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Self similarity sets via fixed point theory with lack of convexity

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Abstract. A well-known theorem of fractal geometry, presented by J. Hutchinson ([16]), says that there exists a unique compact self similar set with respect to any finite set of contractions on a complete metric space. Motivated by this result, in this paper, we prove fixed set theoretical theorems in order to obtain useful variations of this important result for Meir-Keeler operators and using the technique of measure of weak-noncompactness for operators acting in Banach spaces and Banach algebras.

1. Introduction

The fixed-point theory, developed around metric and topological spaces, plays an important role in mathematics and other disciplines like engineering and game theory.
In metric space context, the fixed point property is usually related to a certain class of mappings described by some metric conditions. For example, in [1], Afshari et al. consider a generalized Geraghgty multi-valued mappings in the context of complete metric spaces endowed with a graph. In quasimetric spaces, Karapinar et al. ,in [19], obtained new results using admissible function and inspired by Proinov type contraction. In [13], Fulga et al. discussed common fixed point theorems on quasi-cone spaces over a divisible Banach algebras. Recently, in [2], Afshari et al. proposed a notion of quasicone Banach spaces over Banach algebras and examined the existence of some common fixed points of two self mappings.
In the theory of fixed points in topological spaces, the convexity of the domain is required, so for the lack of convexity of the domain, we remark that there does not exist a fixed point any longer. That is why we study the existence of fixed sets that appear in various branches of applied mathematical analysis. For instance, the global attractors of a semiflow is defined as a maximal attracting set which is fixed under every member of the flow in dynamical systems. Another example is provided by fractal geometry, where a self-similar set is defined as a compact set fixed under the union of a finite set of contractions.
The existence of a fixed set is often treated in a particular context that is developing but many authors sketched a general fixed set theory for set-valued maps. A famous fixed point principle which is useful for solving certain nonlinear functional equations while studying their stability is the Krasnoselskii fixed point theorem stated in [20]. In [21], Ok, offered suitable modifications of this result when the hypothesis of convexity was relaxed. Obviously, the conclusion of the theorem did not need to hold in this case. Indeed, he started by observing that when the lack of a fixed point of an operator was due to the nonconvexity of the domain, we could still find a fixed (invariant) set of that operator (see [22]). In [4], Al-Thagafi and Shahzad

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extended and improved several fixed-point and fixed-set results including that given by Ok [21], and a positive answer to Ok's question is provided. Using the technique of measure of weak noncompacteness, in [8], Ben Amar et al. proved some fixed set results of Krasnosel'skii type for the sum of two multivalued operators in the setting of a weak topology, without the assumption of the convexity of their common domain. In [7], Ben Amar et al. proved some fixed set results for the sum and the product of three multivalued mappings acting on Banach algebras satisfying a certain sequential condition in the weak topology setting. Their hybrid theorems improved and generalized those in [8].
In 2015, Aghajani, Mursaleen and Shole Haghighi (in [3]) introduced the definition of Meir-Keeler condensing operators and proved a theorem that guarantee the existence of a fixed point for single valued mappings. In [5], Ben Amar et al. introduced the concept of Meir-Keeler condensing operators in a Banach space via an arbitrary measure of weak noncompactness and proved some generalizations of Darbo's fixed point theorem by considering a measure of weak noncompactness which does not necessary have the maximum property. Recently, in [6], Belhaj M. et al. have established a new fixed point theorem for multivalued Meir-Keeler condensing mappings via an arbitrary measure of weak noncompactness which in turn included the fixed point theorems of Krasnoselskii and Dhage as special cases in non separable spaces. Meir-Keeler operators were also treated in metric context and recently, in [18], Karapinar and Fulga introduced the notion of hybrid Juggi-Meir-Keeler type contraction and extend a number of existing results in the literature.
In this paper we extend several fixed point and fixed set results without the assumption of the convexity including those given in [7], [8], [9], [14], [10] and [15] under weak toplogy settings. Applications to the theory of self-similarity are also given.

2. Preliminaries

We present in this section some notations and definitions which we will need in what follows.

