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Subgradient extragradient method with double inertial steps for quasi-monotone variational inequalities

Haiying Li[a,*], Xingfang Wang[a]

[a]*College of Mathematics and Information Science, Henan Normal University, Xinxiang 453007.*

Abstract. In this paper, we present a modified subgradient extragradient method with double inertial extrapolation terms and a non-monotonic adaptive step size for solving quasi-monotone and Lipschitz continuous variational inequalities in real Hilbert spaces. Under some suitable conditions, we obtain the weak convergence theorem of our proposed algorithm. Moreover, strong convergence is obtained when the cost operator is strongly pseudo-monotone and Lipschitz continuous. Finally, several numerical results illustrate the effectiveness and competitiveness of our algorithm.

1. Introduction

Throughout this paper, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. It is well known that the classical variational inequality problem (VIP, for short) is defined as: find $v \in C$ such that

$$\langle Av, y - v \rangle \geq 0, \ \forall y \in C, \tag{1}$$

where C is a nonempty, closed, and convex subset of H and $A : H \to H$ is a continuous mapping. The solution set of VIP is denoted by S.

The variational inequality problem has been widely used to problems such as nonlinear programming, network equilibrium problems and complementarity problems (see, for example, [9, 16, 24] and the references therein). So far, a number of iterative algorithms have been proposed for solving the VIP (see, for example, [1, 6, 7, 22, 23, 32, 38, 40]), we mainly focus on several projection methods that inspire us to investigate new algorithms.

One of the most popular early algorithms for solving the problem (1) is the extragradient method (EGM), which was presented by Korpelevich in [18]. This method was first introduced to solve saddle point problems in finite dimensional spaces. It is of the form: Choose $x_1 \in C, L > 0, \lambda_n \in (0, \frac{1}{L})$,

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n) \\ x_{n+1} = P_C(x_n - \lambda_n A y_n). \end{cases} \tag{2}$$

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*Corresponding author: Haiying Li
Email addresses: haiyingli2020@163.com (Haiying Li), xingfangwang0902@163.com (Xingfang Wang)

It was shown that $\{x_n\}$ converges weakly to a solution of VIP when the operator A is monotone and L-Lipschitz continuous. In recent years, the extragradient method has been extend to infinite spaces in various ways (see, for example, [2, 11, 27, 28, 34]). However, the extragradient method needs to calculate two projection values onto the feasible set in each iteration, which might be difficult if the feasible set C is a general closed and convex set.

To overcome the major drawback of algorithm (2), Censor et al.[4] introduced the following subgradient extragradient method (SEGM), which uses a projection onto a specific half-space in place of the second projection onto C in the EGM. It is of the form: Given $x_0 \in C, L > 0, \lambda_n \in (0, \frac{1}{L})$,

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n) \\ T_n = \{w \in H : \langle x_n - \lambda_n A x_n - y_n, w - y_n \rangle \le 0\} \\ x_{n+1} = P_{T_n}(x_n - \lambda_n A y_n), \end{cases} \tag{3}$$

the weak convergence has been obtained when the operator A is monotone and L-Lipschitz continuous. Several improved versions of the SEGM have been proposed (see, for example, [3, 30, 31, 33, 36]).

Quite recently, many authors are committed to investigating algorithms with inertial extrapolation terms, which can be used to speed up the convergence of iterative methods for variational inequalities effectively (see, for example [8, 10, 14, 29, 35]). Shehu et al.[26] presented a modified subgradient extragradient method with single inertial step and self-adaptive step sizes to solve VIP: Given $\lambda_1 > 0, x_0, x_1 \in H$ and $\mu \in (0, 1)$,

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ y_n = p_c(w_n - \lambda_n a w_n) \\ t_n = \{w \in h : \langle w_n - \lambda_n a w_n - y_n, w - y_n \rangle \le 0\} \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n p_{t_n}(w_n - \lambda_n a y_n), \end{cases} \tag{4}$$

where

$$\lambda_{n+1} = \begin{cases} \min\{\dfrac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \lambda_n\}, & if Aw_n \ne Ay_n, \\ \lambda_n, & otherwise, \end{cases}$$

and $0 \le \theta_n \le \theta_{n+1} \le 1$, $0 < \alpha \le \alpha_n \le \alpha_{n+1} \le \frac{1}{2+\delta}(\delta > 0)$. If the operator A is monotone and L-Lipschitz continuous, then the sequence $\{x_n\}$ generated by algorithm (4) converges weakly to a solution of VIP.

Inspired by Shehu et al.[26], Yao et al.[37] proposed a relaxed SEGM with double inertial extrapolation steps, that is, added an inertia to the algorithm (4). It is of the form: Given $\lambda_1 > 0, x_0, x_1 \in H$ and $\mu \in (0, 1)$,

$$\begin{cases} z_n = x_n + \delta(x_n - x_{n-1}) \\ w_n = x_n + \theta_n(x_n - x_{n-1}) \\ y_n = P_C(w_n - \lambda_n A w_n) \\ T_n := \{w \in H : \langle w_n - \lambda_n A w_n - y_n, w - y_n \rangle \le 0\} \\ x_{n+1} = (1 - \alpha_n)z_n + \alpha_n P_{T_n}(w_n - \lambda_n A y_n), \end{cases} \tag{5}$$

where

$$\lambda_{n+1} = \begin{cases} \min\{\dfrac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \lambda_n\}, & if Aw_n \ne Ay_n, \\ \lambda_n, & otherwise, \end{cases}$$

and $0 \le \theta_n \le \theta_{n+1} \le 1$, $0 \le \delta < \min\{\frac{\epsilon - \sqrt{2\epsilon}}{\epsilon}, \theta_1\}$, $0 < \alpha \le \alpha_n \le \alpha_{n+1} < \frac{1}{1+\epsilon}(2 < \epsilon < \infty)$. $\{x_n\}$ converges weakly to a solution of VIP when A is pseudo-monotone and L-Lipschitz continuous. Moreover, in the Numerical

Examples section of Yao et al.[37], it has been shown that this method is more efficient and implementable than the algorithm (4).

It is worth mentioning that very few authors consider the existing methods when the operator A is quasi-monotone, which is weaker than the usual monotone (or pseudo-monotone) condition. Obviously, convergence results obtained under quasi-monotone can be generalized to the results under monotone (or pseudo-monotone), but the reverse may not be true.

Our contributions:

• We introduce a modified subgradient extragradient method with double inertial extrapolation steps to solve the variational inequality problem in real Hilbert space, our method accelerate the convergence rates of the methods in [26, 37] effectively. Compared with the method in [37], our algorithm has the following advantages: (1) we obtain weak convergence result under a weaker operator condition (A is quasi-monotone rather than pseudo-monotone); (2)we take a modified non-monotonic step size, which accelerates the convergence rate effectively; (3) δ_n is variable in the inertia term $z_n = x_n + \delta_n(x_n - x_{n-1})$ in our paper.

• We obtain $\{x_n\}$ generated by Algorithm 1 converges strongly to a point of S when the operator A is strongly pseudo-monotone; We obtain $\{x_n\}$ generated by Algorithm 2 converges weakly to a point of $Fix(T) \cap S$ when the operator A is monotone, T is quasi-nonexpansive and $I - T$ is demiclosed.

• We give numerical simulations to show that our proposed method is more efficient and faster than the related methods.

Our paper is organized as follows: Several definitions and lemmas are given in Sect. 2. In Sect. 3, we present our method and analyse the weak convergence of our method. In Sect. 4, we analyse the strong convergence of our method. We give numerical experiments to illustrate the feasibility of our methods in Sect. 5.

2. Preliminaries

Definition 2.1. *The operator $A : H \to H$ is said to be*

(i) L-Lipschitz continuous, if there exists a constant $L > 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \ \forall x, y \in H.$$

(ii) ϱ-strongly monotone, if there exists a constant $\varrho > 0$ such that

$$\langle Ay - Ax, y - x\rangle \geq \varrho\|y - x\|^2, \ \forall x, y \in H.$$

(iii) monotone, if

$$\langle Ay - Ax, y - x\rangle \geq 0, \ \forall x, y \in H.$$

(iv) η-strongly pseudo-monotone, if there exists a constant $\eta > 0$ such that

$$\langle Ax, y - x\rangle \geq 0 \Rightarrow \langle Ay, y - x\rangle \geq \eta\|y - x\|^2, \ \forall x, y \in H.$$

(v) pseudo-monotone, if

$$\langle Ax, y - x\rangle \geq 0 \Rightarrow \langle Ay, y - x\rangle \geq 0, \ \forall x, y \in H.$$

(vi) quasi-monotone, if

$$\langle Ax, y - x\rangle > 0 \Rightarrow \langle Ay, y - x\rangle \geq 0, \ \forall x, y \in H.$$

Clearly, $(ii) \Rightarrow (iii) \Rightarrow (v) \Rightarrow (vi)$ and $(ii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi)$, but the converses are not always true.
Minty formulation of VIP (shortly, MVIP) is defined as:

$$\text{find } v \in C \text{ such that } \langle Ay, y - v\rangle \geq 0, \forall y \in C.$$

The solution set of MVIP is denoted by S_D. When A is quasi-monotone, we have S_D is a closed and convex subset of C. Furthermore, since C is convex and A is continuous, we have $S_D \subset S$.

