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A sinc-collocation approximation solution for strongly nonlinear class of weakly singular two-point boundary value problems

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Abstract. In this study, an efficient collocation method based on Sinc function coupled with double exponential transformation is developed. This approach is used for solving a class of strongly nonlinear regular or weekly singular two-point BVPs with homogeneous or non homogeneous boundary conditions. The properties of the Sinc-collocation scheme were used to reduce the computations of the problem to the nonlinear system of equations. To use the Newton method in solving the nonlinear system, its vector-matrix form was obtained. The convergence analysis of the method is discussed. The analysis show that the method is convergent exponential. In order to investigate the capability and accuracy of the method, it is applied to solve several existing problems chosen from the open literature. The numerical results compared with other existing methods. The obtained results indicate high capacity and rapid convergence of the proposed method.

1. Introduction

This paper considers the following class of strongly nonlinear two-point boundary value problems:

$$\begin{cases} u''(x) + r(x)R\Big(u(x)\Big)u'(x) = S\Big(x, u(x)\Big), \quad a \le x \le b, \\ \\ u(a) = \alpha, \quad u(b) = \beta, \end{cases} \tag{1}$$

where $R(u)$ is a function of u, and $S(u, x)$ is an analytic function of u and x. r is a continuous function of x in the neighborhood (a, b) that maybe singular at a or b or both. Also α, and β are given constants. Agarval et al. [1] have discussed the existence and uniqueness of the problem in detail.

Since there is no certain analytical method to obtain an exact close form solution for this class of problems, we have to apply numerical approaches. Many researchers have considered special models for problem (1). See et al. [17] consider the special case of problem (1) as $R(u) = u$. They applied numerical method based on Adams Multon type. In [8], Ha used a new nonlinear shooting method for solving two-point BVPs. Wang et al. [25] study another case of problem (1) as $R(u) = 1$ and $S(u, x) = g(x) + N(u)$. They introduce three new algorithms based on reproducing kernel space for solving their problem. They could find the

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exact solution in the form of series representation. Chun et al. [3] study the homotopy perturbation method (HPM) to solve several linear and nonlinear BVPs. They compare HPM with Adomian decomposition method and shooting method in numerical point of view. El-Gamel et al. [5] developed Sinc-Galerkin method coupled with single exponential transformation to solve problem (1) where $R(u) = u$. Ghorbani et al. [7] proposed different version of the variational iteration method (VIM) to solve BVPs. They also obtained the error bound of their method. Roul et al. [14] suggested a new iterative technique as discrete organized homotopy analysis method based on domain decomposition approach for solving an special case of problem (1) with form of $S(u, x) = \frac{u(x)}{1-\alpha u(x)}$ which arises from fluid in an iron drog configuration in circular cylindrical conduit. Rashidinia et al. [15] considered the Sinc-Galerkin method coupled with double exponential transformation for solving this problem. The authors serious attention to these issue led us to use the another simple, fast and powerful Sinc method as Sinc collocation method to obtain numerical solution of the general case of problem (1).

The Sinc function was used as a based numerical solution of linear BVPs by Frank Stenger more than thirty years ago [18]. Subsequently the use of the Sinc function coupled with collocation and Galerkin process for solving various problems was developed. Many authors have researched the properties and uses of this function in various fields of mathematics. They were able to prove that the convergence rate of the method is exponentially form [18], [19], [16].

The original domain for Sinc function is real numbers, therefore for applying in different domains, it is necessary to use appropriate transformation. This transformation is very effective in convergence rate of method for more information see [19].