Definition 2.1. *Let M be a nonempty subset of a Banach space E. Let $T : M \to \mathcal{P}(E)$ be a multivalued mapping. We say that:*

1. *T has a weakly sequentially closed graph if for every sequence $\{x_n\} \subset M$ with $x_n \rightharpoonup x$ in M and for every sequence $\{y_n\}$ with $y_n \in T(x_n)$, for all $n \in \mathbb{N}$, $y_n \rightharpoonup y$ in E implies $y \in T(x)$.*
2. *T is weakly completely continuous if T has a weakly sequentially closed graph and $T(A)$ is a relatively weakly compact subset of E, for any bounded subset A of M.*
3. *T is sequentially weakly upper semicompact in M (s.w.u.sco., for short) if for any sequence $\{x_n\} \subset M$ with $x_n \rightharpoonup x$ and for every sequence $\{y_n\}$ with $y_n \in T(x_n)$, for all $n \in \mathbb{N}$, the sequence $\{y_n\}$ has a weakly convergent subsequence in E.*

Definition 2.2. *Let M be a nonempty subset of a Banach space E. Let $T : M \to E$ be a mapping. We say that T is weakly sequentially continuous if for every sequence $\{x_n\} \subset M$ with $x_n \rightharpoonup x$ in M, the sequence $T(x_n) \rightharpoonup T(x)$.*

Definition 2.3. *We say that the Banach algebra X satisfies condition $(\mathcal{P})$ if the operation of multiplication $(x, y) \to x.y$ is sequentially weakly continuous; i.e., if $\{x_n\}$ and $\{y_n\}$ are two sequences of X such that $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$, then $x_n.y_n \rightharpoonup x.y$.*

Let E be a Hausdorff linear topological space, then we define

$$\mathcal{P}(E) = \{A \subset E; A \neq \emptyset\}$$

and

$$\mathcal{P}_{bd}(E) = \{A \subset E; A \neq \emptyset, \text{bounded}\}.$$

We recall that a function $\omega : \mathcal{P}_{bd}(E) \to \mathbb{R}_+$ is said to be a Measure of Weak Noncompactness (MWNC, for short) on E if it satisfies the following properties:

(1) For any bounded subset Ω_1, Ω_2 of E, we have $\Omega_1 \subseteq \Omega_2$ implies $\omega(\Omega_1) \leq \omega(\Omega_2)$.

(2) $\omega(\overline{conv}(\Omega)) = \omega(\Omega)$, for all bounded subsets $\Omega \subset E$.

(3) $\omega(\Omega \cup \{a\}) = \omega(\Omega)$ for all $a \in E$, $\Omega \in \mathcal{P}_{bd}(E)$.

(4) $\omega(\Omega) = 0$ if and only if Ω is relatively weakly compact in E.

(5) If $(X_n)_{n\geq 1}$ is a decreasing sequence of nonempty bounded and weakly closed subsets of E with $\lim_{n\to+\infty} \omega(X_n) = 0$, then $\cap_{n=1}^{\infty} X_n$ is non empty and $\omega(\cap_{n=1}^{\infty} X_n) = 0$.

The MWNC ω is said to be

(i) positive homogeneous if $\omega(\lambda\Omega) = \lambda\omega(\Omega)$, for all $\lambda > 0$ and $\Omega \in \mathcal{P}_{bd}(E)$,

(ii) subadditive if $\omega(\Omega_1 + \Omega_2) \leq \omega(\Omega_1) + \omega(\Omega_2)$, for all $\Omega_1, \Omega_2 \in \mathcal{P}_{bd}(E)$.

As an example of MWNC, we have the De Blasi measure of weak noncompactness [11], defined on $\mathcal{P}_{bd}(E)$ by :

$$\mu(M) = inf\,\{\varepsilon > 0; \textit{there exists } K \textit{ weakly compact such that}: \ M \subset K + B_\varepsilon\}.$$

It is well known that μ is homogeneous, subadditive, and satisfies the set additivity property:

$$\mu(M \cup N) = max\,\{\mu(M), \mu(N)\},\ for\ all\ M,\ N \in \mathcal{P}_{bd}(X).$$

For more properties of the MWNC, we can refer to [11].

Definition 2.4. *Let C be a nonempty subset of a Banach space E. A multivalued map $T : C \to \mathcal{P}(C)$ is called Meir-Keeler condensing if for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon \leq \omega(A) < \varepsilon + \delta \Rightarrow \omega(T(A)) < \varepsilon, \tag{1}$$

for all bounded subset A of C.

Definition 2.5. *Let C be a nonempty subset of a Banach space E. A multivalued map $T : C \to \mathcal{P}(C)$ is called countably D-set-Lipchitzian if:*

1. *$T(C)$ is bounded,*
2. *$\omega(T(B)) \leq \phi(\omega(B))$ for any countable bounded subset B of C with $\omega(B) > 0$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous non decreasing function with $\phi(0) = 0$. The function ϕ is called a D-function of T on E.*

3. Fixed-set results for the sum of two multi-valued maps

We establish here a generalization of Krasnoselskii's theorem in the setting of weak topology without the assumption of the convexity of the common domain of maps T and S.

