Filomat 37:29 (2023), 10025–10031
https://doi.org/10.2298/FIL2329025Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Function characterizations of some topological spaces

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Abstract. We present some new characterizations of some spaces such as stratifiable spaces, MCP-spaces in terms of real-valued functions.

1. Introduction and preliminaries

A space always means a topological space. The set of all positive integers is denoted by $\mathbb{N}$. For a space X, denote by $\mathcal{C}_X$ the family of all compact subsets of X. τ and τ^c denote the topology of X and the family of all closed subsets of X, respectively. For a subset A of a space, we write $\overline{A}$ for the closure of A. Also, we use χ_A to denote the characteristic function of A.

Let f be a real-valued function on a space X and r a real number, denote $\{f > r\} = \{x \in X : f(x) > r\}$ and analogously for the others. f is called lower (upper) semi-continuous [4] if for any real number r, the set $\{f > r\}$ ($\{f < r\}$) is open. We write $L(X)$ ($U(X)$) for the set of all lower (upper) semi-continuous functions from X into $[0, 1]$. $C(X)$ is the set of all continuous functions from X into $[0, 1]$.

Definition 1.1. A space X is called *stratifiable* [2] (*semi-stratifiable* [3]) if there is a map $\rho : \mathbb{N} \times \tau^c \to \tau$ such that

(1) $F = \bigcap_{n\in\mathbb{N}} \rho(n, F) = \bigcap_{n\in\mathbb{N}} \overline{\rho(n, F)}$ ($F = \bigcap_{n\in\mathbb{N}} \rho(n, F)$) for each $F \in \tau^c$;

(2) if $F, H \in \tau^c$ and $F \subset H$, then $\rho(n, F) \subset \rho(n, H)$ for all $n \in \mathbb{N}$.

X is called k-semi-stratifiable [10] if there is a map $\rho : \mathbb{N} \times \tau^c \to \tau$ satisfies the conditions for a semi-stratifiable space and

(3) For each $K \in \mathcal{C}_X$ and $F \in \tau^c$ with $K \cap F = \emptyset$, there is $m \in \mathbb{N}$ such that $K \cap \rho(m, F) = \emptyset$.

The map ρ is called the stratification (semi-stratification, k-semi-stratification) for X.

Definition 1.2. ([5]) A space X is called an MCP-spaces (MCM-spaces) if there is an operator U assigning to each decreasing sequence $\langle F_j \rangle$ of closed subsets of X with empty intersection, a sequence of open sets $\{U(n, \langle F_j \rangle) : n \in \mathbb{N}\}$ such that

(1) $F_n \subset U(n, \langle F_j \rangle)$ for each $n \in \mathbb{N}$,

(2) if $\langle F_j \rangle \preceq \langle G_j \rangle$, then $U(n, \langle F_j \rangle) \subset U(n, \langle G_j \rangle)$ for all $n \in \mathbb{N}$,

(3) $\bigcap_{n\in\mathbb{N}} \overline{U(n, \langle F_j \rangle)} = \emptyset$ ($\bigcap_{n\in\mathbb{N}} U(n, \langle F_j \rangle) = \emptyset$).

2020 *Mathematics Subject Classification*. Primary 54C08; Secondary 54C30, 54D20, 54E20.
Keywords. Real-valued functions; Stratifiable spaces; MCP-spaces.
Received: 21 March 2023; Revised: 07 June 2023; Accepted: 09 June 2023
Communicated by Ljubiša D.R. Kočinac
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Definition 1.3. ([11]) A space X is called a k-MCM space if there is an operator U assigning to each decreasing sequence $\langle F_j\rangle$ of closed subsets of X with empty intersection, a decreasing sequence $\{U(n,\langle F_j\rangle)\}_{n\in\mathbb{N}}$ of open subsets of X such that

(1) $F_n \subset U(n,\langle F_j\rangle)$ for each $n\in\mathbb{N}$,

(2) if $\langle F_j\rangle \leq \langle H_j\rangle$, then $U(n,\langle F_j\rangle)\subset U(n,\langle H_j\rangle)$ for each $n\in\mathbb{N}$,

(3) for each $K\in C_X$, there exists $m\in\mathbb{N}$ such that $K\cap U(m,\langle F_j\rangle)=\emptyset$.

Characterizations of some topological spaces such as stratifiable spaces, MCP-spaces in terms of real-valued functions which are known to be insertion theorems were studied extensively in literature [8, 9, 12–15]. Various conditions formulated with real-valued functions which guarantee a space possesses some corresponding properties were presented. For example.