Lemma 2.2. *([39]) Let C be a nonempty closed and convex subset of H. If either*

(i) A is pseudomonotone on C and $S \neq \emptyset$*,*
(ii) A is the gradient of G, where G is a differential quasiconvex function on an open set $K \supset C$
and attains its global minimum on C,
(iii) A is quasimonotone on C, $A \neq 0$ *on C and C is bounded,*
(iv) A is quasimonotone on C, $A \neq 0$ *on C and there exists a positive number r such that, for every* $v \in C$ *with* $\|v\| \geq r$*, there exists* $y \in C$ *such that* $\|y\| \leq r$ *and* $\langle Av, y - v\rangle \leq 0$*,*
(v) A is quasimonotone on C, intC is nonempty and there exists $v^* \in S$ *such that* $Av^* \neq 0$*.*

Then, S_D *is nonempty.*

Lemma 2.3. *The following statements hold in H:*

(i) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \ \forall\, x, y \in H.$
(ii)$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \ \forall\, x, y \in H.$
(iii)$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2, \ \forall\, x, y \in H, \ \lambda \in R.$

Lemma 2.4. *Let C be a nonempty closed and convex subset of H and* P_C *be the matric projection from H onto C. Then for any* $x, y \in H$ *and* $z \in C$*, the following hold:*

(i) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle.$
(ii)$\|P_C x - z\|^2 \leq \|x - z\|^2 - \|P_C x - x\|^2.$

Lemma 2.5. *For any* $x \in H$ *and* $z \in C$*, then* $z = P_C(x)$ *if and only if*

$$\langle x - z, y - z\rangle \leq 0, \forall y \in C.$$

Lemma 2.6. *([21], Lemma 2.2) Let* $\{\phi_n\}, \{\delta_n\}$ *and* $\{\theta_n\}$ *be sequences in* $[0, +\infty)$ *such that*

$$\phi_{n+1} \leq \phi_n + \theta_n(\phi_n - \phi_{n-1}) + \delta_n, \forall n \geq 1, \sum_{n=1}^{\infty} \delta_n < +\infty,$$

and exists a real number θ *with* $0 \leq \theta_n \leq \theta < 1$ *for all* $n \in N$*. Then the following assertions hold:*

(i) $\sum_{n=1}^{\infty} [\phi_n - \phi_{n-1}]_+ < +\infty$ *where* $[t]_+ = \max\{t, 0\}$ *for any* $t \in R$*;*
(ii) there exists $\phi^* \in [0, +\infty)$ *such that* $\lim_{n\to\infty} \phi_n = \phi^*$*.*

Lemma 2.7. *([25]) Let C be a nonempty subset of H and let* $\{x_n\}$ *be a sequence in H such that the following two conditions:*

(i) for each $x \in C$*, the limit of sequence* $\{\|x_n - x\|\}$ *exists;*
(ii) any weak cluster point of sequence $\{x_n\}$ *is in C.*

Then there exists $x^* \in C$ *such that* $\{x_n\}$ *converges weakly to* x^**.*

Lemma 2.8. *([5]) Consider the problem VIP with C being a nonempty, closed, convex subset of a real Hilbert space H and* $F : C \to H$ *being pseudomonotone and continuous. Then,* x^* *is a solution of VIP if and only if*

$$\langle Fx, x - x^*\rangle \geq 0, \forall x \in C.$$

Lemma 2.9. *([19], Lemma 3.2) Let* $A : H \to H$ *be a monotone and L-Lipschitz continuous mapping. Let* $T = P_C(I - \lambda A)$*, where* $\lambda > 0$*. If* $\{x_n\}$ *is a sequence in H satisfying* $x_n \rightharpoonup q$ *and* $x_n - Tx_n \to 0$*, then* $q \in S$*.*

Lemma 2.10. *([12, 17]) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable and pseudo-convex on C. Then $x^* \in C \subset \mathbb{R}^n$ satisfies*

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0,\ \forall x \in C$$

if and only if x^* is a minimum of f in C.

3. Weak convergence

In this section, we show that the sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to a point in $S_D \subset S$ under the following conditions:
(C1) The feasible set C is a nonempty, closed and convex subset of H;
(C2) The operator $A : H \to H$ is quasi-monotone;
(C3) The operator $A : H \to H$ is L-Lipschitz continuous and satisfies the following condition:

$$\text{if } \{x_n\} \subset H,\ x_n \rightharpoonup v^* \text{ and } \liminf_{n\to\infty} \|Ax_n\| = 0, \text{ then } Av^* = 0;$$

(C4) $S_D \neq \emptyset$;
(C5) $0 \leq \theta_n \leq \theta_{n+1} \leq 1$;
(C6) $0 \leq \delta_n \leq \delta_{n+1} \leq \delta < \min\{\frac{\epsilon - \sqrt{2\epsilon}}{\epsilon}, \theta_1\}$, $\epsilon \in (2, +\infty)$;
(C7) $0 < \alpha \leq \alpha_n \leq \alpha_{n+1} < \frac{1}{1+\epsilon}$, $\epsilon \in (2, +\infty)$.

Algorithm 1

Iterative step:
1. Take the parameters $\mu \in (0, 1)$ and $\lambda_1 > 0$. Choose a nonnegative real sequence $\{a_n\}$ such that $\sum_{n=1}^{\infty} a_n < +\infty$. Let $x_0, x_1 \in H$ be given starting points. Set n:=1.
2. Compute

$$\begin{cases} z_n = x_n + \delta_n(x_n - x_{n-1}) \\ w_n = x_n + \theta_n(x_n - x_{n-1}) \\ y_n = P_C(w_n - \lambda_n A w_n) \\ u_n = P_{T_n}(w_n - \lambda_n A y_n), \end{cases} \tag{6}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu(\|w_n - y_n\|^2 + \|u_n - y_n\|^2)}{2\langle Aw_n - Ay_n, u_n - y_n\rangle}, \lambda_n + a_n\right\}, & if \langle Aw_n - Ay_n, u_n - y_n\rangle > 0, \\ \lambda_n + a_n, & otherwise, \end{cases} \tag{7}$$

and

$$T_n := \{w \in H : \langle w_n - \lambda_n A w_n - y_n, w - y_n \rangle \leq 0\}.$$

If $w_n = y_n = x_n$, STOP. Otherwise
3. Compute

$$x_{n+1} = (1 - \alpha_n) z_n + \alpha_n u_n, n \geq 1. \tag{8}$$

4. Set $n \leftarrow n + 1$, and go to 2.

Remark 3.1. *Observe that if $x_n = w_n = y_n$, then (6) implies $x_n = P_C(x_n - \lambda_n A x_n)$, from Lemma 2.5 we know $x_n \in S$.*

Remark 3.2. *It is obvious that* $\langle Aw_n - Ay_n, u_n - y_n\rangle \le \frac{\mu}{2\lambda_{n+1}}\left(\|w_n - y_n\|^2 + \|u_n - y_n\|^2\right)$.

Lemma 3.3. ([20], Lemma 3.3) Suppose that Condition (C3) holds, then the sequence $\{\lambda_n\}$ generated by (7) is well defined and $\lim_{n\to\infty}\lambda_n = \lambda$ and $\lambda \in [\min\{\frac{\mu}{L}, \lambda_1\}, \lambda_1 + \sum_{n=1}^{\infty} a_n]$.

Lemma 3.4. *Suppose that* $\{x_n\}$ *is generated by Algorithm 1 and Condition (C1)-(C7) hold. Then* $\{x_n\}$ *is bounded and* $\lim_{n\to\infty}\|x_n - x^*\|$ *exists, where* $x^* \in S_D$.

Proof. Pick a point $x^* \in S_D$, we have $\langle Ay_n, y_n - x^*\rangle \ge 0$, then $\langle Ay_n, y_n - x^* + u_n - u_n\rangle \ge 0$ and thus

$$\langle Ay_n, x^* - u_n\rangle \le \langle Ay_n, y_n - u_n\rangle, \tag{9}$$

by $u_n \in T_n$, we get $\langle w_n - \lambda_n Aw_n - y_n, u_n - y_n\rangle \le 0$. Therefore

$$\begin{aligned}\langle w_n - \lambda_n Ay_n - y_n, u_n - y_n\rangle &= \langle w_n - \lambda_n Aw_n - y_n, u_n - y_n\rangle + \lambda_n\langle Aw_n - Ay_n, u_n - y_n\rangle\\ &\le \lambda_n\langle Aw_n - Ay_n, u_n - y_n\rangle.\end{aligned} \tag{10}$$

From $y_n = P_C(w_n - \lambda_n Aw_n)$ and Lemma 2.5, we have

$$\langle w_n - \lambda_n Aw_n - y_n, y - y_n\rangle \le 0, \ \forall y \in C. \tag{11}$$

By (11) and the definition of T_n, we know $y \in T_n$, therefore $x^* \in C \subset T_n$.
Using Lemma 2.4 and (9), we have

$$\begin{aligned}\|u_n - x^*\|^2 &\le \|w_n - \lambda_n Ay_n - x^*\|^2 - \|w_n - \lambda_n Ay_n - u_n\|^2\\ &= \|w_n - x^*\|^2 - \|w_n - u_n\|^2 + 2\lambda_n\langle Ay_n, x^* - u_n\rangle\\ &\le \|w_n - x^*\|^2 - \|w_n - u_n\|^2 + 2\lambda_n\langle Ay_n, y_n - u_n\rangle\\ &= \|w_n - x^*\|^2 - \|w_n - u_n + y_n - y_n\|^2 + 2\lambda_n\langle Ay_n, y_n - u_n\rangle\\ &= \|w_n - x^*\|^2 - \|w_n - y_n\|^2 - \|y_n - u_n\|^2 + 2\langle w_n - \lambda_n Ay_n - y_n, u_n - y_n\rangle.\end{aligned} \tag{12}$$

Using (10) and (12), we obtain

$$\begin{aligned}\|u_n - x^*\|^2 &\le \|w_n - x^*\|^2 - \|w_n - y_n\|^2 - \|y_n - u_n\|^2 + 2\lambda_n\langle Aw_n - Ay_n, u_n - y_n\rangle\\ &\le \|w_n - x^*\|^2 - \|w_n - y_n\|^2 - \|y_n - u_n\|^2 + \frac{\lambda_n\mu}{\lambda_{n+1}}\left(\|w_n - y_n\|^2 + \|u_n - y_n\|^2\right)\\ &= \|w_n - x^*\|^2 - \left(1 - \frac{\lambda_n\mu}{\lambda_{n+1}}\right)\|w_n - y_n\|^2 - \left(1 - \frac{\lambda_n\mu}{\lambda_{n+1}}\right)\|u_n - y_n\|^2.\end{aligned} \tag{13}$$