Theorem 3.1. *Let M be a non-empty, weakly closed subset of a Banach space E and ω a subadditive MNWC on E. Let assume that $S : M \to \mathcal{P}(E)$ and $T : M \to \mathcal{P}(E)$ are two maps satisfying the following conditions:*

1. *S is weakly completely continuous.*
2. *T is Meir-Keeler condensing with respect to measure ω and has a weakly sequentially closed graph.*
3. *$S(M) + T(M)$ is a bounded set of M.*

Then
i) there exists a minimal C weakly compact subset of M such that $C = S(C) + T(C)$;
ii) there exists a maximal $A \in \mathcal{P}(M)$ such that $A = S(A) + T(A)$.

Proof We consider the sequence $\{K_n\}_n$ of subsets of M defining by $K_0 = M$ and $K_{n+1} = S(K_n)+T(K_n), n \geq 0$. Defining $\varepsilon_n = \omega(K_n)$, we have

$$\varepsilon_{n+1} = \omega(K_{n+1}) = \omega(S(K_n) + T(K_n)) \leq \omega(S(K_n)) + \omega(T(K_n)).$$

Since S is weakly relatively compact and T is Meir-Keeler condensing, then $\varepsilon_{n+1} < \omega(K_n) = \varepsilon_n$. Now $\{\varepsilon_n\}_{n\geq 0}$ is a positive decreasing sequence of real numbers, then there exists $\beta \geq 0$ such that $\varepsilon_n \longrightarrow \beta$ as $n \mapsto \infty$. Supposing that $\beta > 0$, then there exists N_0 such that

$$n > N_0 \Rightarrow \beta \leq \varepsilon_n < \beta + \delta(\beta).$$

In addition, by the definition of Meir-Keeler condensing operators, we get $\varepsilon_{n+1} < \beta$. This is absurd, so $\beta = 0$. As a result, $(\overline{K_n}^w)$ is a decreasing sequence of nonempty, bounded and weakly closed subsets with $\omega(K_n) \to 0$ as $n \mapsto +\infty$. Consequently, using condition (5) in the definition of the measure of weak noncompactness, we deduce that the set $K_\infty = \cap_{n\geq 1}\overline{K_n}^w \in Ker\ \omega$, so we can deduce that K_∞ is weakly compact.
Let now

$$\mathcal{H} = \{H \text{ weakly compact};\ H \subset K_\infty \text{ and } S(H) + T(H) \subset H\}.$$

We have $K_\infty \in \mathcal{H}$, so $\mathcal{H}$ is nonempty. Any chain in $(\mathcal{H}, \supseteq)$ has the finite intersection property, so as K_∞ is weakly compact, the intersection of all members of any chain in $(\mathcal{H}, \supseteq)$ is nonempty. Then, any chain in $(\mathcal{H}, \supseteq)$ has a lower bound in $\mathcal{H}$. Therefore, Zorn's lemma shows that $(\mathcal{H}, \supseteq)$ has a minimal element, say C which is weakly compact. It also shows that, $C \subset K_\infty$ and that $S(C) + T(C) \subset C$. Let

$$L = \overline{S(C) + T(C)}^w.$$

Such that L is also weakly compact and $L \subset C$. Hence,

$$S(L) + T(L) \subset S(C) + T(C) \subset \overline{S(L) + T(L)}^w = L,$$

so $L \in \mathcal{H}$. It follows that $L = C = \overline{S(C) + T(C)}^w$. Now let $x \in \overline{S(C) + T(C)}^w$, using the Eberlein-Smulian theorem (see [12] Theorem 8.12.4 p.549), there exists a sequence $\{x_n\} \subset S(C) + T(C)$ such that $x_n \rightharpoonup x$. Then, there exist $\{\alpha_n\} \subset S(C)$ and $\{\beta_n\} \subset T(C)$ such that

$$x_n = \alpha_n + \beta_n.$$

Let $y_n, z_n \in C$ with $\alpha_n \in S(y_n)$ and $\beta_n \in T(z_n)$. Since S is weakly completely continuous, there exists a subsequence $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$ that converges weakly to α; and since C is weakly compact, there exists subsequence $\{y_{n_k}\}$ of $\{y_n\}$ that converges weakly to $y \in C$ (by the Eberlein-Smulian Theorem). Since S has a weakly sequentially closed graph, then $\alpha \in S(y)$ and so $\beta_{n_k} \rightharpoonup x - \alpha$. Since T has weakly sequentially closed graph, we get $\beta \in T(z)$ and $x = \alpha + \beta \in S(y) + T(z) \subset S(C) + T(C)$. Hence,