Theorem 1.4. *For a space X, the following are equivalent.*

(a) *X is stratifiable.*

(b) *([8]) There exists an order-preserving map $\phi : L(X)\to C(X)$ such that for each $h\in L(X)$, $\phi(h)\leq h$ and $0<\phi(h)(x)<h(x)$ whenever $h(x)>0$.*

(c) *([12]) There exist two order-preserving maps $\phi : L(X)\to U(X)$ and $\psi : L(X)\to L(X)$ such that for each $h\in L(X)$, $\psi(h)\leq\phi(h)\leq h$ and $0<\psi(h)(x)\leq\phi(h)(x)<h(x)$ whenever $h(x)>0$.*

In this paper, we shall present some new characterizations of some spaces such as stratifiable spaces, MCP-spaces in terms of real-valued functions. Conditions in these results are simplified and are easier to be verified.

The following lemma will be frequently used in the proof of the main results.

Lemma 1.5. *Let $\{A_n : n\in\mathbb{N}\}$ be a decreasing sequence of subsets of a space X. For each $x\notin\bigcap_{n\in\mathbb{N}}A_n$, let $n_x=\min\{n\in\mathbb{N} : x\notin A_n\}$. Define a map $h : X\to[0,1]$ by letting for each $x\in X$, $h(x)=0$ whenever $x\in\bigcap_{n\in\mathbb{N}}A_n$ and $h(x)=\frac{1}{n_x}$ whenever $x\notin\bigcap_{n\in\mathbb{N}}A_n$.*

(1) If A_n is open for each $n\in\mathbb{N}$ then $h\in U(X)$.

(2) If A_n is closed for each $n\in\mathbb{N}$ then $h\in L(X)$.

Proof. (1) Let $r\in\mathbb{R}$ and $h(x)<r$.

Case 1. $x\in\bigcap_{n\in\mathbb{N}}A_n$. Then $0=h(x)<r$. There exists $m\in\mathbb{N}$ such that $\frac{1}{m}<r$. Then A_m is an open neighborhood of x. For each $y\in A_m$, if $y\in\bigcap_{n\in\mathbb{N}}A_n$, then $h(y)=0<r$. If $y\notin\bigcap_{n\in\mathbb{N}}A_n$, then $m<n_y$ and thus $h(y)=\frac{1}{n_y}<\frac{1}{m}<r$.

Case 2. $x\notin\bigcap_{n\in\mathbb{N}}A_n$. If $n_x=1$, then $1=h(x)<r$. For each $y\in X$, $h(y)\leq 1<r$. If $n_x>1$, then A_{n_x-1} is an open neighborhood of x. For each $y\in A_{n_x-1}$, if $y\in\bigcap_{n\in\mathbb{N}}A_n$, then $h(y)=0<\frac{1}{n_x}=h(x)<r$. If $y\notin\bigcap_{n\in\mathbb{N}}A_n$, then $n_x-1<n_y$ and thus $h(y)=\frac{1}{n_y}\leq\frac{1}{n_x}=h(x)<r$.

The above argument shows that h is upper semi-continuous.

(2) Let $r\in\mathbb{R}$ and $h(x)>r$.

Case 1. $x\in\bigcap_{n\in\mathbb{N}}A_n$. Then $r<h(x)=0$. For each $y\in X$, $h(y)\geq 0>r$.

Case 2. $x\notin\bigcap_{n\in\mathbb{N}}A_n$. Then $W=X\setminus A_{n_x}$ is an open neighborhood of x. For each $y\in W$, $n_y\leq n_x$ from which it follows that $h(x)=\frac{1}{n_x}\leq\frac{1}{n_y}=h(y)$. Thus $h(y)>r$.

The above argument shows that h is lower semi-continuous. □

2. On stratifiable spaces

This section is devoted to the characterizations of spaces having stratifications, such as stratifiable spaces and k-semi-stratifiable spaces.

Theorem 2.1. *For a space X, the following are equivalent.*

(a) *X is a stratifiable space.*

(*b*) *There exist two order-preserving maps* $\phi : L(X) \to U(X)$ *and* $\psi : L(X) \to L(X)$ *such that for each* $h \in L(X)$, $\psi(h) \leq \phi(h)$ *and* $h(x) = 0$ *if and only if* $\phi(h)(x) = 0 = \psi(h)(x)$.

(*c*) *There exists an order-preserving map* $\phi : L(X) \to U(X)$ *such that for each* $h \in L(X)$ *and* $x \in X$, *if* $h(x) = 0$ *then* $\phi(h)(x) = 0$, *and if* $h(x) \neq 0$ *then there exists an open neighborhood* V_x *of* x *such that* $\inf\{\phi(h)(y) : y \in V_x\} > 0$.