Since $\lim_{n\to\infty}\left(1 - \frac{\lambda_n\mu}{\lambda_{n+1}}\right) = 1 - \mu > 0$, there exists a natural number $N \ge 1$ such that

$$\|u_n - x^*\|^2 \le \|w_n - x^*\|^2, \ \forall n \ge N. \tag{14}$$

From Algorithm 1 and (14) we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|(1 - \alpha_n)(z_n - x^*) + \alpha_n(u_n - x^*)\|^2\\ &= (1 - \alpha_n)\|z_n - x^*\|^2 + \alpha_n\|u_n - x^*\|^2 - \alpha_n(1 - \alpha_n)\|z_n - u_n\|^2\\ &\le (1 - \alpha_n)\|z_n - x^*\|^2 + \alpha_n\|w_n - x^*\|^2 - \alpha_n(1 - \alpha_n)\|z_n - u_n\|^2, \ \forall n \ge N.\end{aligned} \tag{15}$$

From $x_{n+1} = (1 - \alpha_n)z_n + \alpha_n u_n$, we have

$$\|u_n - z_n\| = \frac{1}{\alpha_n}\|x_{n+1} - z_n\|, \ \forall n \ge 1. \tag{16}$$

Combining (15) and (16) we get

$$\|x_{n+1}-x^*\|^2 \le (1-\alpha_n)\|z_n-x^*\|^2+\alpha_n\|w_n-x^*\|^2-\frac{1-\alpha_n}{\alpha_n}\|x_{n+1}-z_n\|^2,\ \forall n\ge N. \tag{17}$$

From Lemma 2.3, we have

$$\begin{aligned}\|w_n-x^*\|^2 &= \|x_n+\theta_n(x_n-x_{n-1})-x^*\|^2\\ &= \|(1+\theta_n)(x_n-x^*)-\theta_n(x_{n-1}-x^*)\|^2\\ &= (1+\theta_n)\|x_n-x^*\|^2-\theta_n\|x_{n-1}-x^*\|^2+\theta_n(1+\theta_n)\|x_n-x_{n-1}\|^2,\ \forall n\ge 1.\end{aligned} \tag{18}$$

$$\begin{aligned}\|z_n-x^*\|^2 &= \|x_n+\delta_n(x_n-x_{n-1})-x^*\|^2\\ &= (1+\delta_n)\|x_n-x^*\|^2-\delta_n\|x_{n-1}-x^*\|^2+\delta_n(1+\delta_n)\|x_n-x_{n-1}\|^2,\ \forall n\ge 1.\end{aligned} \tag{19}$$

Also,

$$\begin{aligned}\|x_{n+1}-z_n\|^2 &= \|x_{n+1}-(x_n+\delta_n(x_n-x_{n-1}))\|^2\\ &= \|(x_{n+1}-x_n)-\delta_n(x_n-x_{n-1})\|^2\\ &= \|x_{n+1}-x_n\|^2+\delta_n^2\|x_n-x_{n-1}\|^2-2\delta_n\langle x_{n+1}-x_n, x_n-x_{n-1}\rangle\\ &\ge \|x_{n+1}-x_n\|^2+\delta_n^2\|x_n-x_{n-1}\|^2-2\delta_n\|x_{n+1}-x_n\|\cdot\|x_n-x_{n-1}\|\\ &\ge \|x_{n+1}-x_n\|^2+\delta_n^2\|x_n-x_{n-1}\|^2-\delta_n\left(\|x_{n+1}-x_n\|^2+\|x_n-x_{n-1}\|^2\right)\\ &= (1-\delta_n)\|x_{n+1}-x_n\|^2+(\delta_n^2-\delta_n)\|x_n-x_{n-1}\|^2.\end{aligned} \tag{20}$$

Substituting (18), (19) and (20) into (17), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le (1-\alpha_n)\left[(1+\delta_n)\|x_n-x^*\|^2-\delta_n\|x_{n-1}-x^*\|^2+\delta_n(1+\delta_n)\|x_n-x_{n-1}\|^2\right]\\ &\quad+\alpha_n\Big[(1+\theta_n)\|x_n-x^*\|^2-\theta_n\|x_{n-1}-x^*\|^2+\theta_n(1+\theta_n)\|x_n-x_{n-1}\|^2\Big]\\ &\quad-\frac{1-\alpha_n}{\alpha_n}\left[(1-\delta_n)\|x_{n+1}-x_n\|^2+(\delta_n^2-\delta_n)\|x_n-x_{n-1}\|^2\right]\\ &= \Big((1-\alpha_n)(1+\delta_n)+\alpha_n(1+\theta_n)\Big)\|x_n-x^*\|^2-\Big(\delta_n(1-\alpha_n)+\alpha_n\theta_n\Big)\|x_{n-1}-x^*\|^2\\ &\quad+\left((1-\alpha_n)\delta_n(1+\delta_n)+\alpha_n\theta_n(1+\theta_n)-\frac{(1-\alpha_n)(\delta_n^2-\delta_n)}{\alpha_n}\right)\|x_n-x_{n-1}\|^2\\ &\quad-\frac{(1-\alpha_n)(1-\delta_n)}{\alpha_n}\|x_{n+1}-x_n\|^2\\ &= \Big(1+\alpha_n\theta_n+\delta_n(1-\alpha_n)\Big)\|x_n-x^*\|^2-\Big(\alpha_n\theta_n+\delta_n(1-\alpha_n)\Big)\|x_{n-1}-x^*\|^2\\ &\quad+\sigma_n\|x_n-x_{n-1}\|^2-\rho_n\|x_{n+1}-x_n\|^2,\ \forall n\ge N,\end{aligned} \tag{21}$$

where

$$\sigma_n := (1-\alpha_n)\delta_n(1+\delta_n)+\alpha_n\theta_n(1+\theta_n)-\frac{(1-\alpha_n)(\delta_n^2-\delta_n)}{\alpha_n},\quad \rho_n := \frac{(1-\alpha_n)(1-\delta_n)}{\alpha_n}.$$

Define $\Gamma_n := \|x_n-x^*\|^2-\Big(\alpha_n\theta_n+\delta_n(1-\alpha_n)\Big)\|x_{n-1}-x^*\|^2+\sigma_n\|x_n-x_{n-1}\|^2,\ \forall n\ge 1.$

By the definition of Γ_n and (21) we have

$$\begin{aligned}\Gamma_{n+1}-\Gamma_n &= \|x_{n+1}-x^*\|^2-\Big(\alpha_{n+1}\theta_{n+1}+\delta_{n+1}(1-\alpha_{n+1})\Big)\|x_n-x^*\|^2+\sigma_{n+1}\|x_{n+1}-x_n\|^2\\ &\quad-\|x_n-x^*\|^2+\Big(\alpha_n\theta_n+\delta_n(1-\alpha_n)\Big)\|x_{n-1}-x^*\|^2-\sigma_n\|x_n-x_{n-1}\|^2\\ &\le \Big(\alpha_n\theta_n+\delta_n(1-\alpha_n)-\alpha_{n+1}\theta_{n+1}-\delta_{n+1}(1-\alpha_{n+1})\Big)\|x_n-x^*\|^2\\ &\quad+\sigma_{n+1}\|x_{n+1}-x_n\|^2-\rho_n\|x_{n+1}-x_n\|^2\\ &= \Big(\alpha_n(\theta_n-\delta_n)-\alpha_{n+1}(\theta_{n+1}-\delta_{n+1})+(\delta_n-\delta_{n+1})\Big)\|x_n-x^*\|^2\\ &\quad+\sigma_{n+1}\|x_{n+1}-x_n\|^2-\rho_n\|x_{n+1}-x_n\|^2,\ \forall n\ge N. \end{aligned}\tag{22}$$

By $0\le\delta_n\le\delta_{n+1}<\theta_1\le\theta_n\le\theta_{n+1}$ and $0<\alpha_n\le\alpha_{n+1}<1$, we have $-\alpha_{n+1}(\theta_{n+1}-\delta_{n+1})\le-\alpha_n(\theta_{n+1}-\delta_{n+1})$ and $-(\delta_{n+1}-\delta_n)\le-\alpha_n(\delta_{n+1}-\delta_n)$.
Thus,

$$\begin{aligned}&\alpha_n(\theta_n-\delta_n)-\alpha_{n+1}(\theta_{n+1}-\delta_{n+1})+(\delta_n-\delta_{n+1})\\ &\le \alpha_n(\theta_n-\delta_n)-\alpha_n(\theta_{n+1}-\delta_{n+1})-\alpha_n(\delta_{n+1}-\delta_n)\\ &=\alpha_n(\theta_n-\theta_{n+1})\\ &\le 0.\end{aligned}$$

Then from (22) we have

$$\Gamma_{n+1}-\Gamma_n\le-(\rho_n-\sigma_{n+1})\|x_{n+1}-x_n\|^2,\ \forall n\ge N.\tag{23}$$

By $\alpha_n<\frac{1}{1+\epsilon}$, we can get $\frac{1-\alpha_n}{\alpha_n}\ge\epsilon$, thus

$$\begin{aligned}\rho_n-\sigma_{n+1} &= \frac{(1-\alpha_n)(1-\delta_n)}{\alpha_n}-(1-\alpha_{n+1})\delta_{n+1}(1+\delta_{n+1})-\alpha_{n+1}\theta_{n+1}(1+\theta_{n+1})\\ &\quad+\frac{(1-\alpha_{n+1})(\delta_{n+1}^2-\delta_{n+1})}{\alpha_{n+1}}\\ &\ge \frac{(1-\alpha_{n+1})(1-\delta_n)}{\alpha_{n+1}}+\frac{(1-\alpha_{n+1})(\delta_{n+1}^2-\delta_{n+1})}{\alpha_{n+1}}\\ &\quad-(1-\alpha_{n+1})\delta_{n+1}(1+\delta_{n+1})-\alpha_{n+1}\theta_{n+1}(1+\theta_{n+1})\\ &\ge \frac{(1-\alpha_{n+1})(\delta_{n+1}^2-\delta_{n+1}-\delta_n+1)}{\alpha_{n+1}}-2(1-\alpha_{n+1})-2\alpha_{n+1}\\ &\ge \frac{(1-\alpha_{n+1})(\delta_{n+1}^2-2\delta_{n+1}+1)}{\alpha_{n+1}}-2(1-\alpha_{n+1})-2\alpha_{n+1}\\ &\ge \epsilon(\delta_{n+1}^2-2\delta_{n+1}+1)-2\\ &= \epsilon\delta_{n+1}^2-2\epsilon\delta_{n+1}+\epsilon-2.\end{aligned}\tag{24}$$

Since $y=\epsilon x^2-2\epsilon x+\epsilon-2$ has two roots: $x_1=\frac{\epsilon-\sqrt{2\epsilon}}{\epsilon}$, $x_2=\frac{\epsilon+\sqrt{2\epsilon}}{\epsilon}$ and $\delta_{n+1}\le\delta<\frac{\epsilon-\sqrt{2\epsilon}}{\epsilon}$, we have $\epsilon\delta_{n+1}^2-2\epsilon\delta_{n+1}+\epsilon-2\ge\epsilon\delta^2-2\epsilon\delta+\epsilon-2>0$. Therefore from (23) and (24) we have

$$\Gamma_{n+1}-\Gamma_n\le-\beta\|x_{n+1}-x_n\|^2,\forall n\ge N.\tag{25}$$

where $\beta:=\epsilon\delta^2-2\epsilon\delta+\epsilon-2>0$. Hence, $\{\Gamma_n\}$ is non-increasing ($n\ge N$).