While studying the integral solution of analytic functions, Mori and Takahasi achieved a suitable transformation that they called double exponential (DE) [24]. This transformation is very powerful to handle end points singularities. In the following, using this transformation in Sinc function, the researchers were able to significantly increase the convergence of the method [15], [20], [21]. In addition to the high order of convergence, the capability to solve singular problems is another strength of the Sinc method. Several numerical methods that, have been designed to solve regular problems, get into trouble in accuracy and they may even divergence when they are applied to singular problems. Whereas the Sinc method has overcome this weakness. When this method is applied to a problem, singularities at the boundary of the problem are easy to handle. Because the method depends only on problems parameters, regardless of the problem being regular or singular. When the Sinc-collocation method is applied to solve a nonlinear differential equation, the properties of the method discretize the nonlinear differential equation into a corresponding strongly nonlinear system of algebraic equations. The obtained nonlinear system can, then, be summarized into a matrix-vector system, which makes it easier to program. These capabilities have made many researchers use this method to solve their problems in recent years. Considering the abilities of the Sinc collocation method, this method has been used recently in solving many linear and nonlinear problems. Solving the fourth-order partial integro-differential equation with the multi-term kernels has been consider in [11]. Sinc−Muntz−Legendre collocation method for solving a class of nonlinear fractional partial differential equations has been studied in [2]. The collocation method using Sinc functions and Chebyshev wavelet method has been implemented to solve linear systems of Volterra integro-differential equations in [12]. A Sinc-Galerkin method has been considered and analyzed for solving the fourth order partial integro-differential equation with a weakly singular kernel in [9]. The exponential convergence of the Sinc-collocation method based on the double exponential (DE) transformation when applied to eighth-order boundary value problems (BVPs) has been proved in [13]. Sinc-Chebyshev collocation method for time-fractional order telegraph equation has been considered in [23]. In this paper The approximate solution of the given problem is expressed as elements of the shifted Chebyshev polynomials of the first kind in time and the Sinc function in space with unknown coefficients. The nonlinear fourth-order time-fractional equation has been studied by the combination of the Sinc-Galerkin method and the double exponential (DE) transformation in [6]. The numerical method based on Sinc function has been applied for the solution of Rosenau−KdV equation in [4]. The considered equation is fully-discretized by using the Sinc collocation method for spatial discretization and the forward finite difference for time discretization. The present study was aimed at developing the Sinc-collocation method by using double exponential transformation to solve problem (1) and compare its results with those other methods.

The paper is organized as follows. Section 2 overview of some properties of the Sinc basis function and notations. Section 3 discusses how the Sinc-collocation method can be applied to discretize problem (1) by an system of a nonlinear algebraic equations. Also solving the resulting nonlinear system by the Newton method is discussed by details. Section 4 deals with the convergence analysis of the our method. In Section 5, the presented method is tested by several examples selected from the open literature, which demonstrates high accuracy of the method. Finally, the study is concluded in Section 6.

2. Preliminary

This section recalls some definitions, properties and notations from [18] and [21].
The Sinc function is defined on $-\infty < x < \infty$ by

$$Sinc(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

For $h > 0$, the translated Sinc function with evenly spaced nodes is given by

$$S(k,h)(x) \equiv Sinc\Big(\frac{x-kh}{h}\Big) \equiv \begin{cases} \frac{\sin((\pi/h)(x-kh))}{(\pi/h)(x-kh)}, & x \neq kh \\ 1, & x = kh, \end{cases} \tag{2}$$

where $k = 0, \pm 1, \pm 2, \ldots$. $S(k,h)(x)$ is the kth Sinc function with step size h evaluated at x.
For $f(x)$, defined over the line, and $h > 0$ the series

$$C(f,h)(x) = \sum_{k=-\infty}^{\infty} f(kh) Sinc\Big(\frac{x-kh}{h}\Big), \tag{3}$$

is called the Whittaker cardinal expansion of f whenever this series converges.
Definition 2.1 Let D_d denotes the infinite strip region with $2d(d > 0)$ in the complex plane:

$$D_d \equiv \Big\{z \epsilon \mathbb{C} \mid |Imz| < d\Big\}. \tag{4}$$

For $0 < \varepsilon < 1$, let $D_d(\varepsilon)$ be defined by

$$D_d(\varepsilon) \equiv \Big\{z\epsilon\mathbb{C} \mid |Rez| < 1/(\varepsilon), |Imz| < d(1-\varepsilon)\Big\}, \tag{5}$$

let $H^1(D_d)$ be the Hardy space over the region D_d, i.e., the set of functions f analytic in D_d such that

$$\lim_{\varepsilon \to 0} \int_{\partial D_d(\varepsilon)} |f(z)||dz| < \infty. \tag{6}$$