$$C = \overline{S(C) + T(C)}^w = S(C) + T(C),$$

which proves i).
Now let

$$\mathcal{K} = \{K \subset M; K \subset S(K) + T(K)\},$$

and let $A = \cup_{K\in\mathcal{K}} K$. We can see that $\mathcal{K}$ is nonempty, since $C \in \mathcal{K}$. We have $A \subset S(A) + T(A)$.
Let $y \in S(A) + T(A)$, then

$$A \cup \{y\} \subset S(A) + T(A) \subset S(A \cup \{y\}) + T(A \cup \{y\}),$$

so $A \cup \{y\} \in \mathcal{K}$ and $y \in A$. Thus $S(A) + T(A) = A$.
If we take $S = 0$, in Theorem 3.1, we obtain the following corollary:

Corollary 3.2. *Let M be a non-empty, weakly closed subset of a Banach space E and ω a subadditive MNWC on E. Assume $T : M \to \mathcal{P}(E)$ satisfying the following conditions:*

1. *T is Meir-Keeler condensing with respect to a measure ω and has weakly sequentially closed graph.*
2. *$T(M)$ is a bounded set of M.*

Then
i) there exists a minimal C weakly compact subset of M such that $C = T(C)$;
ii) there exists a maximal $A \in \mathcal{P}(M)$ such that $A = T(A)$.

Corollary 3.3. *Let M be a non-empty, weakly closed subset of a Banach space E and ω a subadditive MNWC on E. Assume that $T : M \to \mathcal{P}(E)$ has weakly sequentially closed graph and $T(M)$ is relatively weakly compact. Then,*

1. *there exists a minimal C weakly compact subset of M such that $C = T(C)$;*
2. *there exists a maximal $A \in \mathcal{P}(M)$ such that $A = T(A)$.*

Remark 3.4. *Since T is Meir-Keeler condensing for any measure of weak noncompactness on E, this result is an immediate consequence of Corollary* 3.2.

Theorem 3.5. *Let M be a non-empty, weakly closed subset of a Banach space E and ω a subadditive MNWC on E. Assume $S : M \to \mathcal{P}(E)$ and $T : M \to E$ satisfying the following conditions:*

1. *S is weakly completely continuous,*
2. *T is Meir-Keeler condensing with respect to a measure ω and weakly sequentially continuous,*
3. *$S(M) + T(M)$ is a bounded set of M.*

Then
i) there exists a minimal C weakly compact subset of M such that $C = S(C) + T(C)$;
ii) there exists a maximal $A \in \mathcal{P}(M)$ such that $A = S(A) + T(A)$.

Proof
As in the proof of Theorem 3.1, we show that there exists a weakly compact C such that $S(C) + T(C) \subset C$ and $\overline{S(C) + T(C)}^{w} = C$. We claim that $\overline{S(C) + T(C)}^{w}$ is weakly closed. Using the Eberlein-Smulian Theorem, the weak compacity of $\overline{S(C) + T(C)}^{w}$ and C and the fact that T is weakly sequentially continuous and S has weakly sequentially closed graph, the result follows easily.

Theorem 3.6. *Let M be a non-empty, weakly closed subset of a Banach space E and ω a subadditive MNWC on E. Assume $S : M \to \mathcal{P}(E)$ and $T : E \to \mathcal{P}(E)$ satisfying the following conditions:*

1. *S is weakly completely continuous,*
2. *T is Meir-Keeler condensing with respect to a measure ω and has weakly sequentially closed graph,*
3. *$x \in T(x) + S(y),\ y \in M \Longrightarrow x \in M$,*
4. *$(I - T)^{-1}S(M)$ is bounded.*

Then,
i) there exists a minimal K weakly compact subset of M such that $(I - T)(K) = S(K)$ and $K \subset S(K) + T(K)$;
ii) there exists a maximal $A \in \mathcal{P}(M)$ such that $A = S(A) + T(A)$.