Furthermore, by Assumption (f) and (g) we know $\delta_n \le \delta < 1,\ \alpha_n < 1$, thus

$$\begin{aligned}\sigma_n &= (1-\alpha_n)\delta_n(1+\delta_n) + \alpha_n\theta_n(1+\theta_n) - \frac{(1-\alpha_n)(\delta_n^2-\delta_n)}{\alpha_n}\\ &= (1-\alpha_n)\delta_n(1+\delta_n - \frac{\delta_n - 1}{\alpha_n}) + \alpha_n\theta_n(1+\theta_n)\\ &> 0.\end{aligned}$$

Therefore,

$$\begin{aligned}\Gamma_n &= \|x_n - x^*\|^2 - \Big(\alpha_n\theta_n + \delta_n(1-\alpha_n)\Big)\|x_{n-1} - x^*\|^2 + \sigma_n\|x_n - x_{n-1}\|^2\\ &\ge \|x_n - x^*\|^2 - \Big(\alpha_n\theta_n + \delta_n(1-\alpha_n)\Big)\|x_{n-1} - x^*\|^2,\ \forall n \ge 1.\end{aligned}$$

So,

$$\begin{aligned}\|x_n - x^*\|^2 &\le \Big(\alpha_n\theta_n + \delta_n(1-\alpha_n)\Big)\|x_{n-1} - x^*\|^2 + \Gamma_n\\ &\le \Big(\alpha_n + \delta_n(1-\alpha_n)\Big)\|x_{n-1} - x^*\|^2 + \Gamma_n\\ &\le \Big(\frac{1}{1+\epsilon} + \delta(1-\alpha)\Big)\|x_{n-1} - x^*\|^2 + \Gamma_n\\ &\le \Big(\frac{1}{1+\epsilon} + \delta(1-\alpha)\Big)\|x_{n-1} - x^*\|^2 + \Gamma_N\\ &= \gamma\|x_{n-1} - x^*\|^2 + \Gamma_N\\ &\le \gamma^2\|x_{n-2} - x^*\|^2 + \gamma\Gamma_N + \Gamma_N\\ &\ \ \vdots\\ &\le \gamma^{n-N}\|x_N - x^*\|^2 + (1+\gamma+\cdots+\gamma^{n-N-1})\Gamma_N\\ &\le \gamma^{n-N}\|x_N - x^*\|^2 + \frac{\Gamma_N}{1-\gamma},\end{aligned} \tag{26}$$

where $\gamma := \frac{1}{1+\epsilon} + \delta(1-\alpha)$.

Now we prove $\gamma \in (0,1)$. By $\frac{\epsilon-\sqrt{2\epsilon}}{\epsilon} - \frac{\epsilon}{1+\epsilon} = \frac{(1+\epsilon)(\epsilon-\sqrt{2\epsilon})-\epsilon^2}{\epsilon(1+\epsilon)} = \frac{\epsilon-(1+\epsilon)\sqrt{2\epsilon}}{\epsilon(1+\epsilon)} < 0$, we have $\delta \le \frac{\epsilon-\sqrt{2\epsilon}}{\epsilon} < \frac{\epsilon}{1+\epsilon}$, so $\gamma := \frac{1}{1+\epsilon} + \delta(1-\alpha) < \frac{1}{1+\epsilon} + \frac{\epsilon}{1+\epsilon} = 1$, so $\gamma \in (0,1)$.

Hence, $\{\|x_n - x^*\|\}$ is bounded and so is $\{x_n\}$.

By the definition of Γ_{n+1} and (26), we have

$$\begin{aligned}-\Gamma_{n+1} &\le \Big(\alpha_{n+1}\theta_{n+1} + \delta_{n+1}(1-\alpha_{n+1})\Big)\|x_n - x^*\|^2\\ &\le \Big(\frac{1}{1+\epsilon} + \delta(1-\alpha)\Big)\|x_n - x^*\|^2\\ &= \gamma\|x_n - x^*\|^2\\ &\le \gamma^{n-N+1}\|x_N - x^*\|^2 + \frac{\gamma\Gamma_N}{1-\gamma}.\end{aligned} \tag{27}$$

By (25) and (27), we get

$$\beta\sum_{k=N}^{n}\|x_{k+1} - x_k\|^2 \le \Gamma_N - \Gamma_{n+1} \le \gamma^{n-N+1}\|x_N - x^*\|^2 + \frac{\Gamma_N}{1-\gamma}.$$

Therefore,

$$\sum_{n=N}^{\infty} \|x_{n+1} - x_n\|^2 \leq \frac{\Gamma_N}{\beta(1-\gamma)} < \infty. \tag{28}$$

Thus,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{29}$$

From (21) we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \Big(1 + \alpha_n\theta_n + \delta_n(1-\alpha_n)\Big)\|x_n - x^*\|^2 - \Big(\alpha_n\theta_n + \delta_n(1-\alpha_n)\Big)\|x_{n-1} - x^*\|^2 \\
&\quad + \sigma_n\|x_n - x_{n-1}\|^2 - \rho_n\|x_{n+1} - x_n\|^2 \\
&\leq \|x_n - x^*\|^2 + \Big(\alpha_n\theta_n + \delta_n(1-\alpha_n)\Big)\Big(\|x_n - x^*\|^2 - \|x_{n-1} - x^*\|^2\Big) \\
&\quad + \sigma_n\|x_n - x_{n-1}\|^2, \ \forall n \geq N.
\end{aligned} \tag{30}$$

$$\begin{aligned}
\sigma_n &= (1-\alpha_n)\delta_n(1+\delta_n) + \alpha_n\theta_n(1+\theta_n) - \frac{(1-\alpha_n)(\delta_n^2 - \delta_n)}{\alpha_n} \\
&\leq (1-\alpha)\delta(1+\delta) + \frac{2}{1+\epsilon} + \frac{1-\alpha}{\alpha}(\delta_n - \delta_n^2) \\
&\leq (1-\alpha)\delta(1+\delta) + \frac{2}{1+\epsilon} + \frac{1-\alpha}{4\alpha}.
\end{aligned}$$

So,

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \|x_n - x^*\|^2 + \Big(\alpha_n\theta_n + \delta_n(1-\alpha_n)\Big)\Big(\|x_n - x^*\|^2 - \|x_{n-1} - x^*\|^2\Big) \\
&\quad + \Big((1-\alpha)\delta(1+\delta) + \frac{2}{1+\epsilon} + \frac{1-\alpha}{4\alpha}\Big)\|x_n - x_{n-1}\|^2, \ \forall n \geq N.
\end{aligned} \tag{31}$$

Note also that $\alpha_n\theta_n + \delta_n(1-\alpha_n) < \frac{1}{1+\epsilon} + \delta(1-\alpha) = \gamma < 1$.
By (28) we get $\sum_{n=N}^{\infty} [(1-\alpha)\delta(1+\delta) + \frac{2}{1+\epsilon} + \frac{1-\alpha}{4\alpha}]\|x_n - x_{n-1}\|^2 < +\infty$.
Invoking Lemma 2.6 in (31), we get $\lim_{n\to\infty} \|x_n - x^*\|$ exists. □

Lemma 3.5. *Let $\{x_n\}$ be generated by Algorithm 1 such that Condition (C1)-(C7) hold. If v^* is one of the weak cluster points of $\{x_n\}$, then we have at least one of the following : $v^* \in S_D$ or $Av^* = 0$.*

Proof. By Lemma 3.4, $\{x_n\}$ is bounded. Hence we can let v^* be a weak cluster point of $\{x_n\}$. Then we can choose a subsequence of $\{x_n\}$, denoted by $\{x_{n_k}\}$ such that $x_{n_k} \rightharpoonup v^* \in H$.
We consider the following two possible cases.
Case I: Suppose that $\limsup_{k\to\infty} \|Ax_{n_k}\| = 0$. Then $\lim_{k\to\infty} \|Ax_{n_k}\| = \liminf_{k\to\infty} \|Ax_{n_k}\| = 0$. Thus we obtain from Condition (C3) that $Av^* = 0$.
Case II: Suppose that $\limsup_{k\to\infty} \|Ax_{n_k}\| > 0$. Then without loss of generality, we can choose a subsequence of $\{Ax_{n_k}\}$ still denoted by $\{Ax_{n_k}\}$ such that $\lim_{k\to\infty} \|Ax_{n_k}\| = M_1 > 0$.
We first prove that $\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|u_n - w_n\| = \lim_{n\to\infty} \|y_n - u_n\| = 0$.

