L(\{J \in Idl(L) \colon \mathbb{I} \notin J^*\}) \\
&= \alpha_L(\{J \in Idl(L) \colon J^* \in \mathbb{I}^*\}) \\
&= \alpha_L(\mu_L(\mathbb{I}^*)).
\end{aligned}$$

By Lemma 3.1 and Lemma 3.6, the maps α_L, $\Phi\alpha_L$ and μ_L all preserve arbitrary meets. Then Lemma 3.8 gives that $\alpha_L \circ \Phi\alpha_L = \alpha_L \circ \mu_L$. □

By Lemma 3.6 and Proposition 3.9, we can obtain the following result.

Proposition 3.10. *Let L be a coframe. Then the pair (L, α_L) is a Φ-algebra of the Scott closed set monad (Φ, μ, η) over* $\mathbf{ConvexS}_0$.

Lemma 3.11. ([1]) *A retract of an injective object is injective.*

The following conclusion can also be found in [10]. For the completeness of this paper, we deduce it here, but with a different proof.

Proposition 3.12. ([10]) *Let (X, α_X) be a Φ-algebra with respect to (Φ, η, μ). Then α_X satisfies the following properties.*

(1) $(X, \le_{\mathfrak{C}_X})$ is a coframe;

(2) For each $A \subseteq X$, $\alpha_X(R_A) = \bigwedge A$, where $R_A = \{C \in \mathfrak{C}_X \colon C \cap A = \emptyset\}$. Hence X is a complete lattice;

(3) α_X preserves finite joins.

Proof. (1) Since (X, α_X) is a Φ-algebra with respect to (Φ, η, μ), we have $\alpha_X \circ \eta_X = id_X$ and $\eta_X \dashv \alpha_X$. Then X is a retract of ΦX and so Lemma 3.11 gives that X is injective in $\mathbf{ConvexS}_0$. Thus, it follows by Corollary 3.3 that $(X, \le_{\mathfrak{C}_X})$ is a coframe.

(2) From Corollary 3.3 and the proof of (1), we have $(X, \le_{\mathfrak{C}_X})$ is a coframe, $\mathfrak{C}_X = Idl((X, \le_{\mathfrak{C}_X}))$ and $\eta_X \dashv \alpha_X$. Then by the uniqueness of right adjoint and Lemma 3.6, we have $\alpha_X(\mathbb{I}) = \bigwedge\{x \in X \colon \downarrow x \notin \mathbb{I}\}$. For each $A \subseteq X$, it is clear that $\uparrow A = \{y \in X \colon \downarrow y \notin R_A\}$ and then $\alpha_X(R_A) = \bigwedge A$.

(3) By (1) and Proposition 3.7. □

Now, by Proposition 3.10 and Proposition 3.12, we can immediately obtain the first main theorem in this paper as below.

Theorem 3.13. *The Eilenberg-Moore algebras of the Scott closed set monad over* $\mathbf{ConvexS}_0$ *are precisely coframes endowed with the Scott convex structure. Moreover, the structured morphism $\alpha_X \colon \Phi X \longrightarrow X$ of an algebra with the underlying S_0-convex space X is given by $\alpha_X(\mathbb{I}) = \bigwedge\{x \in X \colon \downarrow x \notin \mathbb{I}\}$ for any $\mathbb{I} \in \Phi X$.*

In the following, we shall present the second main theorem in this paper.

Theorem 3.14. $\mathbf{CoFrm}$ *is monadic over* $\mathbf{ConvexS}_0$.

Proof. We only need to show that a map $f\colon L \longrightarrow M$ between coframes is a coframe homomorphism iff f is CP and $f \circ \alpha_L = \alpha_M \circ \Phi f$.

Sufficiency: By Lemma 3.5, f preserves finite joins. By Proposition 3.12, we have $f(\bigwedge A) = f(\alpha_L(R_A)) = \alpha_M(\Phi f(R_A)) = \alpha_M(R_{f(A)}) = \bigwedge f(A)$.

Necessity: By Lemma 3.5, it suffices to show that $f \circ \alpha_L = \alpha_M \circ \Phi f$. We first prove that $\mathbb{I} = R_B$ for any $\mathbb{I} \in \Phi L$, where $B = X\setminus \bigcup \mathbb{I}$. It is clear that $\mathbb{I} \subseteq R_B$. Let $I \in R_B$. Then $I \subseteq \bigcup \mathbb{I}$. Since I is an ideal of L and $\mathbb{I}$ is Scott closed in $Idl(L)$, we have $I \in \mathbb{I}$ and so $R_B \subseteq \mathbb{I}$. For any $A \subseteq L$, we have $f(\alpha_L(R_A)) = f(\bigwedge A) = \bigwedge f(A) = \alpha_L(R_{f(A)}) = \alpha_L(\Phi f(R_A))$. From Lemma 3.1 and Lemma 3.6, the maps α_L, α_M and Φf all preserve arbitrary meets. Thus, $f \circ \alpha_L = \alpha_M \circ \Phi f$. □

Remark 3.15. Obviously, the comparison functor $K\colon \mathbf{CoFrm} \longrightarrow \mathbf{ConvexS}_0^{\Phi}$ is defined by $L \mapsto (\mathbf{Idl}(L), \alpha_L)$ and $(L \xrightarrow{f} M) \mapsto (\mathbf{Idl}(L) \xrightarrow{f} \mathbf{Idl}(M))$. Let $\mathbf{Idl}\colon \mathbf{CoFrm} \longrightarrow \mathbf{ConvexS}_0$ denote the functor defined by $L \mapsto (L, Idl(L))$ and $(L \xrightarrow{f} M) \mapsto (\mathbf{Idl}(L) \xrightarrow{f} \mathbf{Idl}(M))$. Then the related adjoint pair is $\Phi \dashv \mathbf{Idl}\colon \mathbf{ConvexS}_0 \longrightarrow \mathbf{CoFrm}$. By Corollary 3.3 and Proposition 3.12, it can be similarly proved that K is an isomorphic functor with the specialization order functor as the inverse. This means that the monadicity of **CoFrm** over $\mathbf{ConvexS}_0$ is in fact strict.

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