Proof
We check that $(I - T)^{-1}S(M) \subset M$. Let $x \in (I - T)^{-1}S(M)$, then there exists $y \in M$ such that $x \in (I - T)^{-1}S(y)$. Therefore, it follows that $x \in S(y) + T(x)$ and, by assumption (3), we get $x \in M$. Hence $(I - T)^{-1}S(M) \subset M$. We define,

$$H := (I - T)^{-1}S : M \to \mathcal{P}(M).$$

We need now to show that H has weakly sequentially closed graph. Let $\{x_n\} \subset M$ with $x_n \rightharpoonup x$ and let $y_n \in H(x_n)$ with $y_n \rightharpoonup y$. Since set $\{x_n\}_{n\in\mathbb{N}}$ is bounded and S is weakly completely continuous, we find that $S(\{x_n\})$ is relatively weakly compact. Then by the Eberlein Smulian theorem, if $s_n \in S(x_n)$, there is a subsequence $\{s_{n_k}\}$ such that $s_{n_k} \rightharpoonup s$. Since S has a weakly sequentially closed graph and using these subsequences, we get $s \in S(x)$. Since T has weakly sequentially closed graph, we get $y - s \in T(y)$, so $y \in (I-T)^{-1}Sx$. We consider the sequence $\{K_n\}_n$ of subsets of M defined by $K_0 = M$ and $K_{n+1} = H(K_n), n \geq 0$. Let $\varepsilon_n = \omega(K_n)$, then we have

$$\begin{aligned}\varepsilon_{n+1} &= \omega(K_{n+1}) = \omega(H(K_n))\\ &\leq \omega(T(I-T)^{-1}S(K_n) + S(K_n))\\ &\leq \omega(T(K_{n+1}) + S(K_n))\\ &\leq \omega(T(K_n) + S(K_n))\\ &\leq \omega(T(K_n)) + \omega(S(K_n)).\end{aligned}$$

Since S is weakly relatively compact and T is Meir-Keeler condensing, then $\varepsilon_{n+1} < \varepsilon_n$. Now $\{\varepsilon_n\}_{n\geq 0}$ is a positive decreasing sequence of real numbers, so there exists $\beta \geq 0$ such that $\varepsilon_n \longrightarrow \beta$ as $n \mapsto \infty$. Supposing that $\beta > 0$, so there exists N_0 such that

$$n > N_0 \Rightarrow \beta \leq \varepsilon_n < \beta + \delta(\beta).$$

Based on the definition of Meir-Keeler condensing operator, we get $w(K_{n+1}) = \varepsilon_{n+1} < \beta$. This is absurd, so $\beta = 0$. Hence $\{\overline{K_n}^w\}_n$ is a decreasing sequence of nonempty, bounded and weakly closed subsets with $\omega(\overline{K_n}^\omega) \to 0$ as $n \mapsto +\infty$. Consequently, by condition (5) (in the definition of the measure of weak noncompactness), we deduce that the set $K_\infty = \cap_{n\geq 1}\overline{K_n}^w \in Ker\ \omega$, so we can deduce that K_∞ is weakly compact. It follows now, by Corollary 3.3, that there exists a minimal K weakly compact subset K_∞ such that $K = H(K) = (I-T)^{-1}S(K)$, so $(I-T)(K) = S(K)$. This gives that $K \subset S(K) + T(K)$. We find also, by the same proof as in Theorem 3.1, that there exists a maximal $A \in \mathcal{P}(K_\infty)$ such that $A = S(A) + T(A)$.

4. Fixed-set results in Banach algebras

In the following, we state some fixed set theorems for multivalued mappings in Banach algebras by using the sequential characterization denoted by $(\mathcal{P})$.

Theorem 4.1. *Let E be a Banach algebra with condition $(\mathcal{P})$ and let S be a nonempty weakly closed subset of E. Let $A, B, C : S \to \mathcal{P}(E)$ be three multivalued mappings satisfying the following properties:*

1. *A, B and C are s.w.u.sco.*
2. *A, B and C are countably D-set Lipchitzian.*
3. *$A(S)B(S) + C(S)$ is a bounded set of S.*
4. *For $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\|A(S)\|\phi_B(r) + \|B(S)\|\phi_A(r) + \phi_A(r)\phi_B(r) + \phi_C(r) < \varepsilon.$$

for all $r \in [\varepsilon, \varepsilon + \delta[$ where ϕ_A, ϕ_B and ϕ_C are the D-functions of A, B and C, respectively.

Then,
i) there exists a minimal K weakly compact subset of S such that $K = A(K)B(K) + C(K)$;
ii) there exists a maximal $L \in \mathcal{P}(S)$ such that $L = A(L)B(L) + C(L)$.