From (29) we know that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, then

$$\|x_{n+1}-z_n\|\le\|x_{n+1}-x_n\|+\|x_n-z_n\|=\|x_{n+1}-x_n\|+\delta_n\|x_n-x_{n-1}\|\to 0, n\to\infty.$$
$$\|x_{n+1}-w_n\|\le\|x_{n+1}-x_n\|+\|x_n-w_n\|=\|x_{n+1}-x_n\|+\theta_n\|x_n-x_{n-1}\|\to 0, n\to\infty.$$
$$\|w_n-z_n\|=\|w_n-x_{n+1}\|+\|x_{n+1}-z_n\|\to 0, n\to\infty.$$
$$\|u_n-z_n\|=\frac{1}{\alpha_n}\|x_{n+1}-z_n\|\le\frac{1}{\alpha}\|x_{n+1}-z_n\|\to 0, n\to\infty.$$
$$\|w_n-u_n\|\le\|w_n-z_n\|+\|u_n-z_n\|\to 0, n\to\infty.$$

By $\{x_n\}$ is bounded and $\|x_{n+1}-w_n\|\to 0, \|w_n-u_n\|\to 0, n\to\infty$, we can get $\{\|w_n-x^*\|\}$ and $\{\|u_n-x^*\|\}$ are bounded.
From (13), we have

$$\begin{aligned}&\left(1-\frac{\lambda_n\mu}{\lambda_{n+1}}\right)\left(\|w_n-y_n\|^2+\|y_n-u_n\|^2\right)\\ &\le\|w_n-x^*\|^2-\|u_n-x^*\|^2\\ &=\left(\|w_n-x^*\|+\|u_n-x^*\|\right)\left(\|w_n-x^*\|-\|u_n-x^*\|\right)\\ &\le M\|w_n-u_n\|\to 0,\ n\to\infty,\end{aligned}$$

for some $M>0$, where the existence of M is from the boundedness of $\left\{\|w_n-x^*\|+\|u_n-x^*\|\right\}$. Noting $\lim_{n\to\infty}\lambda_n=\lambda$, we see that $\|w_n-y_n\|\to 0, \|u_n-y_n\|\to 0, n\to\infty$.
By $\|x_{n+1}-w_n\|\to 0$, $\|w_n-y_n\|\to 0$, and $\|x_{n+1}-x_n\|\to 0$, we have $\|x_n-y_n\|\to 0, n\to\infty$.
From $u_n=P_{T_n}(w_n-\lambda_nAy_n)$ and $C\subset T_n$, we get

$$\langle w_n-\lambda_nAy_n-u_n, y-u_n\rangle\le 0,\ \forall y\in C.$$

So,

$$\begin{aligned}0&\le\langle u_{n_k}-w_{n_k}+\lambda_{n_k}Ay_{n_k}, y-u_{n_k}\rangle\\ &=\langle u_{n_k}-w_{n_k}, y-u_{n_k}\rangle+\lambda_{n_k}\langle Ay_{n_k}, y-u_{n_k}\rangle\\ &=\langle u_{n_k}-w_{n_k}, y-u_{n_k}\rangle+\lambda_{n_k}\langle Ay_{n_k}, y-y_{n_k}\rangle+\lambda_{n_k}\langle Ay_{n_k}, y_{n_k}-u_{n_k}\rangle,\ \forall y\in C.\end{aligned}\tag{32}$$

From $\lim_{n\to\infty}\|w_n-u_n\|=\lim_{n\to\infty}\|u_n-y_n\|=0$, we obtain

$$0\le\liminf_{k\to\infty}\langle Ay_{n_k}, y-y_{n_k}\rangle\le\limsup_{k\to\infty}\langle Ay_{n_k}, y-y_{n_k}\rangle<\infty,\ \forall y\in C.\tag{33}$$

Based on (33), we consider the following two cases under case II:
Case 1: Suppose that $\limsup_{k\to\infty}\langle Ay_{n_k}, y-y_{n_k}\rangle>0, \forall y\in C$. Then we can choose a subsequence of $\{y_{n_k}\}$ denoted by $\{y_{n_{k_j}}\}$ such that $\lim_{j\to\infty}\langle Ay_{n_{k_j}}, y-y_{n_{k_j}}\rangle>0$. Thus, there exists $j_0\ge 1$ such that $\langle Ay_{n_{k_j}}, y-y_{n_{k_j}}\rangle>0, \forall j\ge j_0$, by the quasimonotonicity of A on H, it implies that $\langle Ay, y-y_{n_{k_j}}\rangle\ge 0, \forall y\in C, j\ge j_0$. By $x_{n_k}\rightharpoonup v^*$ and $\|x_n-y_n\|\to 0, n\to\infty$, we have $y_{n_k}\rightharpoonup v^*\in C$. As $j\to\infty$, we get $\langle Ay, y-v^*\rangle\ge 0, \forall y\in C$. Therefore, $v^*\in S_D$.
Case 2: Suppose that $\limsup_{k\to\infty}\langle Ay_{n_k}, y-y_{n_k}\rangle=0, \forall y\in C$. Then by (33), we get

$$\lim_{k\to\infty}\langle Ay_{n_k}, y-y_{n_k}\rangle=0,\ \forall y\in C,\tag{34}$$

from which we get

$$\langle Ay_{n_k}, y-y_{n_k}\rangle+|\langle Ay_{n_k}, y-y_{n_k}\rangle|+\frac{1}{k+1}>0,\ \forall y\in C.\tag{35}$$

Since $\lim_{k\to\infty} \|Ax_{n_k}\| = \lim_{k\to\infty} \|Ay_{n_k}\| = M_1 > 0$, we can find $k_0 \geq 1$ such that $\|Ay_{n_k}\| \geq \frac{M_1}{2}, \forall k \geq k_0$.
We set $b_{n_k} = \frac{Ay_{n_k}}{\|Ay_{n_k}\|^2}$, then $\langle Ay_{n_k}, b_{n_k}\rangle = 1$. Therefore, by (35), we get

$$\Big\langle Ay_{n_k}, y + b_{n_k}\Big[|\langle Ay_{n_k}, y - y_{n_k}\rangle| + \frac{1}{k+1}\Big] - y_{n_k}\Big\rangle > 0,\ \forall y \in C. \tag{36}$$

Using the quasi-monotonicity of A on H, we obtain

$$\Big\langle A\Big(y + b_{n_k}\Big[|\langle Ay_{n_k}, y - y_{n_k}\rangle| + \frac{1}{k+1}\Big]\Big), y + b_{n_k}\Big[|\langle Ay_{n_k}, y - y_{n_k}\rangle| + \frac{1}{k+1}\Big] - y_{n_k}\Big\rangle \geq 0,\ \forall y \in C.$$

This implies that

$$\begin{aligned}
&\Big\langle Ay, y + b_{n_k}\Big[|\langle Ay_{n_k}, y - y_{n_k}\rangle| + \frac{1}{k+1}\Big] - y_{n_k}\Big\rangle \\
&\geq \Big\langle Ay - A(y + b_{n_k}\Big[|\langle Ay_{n_k}, y - y_{n_k}\rangle| + \frac{1}{k+1}\Big]), y + b_{n_k}\Big[|\langle Ay_{n_k}, y - y_{n_k}\rangle| + \frac{1}{k+1}\Big] - y_{n_k}\Big\rangle \\
&\geq -\|Ay - A(y + b_{n_k}\Big[|\langle Ay_{n_k}, y - y_{n_k}\rangle| + \frac{1}{k+1}\Big])\| \cdot \|y + b_{n_k}\Big[|\langle Ay_{n_k}, y - y_{n_k}\rangle| + \frac{1}{k+1}\Big] - y_{n_k}\| \\
&\geq -L\|b_{n_k}\Big[|\langle Ay_{n_k}, y - y_{n_k}\rangle| + \frac{1}{k+1}\Big]\| \cdot \|y + b_{n_k}\Big[|\langle Ay_{n_k}, y - y_{n_k}\rangle| + \frac{1}{k+1}\Big] - y_{n_k}\| \\
&= \frac{-L}{\|Ay_{n_k}\|}\Big(|\langle Ay_{n_k}, y - y_{n_k}\rangle| + \frac{1}{k+1}\Big) \cdot \|y + b_{n_k}\Big[|\langle Ay_{n_k}, y - y_{n_k}\rangle| + \frac{1}{k+1}\Big] - y_{n_k}\| \\
&\geq \frac{-2L}{M_1}\Big(|\langle Ay_{n_k}, y - y_{n_k}\rangle| + \frac{1}{k+1}\Big)M_2,\ \forall y \in C, k \geq k_0,
\end{aligned} \tag{37}$$

for some $M_2 > 0$, where the existence of M_2 is from the boundedness of $\Big\{y + b_{n_k}\Big[|\langle Ay_{n_k}, y - y_{n_k}\rangle| + \frac{1}{k+1}\Big] - y_{n_k}\Big\}$.
From (34) we have $\lim_{k\to\infty}\Big(|\langle Ay_{n_k}, y - y_{n_k}\rangle| + \frac{1}{k+1}\Big) = 0, \forall y \in C$. Thus, as $k \to \infty$ in (37), we get $\langle Ay, y - v^*\rangle \geq 0, \forall y \in C$. Therefore, $v^* \in S_D$. □

Theorem 3.6. *Let $\{x_n\}$ be generated by Algorithm 1 such that Condition (C1)-(C7) hold and $Ax \neq 0, \forall x \in C$ (otherwise, $x \in S$). Then $\{x_n\}$ converges weakly to an element of S_D.*

Proof. by Lemma 3.4, $\{x_n\}$ is bounded, hence, let z be a weak cluster point of $\{x_n\}$. Then there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$, such that $x_{n_k} \rightharpoonup z, k \to \infty$, also from $\|x_n - y_n\| \to 0, n \to \infty$, we get $y_{n_k} \rightharpoonup z, k \to \infty$. Since C is closed, we have that $z \in C$. Since $Ax \neq 0, \forall x \in C$, we get $Az \neq 0$. By Lemma 3.5, we get $z \in S_D$. Therefore,
(1) by Lemma 3.5, $\lim_{n\to\infty} \|x_n - z\|$ exists for any $z \in S_D$,
(2) every sequential weak cluster point of $\{x_n\}$ is in S_D.
Using Lemma 2.7, we get $\{x_n\}$ converges weakly to a point in S_D. □

Remark 3.7. *Obviously, all the results in this section can still be derived from the operator A is monotone (or pseudo-monotone) rather than quasi-monotone.*

4. Strong convergence

Theorem 4.1. *Assume that A is η-strongly pseudo-monotone and Condition (C1), (C3)-(C7) hold. Let $\{x_n\}$ be generated by Algorithm 1 , then $\{x_n\}$ converges strongly to an element of S.*