Theorem 2.2 [21] Assume, with positive constants α, β, γ and d, that
1) f belongs to $H^1(D_d)$,
2) f decays double exponentially on the real line, that is,

$$|f(x)| \leq \alpha \exp(-\beta \exp(\gamma |x|)), \quad for\ all\ x \epsilon \mathbb{R}, \tag{7}$$

then we have

$$\sup_{-\infty < x < \infty} \Big| f(x) - \sum_{j=-N}^{N} f(jh) S(j,h)(x) \Big| \leqslant C \exp\Big[\frac{-\pi d \gamma N}{log(\pi d \gamma N/\beta)}\Big], \tag{8}$$

for some C, where the mesh size h is taken as:

$$h = \frac{log(\pi d\gamma N/\beta)}{\gamma N}.$$

The Theorem 2.2 presents the exponentially convergence rate of Sinc approximation on $\mathbb{R}$, proved by Sugihara [21].

Note that the domain of problem (1) is finite interval (a, b) and the original domain of Sinc function is the whole real line. To apply this approximation in solving problem (1), it is necessary to have the same domain for both. For this purpose, the problem domain is changed to $\mathbb{R}$. This transformation can be made with several changes in variables, among which the following transformation, known as double exponential (DE) transformation, is selected

$$x = \psi(t) = \frac{b-a}{2}\tanh\Big(\frac{\pi}{2}\sinh(t) + (\frac{b+a}{2})\Big), \tag{9}$$

$$t = \phi(x) = \psi^{-1}(t) = \log\Big(\frac{1}{\pi}\log(\frac{x-a}{b-x}) + \sqrt{1 + \{\frac{1}{\pi}\log(\frac{x-a}{b-x})\}^2}\Big). \tag{10}$$

By combining the transformation with the Sinc approximation, the convergence rate of the generated method will remain exponential as in Theorem 2.3.

Theorem 2.3 [21] Assume that, for a variable transformation $x = \psi(t)$, the transformed function $f(\psi(t))$ satisfies assumptions 1 and 2 in Theorem 2.2 with some positive constants α, β, γ and d. Then,

$$\sup_{a<x<b}\Big| f(x) - \sum_{j=-N}^{N} f(\psi(jh))S(j,h)(\psi^{-1}(x))\Big| \leqslant C\exp\Big(\frac{-\pi d\gamma N}{log(\pi d\gamma N/\beta)}\Big), \tag{11}$$

for some C, where the mesh size h is taken as:

$$h = \frac{log(\pi d\gamma N/\beta)}{\gamma N}. \tag{12}$$

The collocation method requires the values of derivations of the Sinc function evaluated at the node point. To simplify the calculations, these values are summarized in following lemma.

Lemma 2.4 [19] Let $S(j, h)(x)$ be the k^{th} Sinc function with step size h, so

$$\delta^{(0)}_{jk} = S(j,h)(kh) = \begin{cases} 1, & j = k \\ 0, & j \neq k, \end{cases}$$

$$\delta^{(1)}_{jk} = h\frac{d}{dz}(S(j,h)(z))(kh) = \begin{cases} 0, & j = k \\ \frac{(-1)^{k-j}}{k-j}, & j \neq k, \end{cases}$$

$$\delta^{(2)}_{jk} = h^2\frac{d^2}{dz^2}(S(j,h)(z))(kh) = \begin{cases} \frac{-\pi^2}{3}, & j = k \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & j \neq k. \end{cases}$$

By considering $I^{(l)} = [\delta^{(l)}_{jk}]$, $l = 0, 1, 2$ which are known as Toeplitz matrix, the derivative at the node can be encapsulated to combine these matrices.