Proof
We consider the sequence $(S_n)_n$ of subsets of S defined by $S_0 = S$ and $S_n = A(S_{n-1})B(S_{n-1}) + C(S_{n-1}), n \geq 1$. Using the same approach introduced by Ben Amar et al. in [9], $n \geq 0$ is fixed and

$$a_n = \sup\{\omega(K); K \text{ is a countable subset of } S_n\}.$$

Now, let K_i^n be a sequence of countable subsets of S_n with $\omega(K_i^n) \to a_n$ as $i \mapsto \infty$. Let $K^n = \cup_{i\geq 1} K_i^n$, and since K^n is a countable subset of S_n, we obtain

$$a_n \geq \omega(K^n) \geq \omega(K_i^n) \to a_n.$$

Then $\omega(K^n) = a_n$. Let $x \in K^n$. There exists $a_x \in S_{n-1}$ such that $x \in A(a_x)B(a_x) + C(a_x)$. Let $\mathcal{M}_n = \cup_{x\in K^n}\{a_x\}$. Since K^n is a countable subset of S_n, we have $\mathcal{M}_n$ is a countable subset of S_{n-1} and then $\mathcal{M}_n \subset K^{n-1}$. Then, $K^n \subset A(\mathcal{M}_n)B(\mathcal{M}_n) + C(\mathcal{M}_n)$. It is worth noting that since $A(S)B(S) + C(S)$ is bounded, so are sets $S_n, \mathcal{M}_n$ and K^n. We find that $\{a_n\}_{n\geq 1}$ is a positive decreasing sequence of real numbers. We assume $a = \lim_{n\mapsto+\infty} a_n$. If $a > 0$, then there exists $n_0 \in \mathbb{N}$ such that

$$a \leq \omega(K^{n_0}) < a + \delta(a),$$

where $\delta(a)$ is chosen according to (1). By the definition of a_n, we have

$$\begin{aligned} a_{n_0+1} &= \omega(K^{n_0+1}) \\ &\leq \omega(A(\mathcal{M}_{n_0+1})B(\mathcal{M}_{n_0+1}) + C(\mathcal{M}_{n_0+1})) \\ &\leq \omega(A(K^{n_0})B(K^{n_0}) + C(K^{n_0})) \\ &\leq \|A(K^{n_0})\|\phi_B(\omega(K^{n_0})) + \|B(K^{n_0})\|\phi_A(\omega(K^{n_0})) \\ &+ \phi_A(\omega(K^{n_0}))\phi_B(\omega(K^{n_0})) + \phi_C(\omega(K^{n_0})) \\ &< a. \end{aligned}$$

Which is a contradiction. Then we deduce that $a = 0$. For $n \geq n_0$, let $\{x_k^n\}_{k\geq 1} \subset S_n$. Since $\{x_k^n\}_{k\geq 1}$ is a countable subset of S_n, we have $\omega(\{x_k^n, k \geq 1\}) \leq a = 0$. Then S_n is relatively weakly sequentialy compact and now the Eberlein-Smulian theorem argument guarantees that S_n is relatively weakly compact. Consequently, by condition (5), (in the definition of the measure of weak noncompactness), we deduce that the set $S_\infty = \cap_{n\geq n_0} \overline{S_n}^w$ is nonempty, weakly closed and $S_\infty \in ker\ \omega$. Let

$$\mathcal{F} = \{X \subset S; X \text{ be a weakly compact subset of } S_\infty \text{ and } A(X)B(X) + C(X) \subset X\}.$$

Therefore, by Zorn's Lemma, and arguing in the same way as in [8], we prove the existence of a weakly compact set $K \subset S_\infty$ verifying $A(K)B(K) + C(K) \subset K$. Let

$$N = \overline{A(K)B(K) + C(K)}^\omega.$$

N is also weakly compact and $N \subset K$. It follows that

$$A(N)B(N) + C(N) \subset A(K)B(K) + C(K) \subset \overline{A(K)B(K) + C(K)}^\omega = N.$$