Proof. (a) ⇒ (b) Let ρ be the stratification for X which is decreasing with respect to n. For each $h \in L(X)$ and $n \in \mathbb{N}$, let $F(n,h) = \{h \leq \frac{1}{n}\}$. Then $\{F(n,h) : n \in \mathbb{N}\}$ is a decreasing sequence of closed subsets of X such that $\bigcap_{n\in\mathbb{N}} F(n,h) = h^{-1}(0)$. For each $x \in X$, if $x \notin \bigcap_{n\in\mathbb{N}} F(n,h)$ then there exists $i \in \mathbb{N}$ such that $x \notin \overline{\rho(i, F(i,h))}$. Let

$$n_x(h) = \min\{n \in \mathbb{N} : x \notin \rho(n, F(n,h))\},\ m_x(h) = \min\{n \in \mathbb{N} : x \notin \overline{\rho(n, F(n,h))}\}.$$

For each $x \in X$, let $\phi(h)(x) = 0 = \psi(h)(x)$ whenever $x \in \bigcap_{n\in\mathbb{N}} F(n,h)$ and $\phi(h)(x) = \frac{1}{n_x(h)}$, $\psi(h)(x) = \frac{1}{m_x(h)}$ whenever $x \notin \bigcap_{n\in\mathbb{N}} F(n,h)$. Since

$$\bigcap_{n\in\mathbb{N}} F(n,h) = \bigcap_{n\in\mathbb{N}} \rho(n, F(n,h)) = \bigcap_{n\in\mathbb{N}} \overline{\rho(n, F(n,h))},$$

by Lemma 1.5, $\phi(h) \in U(X)$ and $\psi(h) \in L(X)$.

Let $x \in X$. If $x \in \bigcap_{n\in\mathbb{N}} F(n,h)$ then $\phi(h)(x) = 0 = \psi(h)(x)$. If $x \notin \bigcap_{n\in\mathbb{N}} F(n,h)$, since $x \notin \overline{\rho(m_x(h), F(m_x(h),h))} \supset \rho(m_x(h), F(m_x(h),h))$, we have that $n_x(h) \leq m_x(h)$. It follows that $\psi(h)(x) = \frac{1}{m_x(h)} \leq \frac{1}{n_x(h)} = \phi(h)(x)$. Therefore, $\psi(h) \leq \phi(h)$.

Let $x \in X$. Then $h(x) = 0$ if and only if $x \in \bigcap_{n\in\mathbb{N}} F(n,h)$ if and only if $\phi(h)(x) = 0 = \psi(h)(x)$.

Suppose that $h_1 \leq h_2$. For each $x \in X$, if $\phi(h_1)(x) = 0$ then $\phi(h_1)(x) \leq \phi(h_2)(x)$. If $\phi(h_1)(x) = \frac{1}{n_x(h_1)}$ then $x \notin \bigcap_{n\in\mathbb{N}} F(n,h_1)$. From $h_1 \leq h_2$ it follows that $F(n,h_2) \subset F(n,h_1)$ for each $n \in \mathbb{N}$ and thus $x \notin \bigcap_{n\in\mathbb{N}} F(n,h_2)$. Since $x \notin \rho(n_x(h_1), F(n_x(h_1),h_1)) \supset \rho(n_x(h_1), F(n_x(h_1),h_2))$, we have $n_x(h_2) \leq n_x(h_1)$ and thus $\phi(h_1)(x) = \frac{1}{n_x(h_1)} \leq \frac{1}{n_x(h_2)} = \phi(h_2)(x)$. Similarly, $\psi(h_1) \leq \psi(h_2)$.

(b) ⇒ (c) Assume (b). If $h(x) \neq 0$ then $\psi(h)(x) \neq 0$ and thus there exists $r > 0$ such that $\psi(h)(x) > r$. Let $V_x = \{\psi(h) > r\}$. Then V_x is an open neighborhood of x. For each $y \in V_x$, $\phi(h)(y) \geq \psi(h)(y) > r$ which implies that $\inf\{\phi(h)(y) : y \in V_x\} > 0$.

(c) ⇒ (a) Assume (c). For each $F \in \tau^c$, let $h_F = 1 - \chi_F$. Then $h_F \in L(X)$.

For each $F \in \tau^c$ and $n \in \mathbb{N}$, let $\rho(n,F) = \{\phi(h_F) < \frac{1}{n}\}$. Then $\rho(n,F) \in \tau$.

If $x \in F$ then $h_F(x) = 0$. It follows that $\phi(h_F)(x) = 0$ and thus $x \in \rho(n,F)$ for each $n \in \mathbb{N}$.

If $x \notin F$ then $h_F(x) \neq 0$. By (c), there exists an open neighborhood V_x of x and $m \in \mathbb{N}$ such that $\phi(h_F)(y) > \frac{1}{m}$ for each $y \in V_x$. It follows that $V_x \cap \{\phi(h_F) < \frac{1}{m}\} = \emptyset$ and thus $x \notin \overline{\rho(m,F)}$. This implies that $\bigcap_{n\in\mathbb{N}} \overline{\rho(n,F)} \subset F$.

Suppose that $F_1 \subset F_2$. For each $x \in X$, if $x \in F_2$ then $h_{F_2}(x) = 0$ and thus $h_{F_2}(x) \leq h_{F_1}(x)$. If $x \notin F_2$ then $x \notin F_1$ and thus $h_{F_2}(x) = 1 = h_{F_1}(x)$. This implies that $h_{F_2} \leq h_{F_1}$. Then $\phi(h_{F_2}) \leq \phi(h_{F_1})$ from which it follows that $\rho(n,F_1) \subset \rho(n,F_2)$ for each $n \in \mathbb{N}$.