Proof. The strong pseudo-monotonicity of A implies that VIP has a unique solution, which is denoted by x^*, hence $\langle Ax^*, y_n - x^*\rangle \geq 0$. By the strong pseudo-monotonicity of A, we have $\langle Ay_n, y_n - x^*\rangle \geq \eta\|y_n - x^*\|^2$. Thus,

$$\langle Ay_n, y_n - u_n + u_n - x^*\rangle \geq \eta\|y_n - x^*\|^2,$$

so

$$\langle Ay_n, x^* - u_n\rangle \leq \langle Ay_n, y_n - u_n\rangle - \eta\|y_n - x^*\|^2. \tag{38}$$

Using Lemma 2.4 and (38), we have

$$\begin{aligned}
\|u_n - x^*\|^2 &\leq \|w_n - \lambda_n Ay_n - x^*\|^2 - \|w_n - \lambda_n Ay_n - u_n\|^2 \\
&= \|w_n - x^*\|^2 - \|w_n - u_n\|^2 + 2\lambda_n\langle Ay_n, x^* - u_n\rangle \\
&\leq \|w_n - x^*\|^2 - \|w_n - u_n\|^2 + 2\lambda_n\langle Ay_n, y_n - u_n\rangle - 2\lambda_n\eta\|y_n - x^*\|^2 \\
&= \|w_n - x^*\|^2 - \|w_n + y_n - y_n - u_n\|^2 + 2\lambda_n\langle Ay_n, y_n - u_n\rangle \\
&\quad - 2\lambda_n\eta\|y_n - x^*\|^2 \\
&= \|w_n - x^*\|^2 - \|w_n - y_n\|^2 - \|y_n - u_n\|^2 - 2\lambda_n\eta\|y_n - x^*\|^2 \\
&\quad + 2\langle w_n - \lambda_n Ay_n - y_n, u_n - y_n\rangle.
\end{aligned} \tag{39}$$

Applying (10) and Remark 3.2 in (39), we obtain

$$\begin{aligned}
\|u_n - x^*\|^2 &\leq \|w_n - x^*\|^2 - \|w_n - y_n\|^2 - \|y_n - u_n\|^2 - 2\lambda_n\eta\|y_n - x^*\|^2 \\
&\quad + 2\lambda_n\langle Aw_n - Ay_n, u_n - y_n\rangle \\
&\leq \|w_n - x^*\|^2 - \|w_n - y_n\|^2 - \|y_n - u_n\|^2 - 2\lambda_n\eta\|y_n - x^*\|^2 \\
&\quad + \frac{\lambda_n\mu}{\lambda_{n+1}}\left(\|w_n - y_n\|^2 + \|u_n - y_n\|^2\right) \\
&= \|w_n - x^*\|^2 - 2\lambda_n\eta\|y_n - x^*\|^2 - \left(1 - \frac{\lambda_n\mu}{\lambda_{n+1}}\right)\left(\|w_n - y_n\|^2 + \|u_n - y_n\|^2\right).
\end{aligned}$$

From Lemma 3.3, we know that there exists a natural number N_1 satisfying $\lambda_n \geq \frac{\lambda}{2}$ and $1 - \frac{\lambda_n\mu}{\lambda_{n+1}} = 1 - \mu > 0$, for all $n \geq N_1$, thus,

$$\|u_n - x^*\|^2 \leq \|w_n - x^*\|^2 - \lambda\eta\|y_n - x^*\|^2, \ \forall n \geq N_1. \tag{40}$$

Therefore,

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1 - \alpha_n)(z_n - x^*) + \alpha_n(u_n - x^*)\|^2 \\
&= (1 - \alpha_n)\|z_n - x^*\|^2 + \alpha_n\|u_n - x^*\|^2 - \alpha_n(1 - \alpha_n)\|z_n - u_n\|^2 \\
&\leq (1 - \alpha_n)\|z_n - x^*\|^2 + \alpha_n\|w_n - x^*\|^2 - \alpha_n\lambda\eta\|y_n - x^*\|^2 \\
&\quad - \alpha_n(1 - \alpha_n)\|z_n - u_n\|^2 \\
&\leq (1 - \alpha_n)\|z_n - x^*\|^2 + \alpha_n\|w_n - x^*\|^2 - \alpha_n(1 - \alpha_n)\|z_n - u_n\|^2 \\
&\quad - \alpha\lambda\eta\|y_n - x^*\|^2, \ \forall n \geq N_1.
\end{aligned}$$

Repeating similar arguments from (16) to (21), we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \left(1 + \alpha_n\theta_n + \delta_n(1 - \alpha_n)\right)\|x_n - x^*\|^2 - \left(\alpha_n\theta_n + \delta_n(1 - \alpha_n)\right)\|x_{n-1} - x^*\|^2 \\
&\quad + \sigma_n\|x_n - x_{n-1}\|^2 - \rho_n\|x_{n+1} - x_n\|^2 - \alpha\lambda\eta\|y_n - x^*\|^2, \ \forall n \geq N_1.
\end{aligned}$$

Thus,

$$\begin{aligned}\alpha\lambda\eta\|y_n - x^*\|^2 &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \sigma_n\|x_n - x_{n-1}\|^2\\ &\quad + \Big(\alpha_n\theta_n + \delta_n(1-\alpha_n)\Big)\Big(\|x_n - x^*\|^2 - \|x_{n-1} - x^*\|^2\Big)\\ &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + M^*\|x_n - x_{n-1}\|^2\\ &\quad + \Big(\alpha_n\theta_n + \delta_n(1-\alpha_n)\Big)\Big(\|x_n - x^*\|^2 - \|x_{n-1} - x^*\|^2\Big),\ \forall n \ge N_1,\end{aligned}$$

where

$$M^* = (1-\alpha)\delta(1+\delta) + \frac{2}{1+\epsilon} + \frac{(1-\alpha)}{4\alpha} \ge \sigma_n.$$

Hence,

$$\begin{aligned}\alpha\lambda\eta\sum_{k=N_1}^{n}\|y_k - x^*\|^2 &\le \|x_{N_1} - x^*\|^2 - \|x_{n+1} - x^*\|^2 + M^*\sum_{k=N_1}^{n}\|x_k - x_{k-1}\|^2\\ &\quad + \Big(\alpha_n\theta_n + \delta_n(1-\alpha_n)\Big)\|x_n - x^*\|^2\\ &\quad - \Big(\alpha_{N_1-1}\theta_{N_1-1} + \delta_{N_1-1}(1-\alpha_{N_1-1})\Big)\|x_{N_1-1} - x^*\|^2.\end{aligned}$$

By Lemma 3.4 and (28), we have $\{x_n\}$ is bounded and $\sum_{k=N_1}^{\infty}\|x_k - x_{k-1}\|^2 < \infty$, from which we deduce that $\sum_{k=N_1}^{\infty}\|y_k - x^*\|^2 < \infty$. Thus, $\lim_{n\to\infty}\|y_n - x^*\| = 0$.
Consequently,

$$\begin{aligned}&\lim_{n\to\infty}\|x_n - x^*\|\\ \le&\lim_{n\to\infty}\Big(\|x_n - w_n\| + \|w_n - y_n\| + \|y_n - x^*\|\Big)\\ =&\ 0.\end{aligned}$$

This completes the proof. □

Next, we put an operator T in Algorithm 1, which leads to the sequence $\{x_n\}$ converges weakly to a point of $Fix(T)\cap S$, where $Fix(T) := \{x \in H : Tx = x\}$ is the nonempty fixed point set of T.
We add the following conditions:
(C8) The operator $A : H \to H$ is monotone;
(C9) T is quasi-nonexpansive, that is

$$Fix(T) \ne \emptyset \text{ and } \|Tx - p\| \le \|x - p\|,\ \forall x \in H, p \in Fix(T);$$

(C10) $I - T$ is demiclosed, that is

$$x_n \rightharpoonup x \text{ and } (I-T)x_n \to 0 \Rightarrow x \in Fix(T);$$

(C11) $Fix(T)\cap S \ne \emptyset$.

Algorithm 2

Iterative step:

1. Take the parameters $\mu \in (0,1)$ and $\lambda_1 > 0$. Choose a nonnegative real sequence $\{a_n\}$ such that $\sum_{n=1}^{\infty} a_n < +\infty$.

Let $x_0, x_1 \in H$ be given starting points. Set n:=1.
2. Compute

$$\begin{cases} z_n = x_n + \delta_n(x_n - x_{n-1}) \\ w_n = x_n + \theta_n(x_n - x_{n-1}) \\ y_n = P_C(w_n - \lambda_n A w_n) \\ u_n = P_{T_n}(w_n - \lambda_n A y_n), \end{cases} \tag{41}$$

where

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\mu(\|w_n - y_n\|^2 + \|u_n - y_n\|^2)}{2\langle Aw_n - Ay_n, u_n - y_n\rangle}, \lambda_n + a_n\}, & if \langle Aw_n - Ay_n, u_n - y_n\rangle > 0, \\ \lambda_n + a_n, & otherwise, \end{cases}$$

and

$$T_n := \{w \in H : \langle w_n - \lambda_n A w_n - y_n, w - y_n\rangle \le 0\}.$$

If $w_n = y_n = x_n = x_{n+1} = z_n$, STOP. Otherwise
3. Compute

$$x_{n+1} = (1 - \alpha_n) z_n + \alpha_n T u_n, n \ge 1. \tag{42}$$

4. Set $n \leftarrow n + 1$, and go to 2.

Remark 4.2. *Using (41) and $x_n = y_n = w_n$, we can deduce that $x_n = P_C(x_n - \lambda_n A x_n)$, which implies that $\langle Ax_n, y - x_n\rangle \ge 0$, $\forall y \in C$, thus, $x_n \in S$. On the other hand, from $C \subset T_n$ and $w_n = y_n$, we get $y_n = P_{T_n}(w_n - \lambda_n A y_n) = u_n$, which together with $x_n = y_n = x_{n+1}$ and (42) implies that $x_n = (1 - \alpha_n)x_n + \alpha_n T x_n$, then, $Tx_n = x_n$, $x_n \in Fix(T)$. Therefore, when $w_n = y_n = x_n = x_{n+1} = z_n$, $x_n \in Fix(T) \cap S$, this algorithm could stop.*

Theorem 4.3. *Let $\{x_n\}$ be generated by Algorithm 2 such that Condition (C1) and (C4)-(C11) hold. Then $\{x_n\}$ converges weakly to a point of $Fix(T) \cap S$.*

Proof. Pick a point of $x^* \in Fix(T) \cap S$. From T is quasi-nonexpansive and (9)-(13), we obtain

$$\begin{aligned} \|Tu_n - x^*\|^2 &\le \|u_n - x^*\|^2 \\ &\le \|w_n - x^*\|^2 - \left(1 - \frac{\lambda_n \mu}{\lambda_{n+1}}\right)\|w_n - y_n\|^2 - \left(1 - \frac{\lambda_n \mu}{\lambda_{n+1}}\right)\|u_n - y_n\|^2. \end{aligned} \tag{43}$$

Repeating the similar arguments from (14)-(31) and use Tu_n in place of u_n, we can deduce that $\|x_{n+1} - x_n\| \to 0$, $n \to \infty$, $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - x^*\|$ exists. By $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$ and the definition of z_n, w_n, we have $\|x_n - z_n\| \to 0$ and $\|x_n - w_n\| \to 0$ as $n \to \infty$.
Hence,

$$\begin{aligned} \|Tu_n - z_n\| &= \frac{1}{\alpha_n}\|x_{n+1} - z_n\| \\ &\le \frac{1}{\alpha_n}\Big(\|x_{n+1} - x_n\| + \|x_n - z_n\|\Big) \to 0. \end{aligned}$$

Combining with $\{x_n\}$ is bounded, we get $\{w_n\}$ and $\{Tu_n\}$ are bounded.