Hence, $N \in \mathcal{F}$. Thus, $N = K = \overline{A(K)B(K) + C(K)}^\omega$. We prove now that $A(K)B(K) + C(K)$ is weakly closed. To begin, let $x \in \overline{A(K)B(K) + C(K)}^\omega$. By the Eberlein-Smulian theorem, there exists a sequence $\{x_n\} \subset A(K)B(K) + C(K)$ such that $x_n \rightharpoonup x$. Accordingly, there exists sequence $\{\alpha_n\} \subset A(K)$, sequence $\{\beta_n\} \subset B(K)$ and sequence $\{\gamma_n\} \subset C(K)$ such that

$$x_n = \alpha_n\beta_n + \gamma_n,$$

with $\alpha_n \in A(y_n), \beta_n \in B(z_n)$ and $\gamma_n \in C(t_n)$, for some y_n, z_n and $t_n \in K$. By the Eberlein-Smulian theorem and according to the weak compactness of K, we find that $y_{n_k} \rightharpoonup y, z_{n_k} \rightharpoonup z$ and $t_{n_k} \rightharpoonup t$, where y_{n_k}, z_{n_k} and t_{n_k}

are subsequences of $\{y_n\}, \{z_n\}$ and $\{t_n\}$, respectively. As A, B and C are s.w.u.sco, we verify the existence of subsequences $\alpha_{n_k}, \beta_{n_k}$ and γ_{n_k} such that

$$\alpha_{n_k} \rightharpoonup \alpha \in A(y), \beta_{n_k} \rightharpoonup \beta \in B(z) \text{ and } \gamma_{n_k} \rightharpoonup \gamma \in C(t).$$

By the condition $(\mathcal{P})$, it follows that $x_{n_k} \rightharpoonup A(y)B(z) + C(t) \subset A(K)B(K) + C(K)$. Thus we prove the result. For *ii*), let

$$\mathcal{L} = \{M \subset S; M \subset A(M)B(M) + C(M)\}$$

and let $L = \bigcup_{M \in \mathcal{L}} M$. Clearly, $\mathcal{L}$ is nonempty since $K \in \mathcal{L}$. We have $L \subset A(L)B(L) + C(L)$.
Let $y \in A(L)B(L) + C(L)$. It follows that

$$L \cup \{y\} \subset A(L)B(L) + C(L) \subset A(L \cup \{y\})B(L \cup \{y\}) + C(L \cup \{y\}).$$

So $L \cup \{y\} \in \mathcal{L}$ and $y \in L$. Thus $A(L)B(L) + C(L) = L$.
When A and C are single valued maps, we deduce the following result.

Corollary 4.2. *Let E be a Banach algebra with condition $(\mathcal{P})$ and let S be a nonempty weakly closed subset of E. Let $B : S \to \mathcal{P}(E)$ and $A, C : S \to E$ be three mappings satisfying the following properties:*

1. *B s.w.u.sco,*
2. *A, B and C are countably D-set Lipchitzian,*
3. *A and C are weakly sequentially continuous,*
4. *$A(S)B(S) + C(S)$ is a bounded set of S,*
5. *For $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\|A(S)\|\phi_B(r) + \|B(S)\|\phi_A(r) + \phi_A(r)\phi_B(r) + \phi_C(r) < \varepsilon,$$

for all $r \in [\varepsilon, \varepsilon + \delta[$ where ϕ_A, ϕ_B, and ϕ_C are the D-functions of A, B and C, respectively.

Then,
i) there exists a minimal K weakly compact subset of S such that $K = A(K)B(K) + C(K)$;
ii) there exists a maximal $L \in \mathcal{P}(S)$ such that $L = A(L)B(L) + C(L)$.

5. Application: Self-similar sets

Let M be a nonempty weakly closed subset of a Banach space and let $\mathcal{F}$ be a family of self maps of M. For any $x \in M$, let

$$\mathcal{F}(x) = \{f(x), f \in \mathcal{F}\}, \quad \mathcal{F}(M) = \cup\{f(M) : f \in \mathcal{F}\}.$$

The study of self-similar sets appeared in 1981, when Hutchinson [16] considered the non-empty compact set $X \subset \mathbb{R}^n$ satisfying:

$$X = f_0(X) \cup f_1(X) \cup ... \cup f_{n-1}(X),$$

where $f_0, f_1, \cdots, f_{n-1}$ are the similarity contraction on $\mathbb{R}^n$. Let remember that a nonempty subset S of M is said to be self similar if $\mathcal{F}(S) = S$. If $\mathcal{F} = \{f_1, \cdots, f_n\}$ is a finite family of self-maps, then $(M, \{f_1, ...f_n\})$ is called an Iterated Function System (IFS). We say that an (IFS) is contraction, ω-condensing, Meir-Keeler condensing,$\cdots$ etc if each f_i is so.
For a Meir-Keeler condensing (IFS) in weak topology circumstances, we show the following results.