Therefore, X is a stratifiable space. □

Theorem 2.2. *A space* X *is* k*-semi-stratifiable if and only if there exists an order-preserving map* $\phi : L(X) \to U(X)$ *such that for each* $h \in L(X)$ *and* $K \in C_X$, $K \cap h^{-1}(0) = \emptyset$ *if and only if* $\inf\{\phi(h)(x) : x \in K\} > 0$.

Proof. Let ρ be the k-semi-stratification for X which is decreasing with respect to n. For each $h \in L(X)$ and $n \in \mathbb{N}$, let $F(n,h) = \{h \leq \frac{1}{n}\}$. Then $\{F(n,h) : n \in \mathbb{N}\}$ is a decreasing sequence of closed subsets of X such that $\bigcap_{n\in\mathbb{N}} F(n,h) = h^{-1}(0)$. For each $x \in X$, if $x \notin \bigcap_{n\in\mathbb{N}} F(n,h)$ then there exists $i \in \mathbb{N}$ such that $x \notin \rho(i, F(i,h))$. Let

$$n_x(h) = \min\{n \in \mathbb{N} : x \notin \rho(n, F(n,h))\}.$$

For each $x \in X$, let $\phi(h)(x) = 0$ whenever $x \in \bigcap_{n\in\mathbb{N}} F(n,h)$ and $\phi(h)(x) = \frac{1}{n_x(h)}$ whenever $x \notin \bigcap_{n\in\mathbb{N}} F(n,h)$. Since $\bigcap_{n\in\mathbb{N}} F(n,h) = \bigcap_{n\in\mathbb{N}} \rho(n, F(n,h))$, by Lemma 1.5 (1), $\phi(h) \in U(X)$.

Let $K \in C_X$. If $K \cap h^{-1}(0) = \emptyset$ then $K \cap \bigcap_{n\in\mathbb{N}} F(n,h) = \emptyset$ and thus $K \cap F(i,h) = \emptyset$ for some $i \in \mathbb{N}$. It follows that $K \cap \rho(m, F(m,h)) = \emptyset$ for some $m \in \mathbb{N}$. Then for each $x \in K$, $n_x(h) \leq m$ and so $\phi(h)(x) \geq \frac{1}{m}$. This implies that $\inf\{\phi(h)(x) : x \in K\} > 0$.

If $\inf\{\phi(h)(x) : x \in K\} > 0$, then for each $x \in K$, $\phi(h)(x) > 0$ which implies that $x \notin \bigcap_{n\in\mathbb{N}} F(n,h) = h^{-1}(0)$. Therefore, $K \cap h^{-1}(0) = \emptyset$.

That ϕ is order-preserving can be shown analogously to the proof of (a) ⇒ (b) of Theorem 2.1.

Conversely, for each $F \in \tau^c$, let $h_F = 1 - \chi_F$. Then $h_F \in L(X)$.

For each $F \in \tau^c$ and $n \in \mathbb{N}$, let $\rho(n,F) = \{\phi(h_F) < \frac{1}{n}\}$. Then $\rho(n,F) \in \tau$.

If $x \in F \in \tau^c$, then $h_F(x) = 0$. It follows that $x \in h_F^{-1}(0)$ and thus $\phi(h_F)(x) = 0$. Hence $x \in \rho(n,F)$ for each $n \in \mathbb{N}$.

Let $K \in C_X$ and $F \in \tau^c$ with $K \cap F = \emptyset$. Then $K \cap h_F^{-1}(0) = \emptyset$ and thus $\inf\{\phi(h_F)(x) : x \in K\} > 0$. It follows that $K \cap \{\phi(h_F) < \frac{1}{m}\} = \emptyset$ for some $m \in \mathbb{N}$. That is, $K \cap \rho(m,F) = \emptyset$.

Suppose that $F_1 \subset F_2$. With a similar argument to the proof of (c) ⇒ (a) of Theorem 2.1, one readily shows that $\rho(n,F_1) \subset \rho(n,F_2)$ for each $n \in \mathbb{N}$.

Therefore, X is k-semi-stratifiable. □

Similarly, we have the following.

Proposition 2.3. *A space X is semi-stratifiable if and only if there exists an order-preserving map $\phi : L(X) \to U(X)$ such that for each $h \in L(X)$ and $x \in X$, $h(x) = 0$ if and only if $\phi(h)(x) = 0$.*

A space X is perfectly normal [4] if and only if there is a map $\rho : \mathbb{N}\times\tau^c \to \tau$ such that $F = \bigcap_{n\in\mathbb{N}} \rho(n,F) = \bigcap_{n\in\mathbb{N}} \overline{\rho(n,F)}$ for each $F \in \tau^c$.