Using (43), we get

$$\begin{aligned}
&\left(1-\frac{\lambda_n\mu}{\lambda_{n+1}}\right)\left(\|w_n-y_n\|^2+\|u_n-y_n\|^2\right)\\
&\le \|w_n-x^*\|^2-\|Tu_n-x^*\|^2\\
&=\left(\|w_n-x^*\|+\|Tu_n-x^*\|\right)\left(\|w_n-x^*\|-\|Tu_n-x^*\|\right)\\
&\le M_3\left(\|w_n-x^*\|-\|Tu_n-x^*\|\right)\\
&\le M_3\|w_n-Tu_n\|\\
&\le M_3\left(\|w_n-z_n\|+\|z_n-Tu_n\|\right)\\
&\le M_3\left(\|w_n-x_n\|+\|x_n-z_n\|+\|z_n-Tu_n\|\right)\to 0,
\end{aligned}$$

for some $M_3 > 0$, where the existence of M_3 is from the boundedness of $\{w_n\}$ and $\{Tu_n\}$. Thus, we get $\|w_n - y_n\| \to 0$, $\|u_n - y_n\| \to 0$ as $n \to \infty$.
Therefore, from

$$\|Tu_n-u_n\|\le\|Tu_n-z_n\|+\|z_n-w_n\|+\|w_n-y_n\|+\|y_n-u_n\|,$$

we have $\|Tu_n - u_n\| \to 0$ as $n \to \infty$.
By $\{x_n\}$ is bounded, we know that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup p \in H$. From $\lim_{n\to\infty}\|x_n - w_n\| = \lim_{n\to\infty}\|w_n - y_n\| = \lim_{n\to\infty}\|u_n - y_n\| = 0$, we also get $u_{n_k} \rightharpoonup p$, which together with the demi-closedness of $I - T$ implies that $p \in Fix(T)$. Moreover, we deduce $w_{n_k} \rightharpoonup p$ from $\lim_{n\to\infty}\|x_n - w_n\| = 0$ and $\|w_{n_k} - P_C(w_{n_k} - \lambda_n A w_{n_k})\| = \|w_{n_k} - y_{n_k}\| \to 0, n \to \infty$, combining Lemma 2.9, we obtain $p \in S$. Thus, $p \in Fix(T) \cap S$. Consequently, using Lemma 2.7 we get $\{x_n\}$ converges weakly to a point of $Fix(T) \cap S$. □

5. Numerical experiments

In this section, we provide some numerical experiments to compare our Algorithm 1 with some existing related algorithms. All the codes were written in MATLAB R2022b and performed on a PC Desktop Intel(R) Core(TM) i5-12500H @ 2.50 GHz, RAM 16.0 GB.

In all these examples, we present numerical comparisons of our proposed Algorithm 1 with Algorithm 1 of Shehu et al. in [26] and Algorithm 1 of Yao et al. in [37].

Example 5.1. *([13]) Consider the following fractional programming problem:*

$$\min f(x)=\frac{x^TQx+a^Tx+a_0}{b^Tx+b_0}$$

subject to $x \in X := \{x \in R^4 : b^T x + b_0 > 0\}$, *where*

$$Q=\begin{pmatrix}5&-1&2&0\\-1&5&-1&3\\2&-1&3&0\\0&3&0&5\end{pmatrix} \tag{44}$$

$a = (1, -2, -2, 1)^T, b = (2, 1, 1, 0)^T, a_0 = -2, b_0 = 4$. *It is easy to verify that* Q *is symmetric and positive definite in* R^4 *and consequently* f *is pseudo-convex and on* $X = \{x \in R^4 : b^T x + b_0 > 0\}$, *Lemma 2.10 reveals the relationship between pseudo-convex optimization and VIP. We minimize* f *over a nonempty, closed and convex subset* $C := \{x \in R^4 : 1 \le x_i \le 10, i = 1, ..., 4\} \subset X$ *by using* $A(x) := \nabla f(x) = ((b^T x + b_0)(2Qx + a) - b(x^T Qx + a^T x + a_0))/(b^T x + b_0)^2$. *This problem has a unique solution* $x^* = (1, 1, 1, 1)^T \in C$. *It is known ([15]) that a differentiable function* f *is pseudo-convex if and only if its gradient is pseudo-monotone, thus* A *is pseudo-monotone, then* A *is quasi-monotone.*

Furthermore, for all $x, y \in C$, $\nabla^2 f$ (the Hessian matrix of f), we have

$$\nabla^2 f(x) = \frac{2Q}{b^T x + b_0} - \frac{(2Qx + a)b^T}{b^T x + b_0} - \frac{b(2Qx + a)^T}{(b^T x + b_0)^2} + \frac{2(x^T Qx + a^T x + a_0)bb^T}{(b^T x + b_0)^3},$$

$$\begin{aligned}\|\nabla^2 f(x)\| =& \frac{2\|Q\|}{b^T x + b_0} + \frac{\|2Qx + a\|\,\|b\|}{b^T x + b_0} + \frac{\|2Qx + a\|\,\|b\|}{(b^T x + b_0)^2} \\ &+ \frac{2(x^T Qx + a^T x + a_0)\|bb^T\|}{(b^T x + b_0)^3} \\ &\leq L_1.\end{aligned}$$

By $b^T x + b_0 \leq b^T x^* + b_0 = 8$, $\|2Qx + a\|$ and $(x^T Qx + a^T x + a_0)$ are bounded on C, we know L_1 exists, thus, $\nabla^2 f$ is bounded. Then for all $x, y \in C$, we have

$$\begin{aligned}\|Ay - Ax\| &= \|\nabla f(y) - \nabla f(x)\| \\ &= \|\int_0^1 \nabla^2 f(x + t(y - x))(y - x)dt\| \\ &\leq \|\int_0^1 \nabla^2 f(x + t(y - x))dt\|\,\|y - x\| \\ &\leq \int_0^1 \|\nabla^2 f(x + t(y - x))\|dt\,\|y - x\| \\ &\leq L_1\|y - x\|. \end{aligned} \tag{45}$$

Thus, A is L_1-Lipschitz continuous.

It is known that strong convergence and weak convergence are equivalent in R^n, thus, in order to prove that A satisfies the condition (C3), we only need prove

$$\text{if } \{x_n\} \subset H,\ x_n \to v^* \text{ and } \liminf_{n\to\infty} \|Ax_n\| = 0, \text{ then } Av^* = 0.$$

From L_1-Lipschitz continuity of A and $x_n \to v^*$, we have

$$\|Ax_n - Av^*\| \leq L_1\|x_n - v^*\| \to 0,\ n \to \infty,$$

then, by $\|Av^*\| = \|Av^* - Ax_n + Ax_n\| \leq \|Av^* - Ax_n\| + \|Ax_n\|$, we have

$$0 \leq \|Av^*\| \leq \liminf_{n\to\infty} \|Ax_n\| = 0.$$

Therefore, we obtain $Av^* = 0$.

From $A(x) = ((b^T x + b_0)(2Qx + a) - b(x^T Qx + a^T x + a_0))/(b^T x + b_0)^2$, we have $A(x^*) = (1, \frac{15}{16}, \frac{7}{16}, \frac{17}{8})^T$. Thus, for all $y = (y_1, y_2, y_3, y_4)^T \in C(1 \leq y_i \leq 10, i = 1, 2, 3, 4)$, we have

$$\langle A(x^*), y - x^*\rangle = (y_1 - 1, y_2 - 1, y_3 - 1, y_4 - 1)(1, \frac{15}{16}, \frac{7}{16}, \frac{17}{8})^T \geq 0.$$

Thus, x^* is a solution of VIP. From Lemma 2.8 we obtain $x^* \in S_D$, that is $S_D \neq \emptyset$.

We now solve VIP with C and A given above. For all methods, we take $x_0 = [10, 10, 10, 10]'$, $x_1 = [10, 20, 30, 40]'$, other parameters are shown in Table 1. We choose the stopping criterion as error $= \|x_n - x^*\| \leq 0.0001$, that is $\log_{10}(error) \leq -4$. The corresponding results are reported in Figure 1 and Table 2.