Theorem 5.1. *Let M be a non-empty weakly closed subset of a Banach space E and μ the De Blasi measure of weak noncompactness on E.*
Let $(M, \{f_1, \cdots, f_n\})$ be a Meir-Keeler condensing (IFS) such that $f_1 \cup \cdots \cup f_n$ has weakly sequentially closed graph and $\cup_{1 \le i \le n} f_i(M)$ is bounded.
Then the (IFS) $(M, \{f_1, \cdots, f_n\})$ has a weakly compact self-similar set.

Proof
Let

$$\begin{aligned} T : & M \to \mathcal{P}(M) \\ & x \mapsto T(x) \end{aligned}$$

where $T(x) = \cup_{i\in\{1,\cdots,n\}} f_i(x)$.
It is clear that T has weakly sequentially closed graph and $T(M)$ is a bounded set of M.
Since each f_i is Meir-Keeler condensing, then for all $\varepsilon_i > 0$ there exists δ_i such that

$$\varepsilon_i \leq \mu(A) < \varepsilon_i + \delta_i \Rightarrow \mu(f_i(A)) < \varepsilon_i$$

for all bounded subset A of M.
Let now A be a bounded set of M and let $\varepsilon > 0$ with $\mu(A) = \varepsilon$ and let δ be chosen as in (1), then

$$\mu(T(A)) = \mu(f_1(A) \cup ... \cup f_n(A)) = max(\mu(f_1(A)), \cdots, \mu(f_n(A))) < \varepsilon.$$

According to Corollary 3.2, we see that there exists a weakly compact self-similar set with respect to $(M, \{f_1, ..f_n\})$.
For a perturbed (IFS) $(M, \{f_1, ..., f_n, f_{n+1}\})$ we obtain the following result which is a direct consequence of Theorem 3.5.

Theorem 5.2. *Let M be a non-empty weakly closed subset of a Banach space E and μ the De Blasi measure of weak noncompactness on E. Let $(M, \{f_1, ..., f_n, f_{n+1}\})$ be a Meir-Keeler condensing (IFS) such that:*

1. *$f_1 \cup ... \cup f_n$ has weakly sequentially closed graph and $f_1(M) \cup ... \cup f_n(M)$ is relatively weakly compact,*
2. *f_{n+1} is weakly sequentially continuous and Meir-Keeler condensing,*
3. *$\cup_{i\in\{1,\cdots,n\}} f_i(M) + f_{n+1}(M)$ is a bounded set of M.*

Then, there exists a weakly compact subset K of M such that

$$\{x - f_{n+1}(x), x \in K\} = \cup_{i\in\{1,...,n\}} f_i(K).$$

Theorem 5.3. *Let M be a non-empty weakly closed subset of a Banach space E and μ the De Blasi measure of weak noncompactness on E. Let $(M, \{Af_1, ..., Af_n, f_{n+1}\})$ be an (IFS) such that:*

1. *$A, f_1, \cdots, f_{n+1}$ are weakly sequentially continuous and countably D-set Lipchitzian,*
2. *For $\varepsilon > 0$, there exists a $\delta > 0$ such that*

   $$\|A(M)\|\phi_i(r) + \|f_i(M)\|\phi_A(r) + \phi_A(r)\phi_i(r) + \phi_{n+1}(r) < \varepsilon,$$

   for all $r \in [\varepsilon, \varepsilon + \delta[$, where ϕ_A and ϕ_i are the D-functions of A and f_i, respectively, and for all $i \in \{1, \cdots, n\}$.
3. *$\cup_{i\in\{1,\cdots,n\}} A(M) f_i(M) + f_{n+1}(M)$ is a bounded set of M.*

Then, there exists a weakly compact subset K of M such that

$$\{x - f_{n+1}(x), x \in K\} = A(K) \cup_{i\in\{1,...,n\}} f_i(K).$$

Proof
We consider the map $B : M \to \mathcal{P}(M), x \mapsto \cup_{i\in\{1,\cdots,n\}} f_i(x)$. By the weak sequential continuity of each f_i, we prove that B is weakly sequentially upper semicompact. Let S be a countable subset of M, by the maximum property of μ, we obtain

$$\begin{aligned} \mu(B(S)) &\leq \max\{\mu(f_1(S)), \cdots, \mu(f_n(S))\} \\ &\leq \max\{\phi_1(\mu(S)), \cdots, \phi_n(\mu(S))\}. \end{aligned}$$

Then B is countably D-set Lipchitzian. Now the result follows by Corollary 4.2.

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