Theorem 2.4. *For a space X, the following are equivalent.*

(*a*) *X is perfectly normal.*

(*b*) *There exist two maps $\phi : L(X) \to U(X)$ and $\psi : L(X) \to L(X)$ such that for each $h \in L(X)$, $\psi(h) \le \phi(h)$ and $h(x) = 0$ if and only if $\phi(h)(x) = 0 = \psi(h)(x)$.*

(*c*) *There exists a map $\phi : L(X) \to U(X)$ such that for each $h \in L(X)$ and $x \in X$, $h(x) = 0$ if and only if $\phi(h)(x) = 0$, and if $h(x) \neq 0$ then there exists an open neighborhood V_x of x such that* $\inf\{\phi(h)(y) : y \in V_x\} > 0$.

Proof. (a) ⇒ (b) Let ρ be the map for a perfectly normal space which is decreasing with respect to n. For each $h \in L(X)$ and $n \in \mathbb{N}$, let $F(n,h) = \{h \le \frac{1}{n}\}$. Then $\{F(n,h) : n \in \mathbb{N}\}$ is a decreasing sequence of closed subsets of X such that $\bigcap_{n\in\mathbb{N}} F(n,h) = h^{-1}(0)$. For each $n \in \mathbb{N}$, let $\sigma(n,F(n,h)) = \bigcap_{j\le n} \rho(n,F(j,h))$. Then $\sigma(n,F(n,h)) \in \tau$. We show that

$$\bigcap_{n\in\mathbb{N}} F(n,h) = \bigcap_{n\in\mathbb{N}} \sigma(n,F(n,h)) = \bigcap_{n\in\mathbb{N}} \overline{\sigma(n,F(n,h))}.$$

For each $n \in \mathbb{N}$ and $j \le n$, $F(n,h) \subset F(j,h) \subset \rho(n,F(j,h))$ which implies that $F(n,h) \subset \bigcap_{j\le n} \rho(n,F(j,h)) = \sigma(n,F(n,h))$. It follows that $\bigcap_{n\in\mathbb{N}} F(n,h) \subset \bigcap_{n\in\mathbb{N}} \sigma(n,F(n,h))$. If $x \notin \bigcap_{n\in\mathbb{N}} F(n,h)$ then $x \notin F(i,h) = \bigcap_{n\in\mathbb{N}} \overline{\rho(n,F(i,h))}$ for some $i \in \mathbb{N}$. Then there exists $m \ge i$ such that

$$x \notin \overline{\rho(m,F(i,h))} \supset \overline{\bigcap_{j\le m} \rho(m,F(j,h))} = \overline{\sigma(m,F(m,h))}.$$

Hence $\bigcap_{n\in\mathbb{N}} \overline{\sigma(n,F(n,h))} \subset \bigcap_{n\in\mathbb{N}} F(n,h)$.

For each $x \in X$, if $x \notin \bigcap_{n\in\mathbb{N}} F(n,h)$ then there exists $i \in \mathbb{N}$ such that $x \notin \overline{\sigma(i,F(i,h))}$. Let

$$n_x(h) = \min\{n \in \mathbb{N} : x \notin \sigma(n,F(n,h))\},\ m_x(h) = \min\{n \in \mathbb{N} : x \notin \overline{\sigma(n,F(n,h))}\}.$$

For each $x \in X$, let $\phi(h)(x) = 0 = \psi(h)(x)$ whenever $x \in \bigcap_{n\in\mathbb{N}} F(n,h)$ and $\phi(h)(x) = \frac{1}{n_x(h)}$, $\psi(h)(x) = \frac{1}{m_x(h)}$ whenever $x \notin \bigcap_{n\in\mathbb{N}} F(n,h)$. Since

$$\bigcap_{n\in\mathbb{N}} F(n,h) = \bigcap_{n\in\mathbb{N}} \sigma(n,F(n,h)) = \bigcap_{n\in\mathbb{N}} \overline{\sigma(n,F(n,h))},$$

by Lemma 1.5, $\phi(h) \in U(X)$ and $\psi(h) \in L(X)$.

Let $x \in X$. If $x \in \bigcap_{n\in\mathbb{N}} F(n,h)$ then $\phi(h)(x) = 0 = \psi(h)(x)$. If $x \notin \bigcap_{n\in\mathbb{N}} F(n,h)$, since $x \notin \overline{\sigma(m_x(h), F(m_x(h),h))} \supset \sigma(m_x(h), F(m_x(h),h))$, we have that $n_x(h) \leq m_x(h)$. It follows that $\psi(h)(x) = \frac{1}{m_x(h)} \leq \frac{1}{n_x(h)} = \phi(h)(x)$. Therefore, $\psi(h) \leq \phi(h)$.