Table 1: Methods parameters for Example 5.1

C our Alg. 1	$\lambda_1 = 0.5$	$\mu = 0.25$	$\theta_n = 1 - \frac{1}{n+1}$	$\delta_n = \frac{1}{2} - \frac{1}{n+2}$
	$\alpha_n = \frac{1}{3} - \frac{1}{n+4}$	$a_n = \frac{1}{n^2}$		
Alg. 1 in [37]	$\lambda_1 = 0.5$	$\mu = 0.25$	$\theta_n = 1 - \frac{1}{n+1}$	$\delta = 0.5$
	$\alpha_n = \frac{1}{3} - \frac{1}{n+4}$			
Alg. 1 in [26]	$\lambda_1 = 0.5$	$\mu = 0.25$	$\theta_n = 1 - \frac{1}{n+1}$	$\alpha_n = \frac{1}{3} - \frac{1}{n+4}$

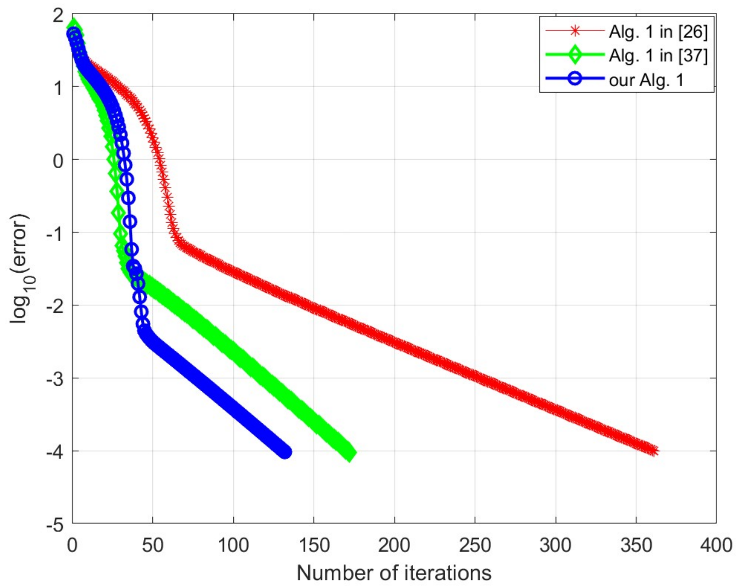

Figure 1: *error* ≤ 0.0001 for Example 5.1

Table 2: Number of iterations for Example 5.1

Algorithm	Alg. 1 in [26]	Alg. 1 in [37]	our Alg. 1
Number of iterations	361	172	132

Example 5.2. *Consider the problem VIP whenever H is the classical* $L^2[0,1]$ *space with the inner product and norm given by*

$$\langle x, y\rangle = \int_0^1 x(t)y(t)\mathrm{d}t,\ \|x\| = \left(\int_0^1 |x(t)|^2\,\mathrm{d}t\right)^{1/2},\quad \forall x, y \in H.$$

Here the unit ball is taken as the feasible set and the nonlinear operator $A : H \to H$ *is given by*

$$A(x) = (1.5 - \|x\|)x,\quad \forall x \in H.$$

It should be noted that the operator A in the example above is pseudo-monotone, and the solution to the variational inequality problem is $x^*(t) = 0$.

Setting an appropriate stopping criterion in numerical algorithms is crucial to prevent computational inefficiencies and conserve resources. The maximum allowed iterations in this study is 100. All the parameters are shown in Table 3. Results of the proposed method against other algorithms are presented in Figure 2 - Figure 5, showing a superior convergence rate, accuracy, and speed. The proposed method is robust and insensitive to the initial guess. The proposed algorithm is computationally efficient and valuable for solving similar problems. With a reasonable stopping criterion and superior performance, it can be applied to a wide range of applications. This constitutes a significant contribution to numerical optimization.

Table 3: Methods parameters for Example 5.2

our Alg. 1	$\lambda_1 = 0.6$	$\mu = 0.9$	$\theta_n = 1 - \frac{1}{n+1}$	$\delta_n = \frac{1}{3} - \frac{1}{n+3}$
	$\alpha_n = 0.5$	$a_n = \frac{1}{n^2}$		
Alg. 1 in [37]	$\lambda_1 = 0.6$	$\mu = 0.9$	$\theta_n = 1 - \frac{1}{n+1}$	$\delta = 0.12$
	$\alpha_n = 0.5$			
Alg. 1 in [26]	$\lambda_1 = 0.6$	$\mu = 0.9$	$\theta_n = 1 - \frac{1}{n+1}$	$\alpha_n = 0.5$

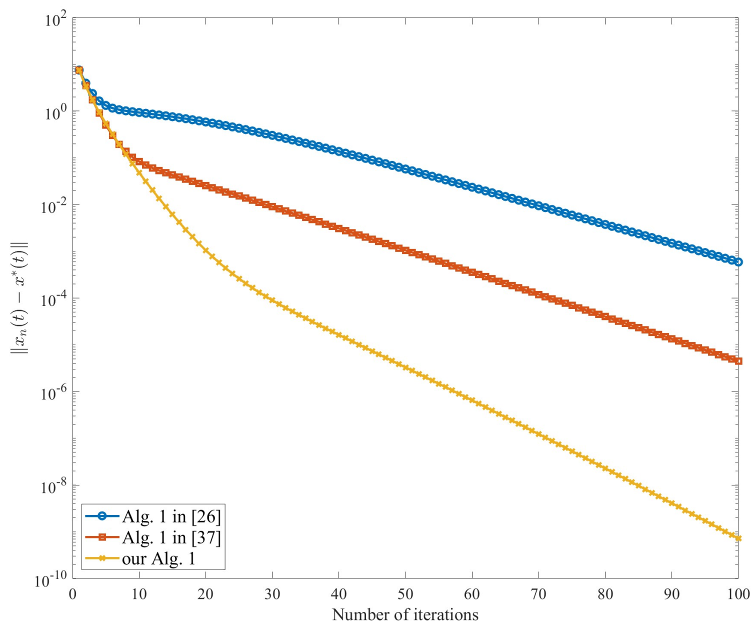

Figure 2: $x_0(t) = x_1(t) = 50t^5$ for Example 5.2

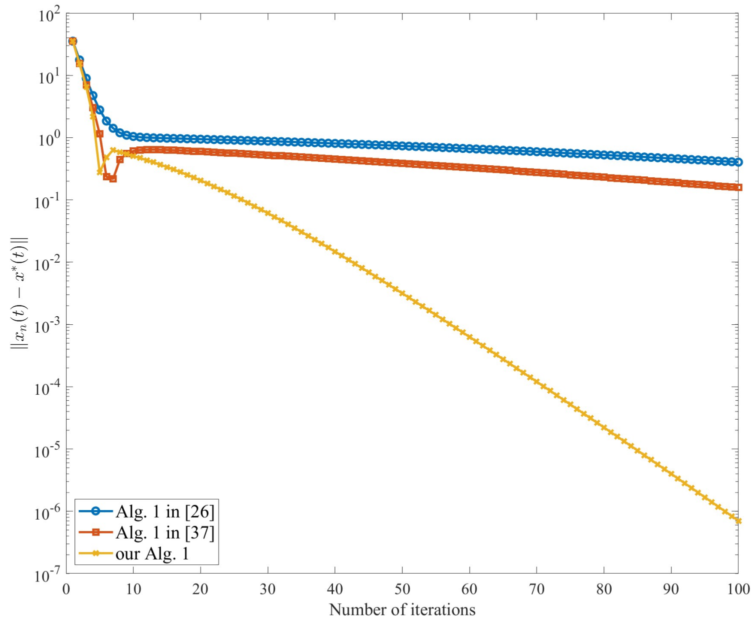

Figure 3: $x_0(t) = x_1(t) = 50 \log t$ for Example 5.2

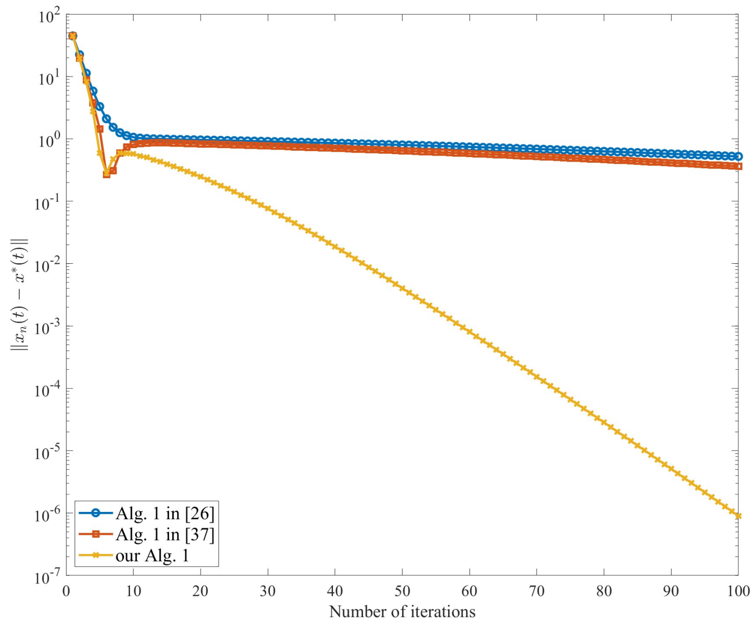

Figure 4: $x_0(t) = x_1(t) = 50 \exp(t)$ for Example 5.2

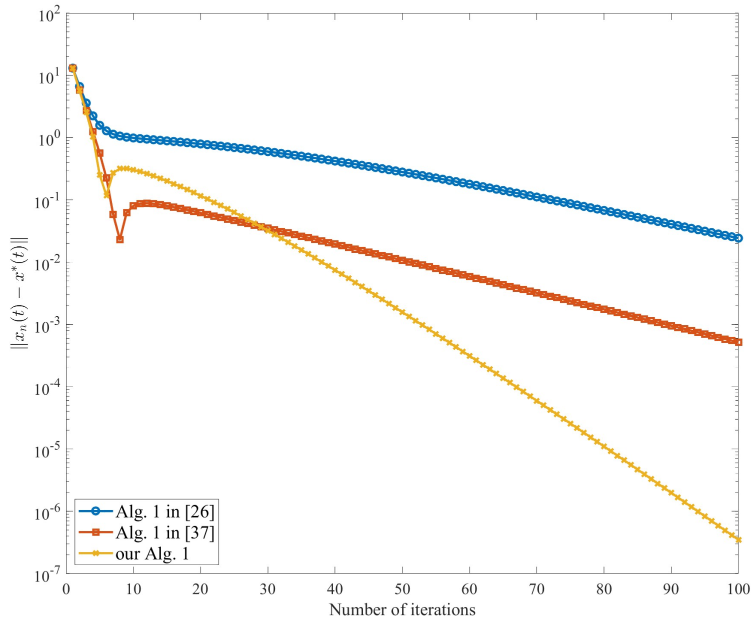

Figure 5: $x_0(t) = x_1(t) = 50 \sin t$ for Example 5.2

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