Let $x \in X$. Then $h(x) = 0$ if and only if $x \in \bigcap_{n\in\mathbb{N}} F(n,h)$ if and only if $\phi(h)(x) = 0 = \psi(h)(x)$.

(b) ⇒ (c) Similar to the proof of (b) ⇒ (c) of Theorem 2.1.

(c) ⇒ (a) Similar to the proof of (c) ⇒ (a) of Theorem 2.1. □

A space X is perfect [4] if and only if there is a map $\rho : \mathbb{N} \times \tau^c \to \tau$ such that $F = \bigcap_{n\in\mathbb{N}} \rho(n,F)$ for each $F \in \tau^c$.

Proposition 2.5. *A space X is perfect if and only if there exists a map $\phi : L(X) \to U(X)$ such that for each $h \in L(X)$ and $x \in X$, $h(x) = 0$ if and only if $\phi(h)(x) = 0$.*

3. On MCP-spaces

In this section, we present characterizations of some weak covering properties, such as MCP-spaces and countably paracompact spaces.

Denote by $L^+(X) = \{h \in L(X) : h > 0\}$ and $U^+(X) = \{h \in U(X) : h > 0\}$.

Theorem 3.1. *For a space X, the following are equivalent.*

(*a*) *X is an MCP-space.*

(*b*) *There exist two order-preserving maps $\phi : L^+(X) \to U(X)$ and $\psi : L^+(X) \to L^+(X)$ such that for each $h \in L^+(X)$, $\psi(h) \leq \phi(h)$ and $\{h \leq \frac{1}{n}\} \subset \{\phi(h) < \frac{1}{n}\}$ for each $n \in \mathbb{N}$.*

(*c*) *There exists an order-preserving map $\phi : L^+(X) \to U(X)$ such that for each $h \in L^+(X)$ and $n \in \mathbb{N}$, $\{h \leq \frac{1}{n}\} \subset \{\phi(h) < \frac{1}{n}\}$ and for each $x \in X$, there exists an open neighborhood V_x of x such that* $\inf\{\phi(h)(y) : y \in V_x\} > 0$.

Proof. (a) ⇒ (b) Let U be the operator in Definition 1.2 such that $\{U(n, \langle F_j\rangle) : n \in \mathbb{N}\}$ is decreasing with respect to n. For each $h \in L^+(X)$ and $n \in \mathbb{N}$, let $F_n(h) = \{h \leq \frac{1}{n}\}$. Then $\langle F_n(h)\rangle$ is a decreasing sequence of closed subsets of X with an empty intersection. For each $x \in X$, let

$$n_x(h) = \min\{n \in \mathbb{N} : x \notin U(n, \langle F_j(h)\rangle)\}, m_x(h) = \min\{n \in \mathbb{N} : x \notin \overline{U(n, \langle F_j(h)\rangle)}\}.$$

Then let $\phi(h)(x) = \frac{1}{n_x(h)}$ and $\psi(h)(x) = \frac{1}{m_x(h)}$. It is clear that $\psi(h) > 0$. By Lemma 1.5, $\phi(h) \in U(X)$ and $\psi(h) \in L(X)$. Moreover, since $n_x(h) \leq m_x(h)$ for each $x \in X$, we have $\psi(h) \leq \phi(h)$.

Let $x \in \{h \leq \frac{1}{n}\}$. Since $x \notin U(n_x(h), \langle F_j(h)\rangle) \supset F_{n_x(h)}(h)$, we have $n < n_x(h)$. It follows that $\phi(h)(x) = \frac{1}{n_x(h)} < \frac{1}{n}$ and thus $x \in \{\phi(h) < \frac{1}{n}\}$.

Suppose that $h_1 \leq h_2$. Then $\langle F_j(h_2)\rangle \leq \langle F_j(h_1)\rangle$ and hence $U(n, \langle F_j(h_2)\rangle) \subset U(n, \langle F_j(h_1)\rangle)$ for each $n \in \mathbb{N}$. Let $x \in X$. Since $x \notin U(n_x(h_1), \langle F_j(h_1)\rangle) \supset U(n_x(h_1), \langle F_j(h_2)\rangle)$, we have that $n_x(h_2) \leq n_x(h_1)$ and thus $\phi(h_1)(x) \leq \phi(h_2)(x)$. Therefore, $\phi(h_1) \leq \phi(h_2)$. Similarly, $\psi(h_1) \leq \psi(h_2)$.

(b) ⇒ (c) Since $\psi(h) > 0$, for each $x \in X$, there exists $r > 0$ such that $\psi(h)(x) > r$. Let $V_x = \{\psi(h) > r\}$. Then V_x is an open neighborhood of x. For each $y \in V_x$, $\phi(h)(y) \geq \psi(h)(y) > r$ which implies that $\inf\{\phi(h)(y) : y \in V_x\} > 0$.

(c) ⇒ (a) For each decreasing sequence $\langle F_j\rangle$ of closed subsets of X with empty intersection and each $x \in X$, let

$$n_x(\langle F_j\rangle) = \min\{n \in \mathbb{N} : x \notin F_n\}, \ \ h_{\langle F_j\rangle}(x) = \frac{1}{n_x(\langle F_j\rangle)}.$$

Then $h_{\langle F_j\rangle} > 0$. By Lemma 1.5 (2), $h_{\langle F_j\rangle} \in L(X)$.

For each $n \in \mathbb{N}$, let

$$U(n, \langle F_j\rangle) = \{\phi(h_{\langle F_j\rangle}) < \frac{1}{n}\}.$$

Then $\langle U(n, \langle F_j \rangle)\rangle$ is a decreasing sequence of open subsets of X. By (c), for each $x \in X$, there exists an open neighborhood V_x of x and $m \in \mathbb{N}$ such that $V_x \cap \{\phi(h_{\langle F_j \rangle}) < \frac{1}{m}\} = \emptyset$. It follows that $x \notin \overline{U(m, \langle F_j \rangle)}$ and thus $\bigcap_{n\in\mathbb{N}} \overline{U(n, \langle F_j \rangle)} = \emptyset$.

Let $n \in \mathbb{N}$ and $x \in F_n$. Then $n < n_x(\langle F_j \rangle)$. It follows that $h_{\langle F_j \rangle}(x) = \frac{1}{n_x(\langle F_j \rangle)} < \frac{1}{n}$. By (c), $x \in \{\phi(h_{\langle F_j \rangle}) < \frac{1}{n}\} = U(n, \langle F_j \rangle)$. Hence $F_n \subset U(n, \langle F_j \rangle)$.

Suppose that $\langle F_j \rangle \leq \langle H_j \rangle$. For each $x \in X$, since $x \notin H_{n_x(\langle H_j \rangle)} \supset F_{n_x(\langle H_j \rangle)}$, we have $n_x(\langle F_j \rangle) \leq n_x(\langle H_j \rangle)$ and thus $h_{\langle H_j \rangle}(x) \leq h_{\langle F_j \rangle}(x)$. This implies that $h_{\langle H_j \rangle} \leq h_{\langle F_j \rangle}$. Hence, $\phi(h_{\langle H_j \rangle}) \leq \phi(h_{\langle F_j \rangle})$ from which it follows that $U(n, \langle F_j \rangle) \subset U(n, \langle H_j \rangle)$ for each $n \in \mathbb{N}$.

By Definition 1.2, X is an MCP-space. □

Theorem 3.2. *X is a k-MCM space if and only if there exists an order-preserving map $\phi : L^+(X) \to U(X)$ such that for each $h \in L^+(X)$ and $n \in \mathbb{N}$, $\{h \leq \frac{1}{n}\} \subset \{\phi(h) < \frac{1}{n}\}$ and for each $K \in \mathcal{C}_X$, $\inf\{\phi(h)(x) : x \in K\} > 0$.*

Proof. Let U be the operator in Definition 1.3 such that $\{U(n, \langle F_j \rangle) : n \in \mathbb{N}\}$ is decreasing with respect to n. For each $h \in L^+(X)$, define a map $\phi(h) \in U(X)$ as that in the proof of (a) ⇒ (b) of Theorem 3.1. Then we only need to show that for each $K \in \mathcal{C}_X$, $\inf\{\phi(h)(x) : x \in K\} > 0$.

Let $K \in \mathcal{C}_X$. Then there exists $m \in \mathbb{N}$ such that $K \cap U(m, \langle F_j(h) \rangle) = \emptyset$. For each $x \in K$, $x \notin U(m, \langle F_j(h) \rangle)$ and thus $n_x(h) \leq m$. It follows that $\phi(h)(x) = \frac{1}{n_x(h)} \geq \frac{1}{m}$. Therefore, $\inf\{\phi(h)(x) : x \in K\} > 0$.

Conversely, for each decreasing sequence $\langle F_j \rangle$ of closed subsets of X with empty intersection, define a decreasing sequence $\langle U(n, \langle F_j \rangle)\rangle$ of open subsets of X as that in the proof of (c) ⇒ (a) of Theorem 3.1. Then we only need to show that for each $K \in \mathcal{C}_X$, there exists $m \in \mathbb{N}$ such that $K \cap U(m, \langle F_j \rangle) = \emptyset$.

Let $K \in \mathcal{C}_X$. Then $\inf\{\phi(h_{\langle F_j \rangle})(x) : x \in K\} > 0$. There exists $m \in \mathbb{N}$ such that $\phi(h_{\langle F_j \rangle})(x) > \frac{1}{m}$ for each $x \in K$. It follows that $K \cap U(m, \langle F_j \rangle) = K \cap \{\phi(h_{\langle F_j \rangle}) < \frac{1}{m}\} = \emptyset$. □

Similarly, we have the following.

Proposition 3.3. *X is an MCM-space if and only if there exists an order-preserving map $\phi : L^+(X) \to U^+(X)$ such that for each $h \in L^+(X)$ and $n \in \mathbb{N}$, $\{h \leq \frac{1}{n}\} \subset \{\phi(h) < \frac{1}{n}\}$.*

A space X is countably paracompact [7] (countably metacompact [6], countably mesocompact [1]) if and only if for each decreasing sequence $\{F_n : n \in \mathbb{N}\}$ of closed subsets of X with empty intersection, there is a decreasing sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets of X such that $F_n \subset U_n$ for each $n \in \mathbb{N}$ and $\bigcap_{n\in\mathbb{N}} \overline{U_n} = \emptyset$ ($\bigcap_{n\in\mathbb{N}} U_n = \emptyset$, for each compact subset $K \subset X$, there exists $m \in \mathbb{N}$ such that $K \cap U_m = \emptyset$).

The following three results can be shown with similar arguments to the proof of Theorem 3.1, Theorem 3.2 and Proposition 3.3 respectively.

Theorem 3.4. *For a space X, the following are equivalent.*

(*a*) *X is countably paracompact.*

(*b*) *There exist two maps $\phi : L^+(X) \to U(X)$ and $\psi : L^+(X) \to L^+(X)$ such that for each $h \in L^+(X)$, $\psi(h) \leq \phi(h)$ and $\{h \leq \frac{1}{n}\} \subset \{\phi(h) < \frac{1}{n}\}$ for each $n \in \mathbb{N}$.*

(*c*) *There exists a map $\phi : L^+(X) \to U(X)$ such that for each $h \in L^+(X)$ and $n \in \mathbb{N}$, $\{h \leq \frac{1}{n}\} \subset \{\phi(h) < \frac{1}{n}\}$ and for each $x \in X$, there exists an open neighborhood V_x of x such that $\inf\{\phi(h)(y) : y \in V_x\} > 0$.*

Theorem 3.5. *X is countably mesocompact if and only if there exists a map $\phi : L^+(X) \to U(X)$ such that for each $h \in L^+(X)$ and $n \in \mathbb{N}$, $\{h \leq \frac{1}{n}\} \subset \{\phi(h) < \frac{1}{n}\}$ and for each $K \in \mathcal{C}_X$, $\inf\{\phi(h)(x) : x \in K\} > 0$.*

Proposition 3.6. *X is countably metacompact if and only if there exists a map $\phi : L^+(X) \to U^+(X)$ such that for each $h \in L^+(X)$ and $n \in \mathbb{N}$, $\{h \leq \frac{1}{n}\} \subset \{\phi(h) < \frac{1}{n}\}$.*

Acknowledgements

The author would like to thank the referees for their valuable comments and suggestions.

References

[1] R. A. Alo, H. L. Shapiro, *Normal topological spaces*, Cambridge Univ. Press, London, 1974.
[2] C. J. R. Borges, *On stratifiable spaces*, Pacific J. Math. **17** (1966), 1–16.
[3] G. D. Creede, *Concerning semi-stratifiable spaces*, Pacific J. Math. **32** (1970), 47–54.
[4] R. Engelking, *General Topology, Revised and Completed Edition*, Heldermann Verlag, Berlin, 1989.
[5] C. Good, R. W. Knight, I. S. Stares, *Monotone countable paracompactness*, Topol. Appl. **101** (2000), 281–298.
[6] Y. Hayashi, *On countably metacompact spaces*, Bull. Univ. Osaka Pref., Ser. A **8** (1960), 161–164.
[7] F. Ishikawa, *On countably paracompact spaces*, Proc. Japan Acad. **31** (1955), 686–687.
[8] E. Lane, P. Nyikos, C. Pan, *Continuous function characterizations of stratifiable spaces*, Acta Math. Hungar. **92** (2001), 219–231.
[9] K. Li, S. Lin, R. Shen, *Insertions of k-semi-stratifiable spaces by semi-continuous functions*, Stud. Sci. Math. Hung. **48** (2011), 320–330.
[10] D. J. Lutzer, *Semimetrizable and stratifiable spaces*, General Top. Appl. **1** (1971), 43–48.
[11] L. Peng, S. Lin, *Monotone spaces and metrization theorems* (in Chinese), Acta Math. Sinica, Chin. Ser. A **46** (2003), 1225–1232.
[12] L. Xie, P. Yan, *Insertions of semi-continuous functions and sequences of sets*, Acta Math. Hungar. **126** (2010), 164–180.
[13] K. Yamazaki, *Locally bounded set-valued mappings and monotone countable paracompactness*, Topol. Appl. **154** (2007), 2817–2825.
[14] E. Yang, *On MCP-spaces and mcb-spaces*, Houston J. Math. **43** (2017), 669–677.
[15] P. Yan, E. Yang, *Semi-stratifiable spaces and the insertion of semi-continuous functions*, J. Math. Anal. Appl. **38** (2007), 429–437.