The notation of "diag", which will be used in the next section, is defined as follows

$$diag\Big(a_1, ..., a_n\Big) = \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & & a_n \end{pmatrix}$$

3. Methodology

In this study, the aim is to develop the DE Sinc collocation method (DE-SCM) to a class of strongly nonlinear two-point BVPs as follow:

$$\begin{cases} u''(x) + r(x)R\Big(u(x)\Big)u'(x) = S\Big(u(x), x\Big) \quad a < x < b, \\ u(a) = \alpha, \qquad u(b) = \beta. \end{cases} \tag{13}$$

As discussed in Selection 2, the DE transformation is used to unify the domain of the problem and the domain of the Sinc approximation. By applying the DE transformation, introduced in Eq.(9), problem (1) is transformed into a new one on real line as

$$\begin{cases} \frac{d^2}{dx^2}u(\psi(t)) + r(\psi(t))R\Big(u(\psi(t))\Big)\frac{d}{dx}u(\psi(t)) = S\Big(u(\psi(t)), \psi(t)\Big), \quad -\infty < t < \infty, \\ \lim_{t\to-\infty} u(\psi(t)) = \alpha, \qquad \lim_{t\to\infty} u(\psi(t)) = \beta. \end{cases} \tag{14}$$

Consider $v(t) = u(\psi(t))$. By applying the chain role of differential, we will have

$$\begin{cases} v''(t) - \frac{\psi''(t)}{\psi'(t)}v'(t) + r(\psi(t))\psi'(t)R\Big(v(t)\Big)v'(t) = (\psi')^2S\Big(v(t), \psi(t)\Big), \quad -\infty < t < \infty, \\ \lim_{t\to-\infty} v(t) = \alpha, \qquad \lim_{t\to\infty} v(t) = \beta. \end{cases} \tag{15}$$

Since basic Sinc functions, introduced in Eq.(2), are vanished at the end of the intervals, so some authors have modified the cardinal expansion to satisfy the boundary conditions. Instead of this modification, the non-homogeneous boundary conditions are converted into homogeneous conditions in this paper. Therefore, the following function is defined

$$\Lambda(t) = c\psi(t) + k, \tag{16}$$

where $c = \frac{\beta-\alpha}{b-a}$, and $k = \frac{b\alpha-a\beta}{b-a}$.
Now, use the transformation of

$$v(t) = w(t) + \Lambda(t). \tag{17}$$

By applied (17), problem (15) is converted into :

$$\begin{cases} w''(t) - \frac{\psi''(t)}{\psi'(t)}w'(t) + r(\psi(t))\psi'(t)R\Big(w(t) + \Lambda(t)\Big)w'(t) \\ \qquad\qquad = (\psi')^2G\Big(w(t) + \Lambda(t), \psi(t)\Big), \quad -\infty < t < +\infty, \\ \lim_{t\to-\infty} w(t) = 0, \qquad \lim_{t\to\infty} w(t) = 0, \end{cases} \tag{18}$$

where

$$G\Big(w(t)+\Lambda(t),\psi(t)\Big)=S\Big(w(t)+\Lambda(t),\psi(t)\Big)-\Lambda'(t)r\Big(\psi(t)\Big)\psi'(t)R\Big(w(t)+\Lambda(t)\Big). \tag{19}$$

Suppose the following Sinc approximation

$$w_m(t)=\sum_{k=-N}^{N}c_jS_j(t), \qquad m=2N+1, \tag{20}$$

where $S_j(t)=S(j,h)(t)$ and unknown coefficients $\{c_j\}_{j=-N}^{N}$ should be determined by the collocation method.

With due attention to the definition of $S_j(t)$, this solution is satisfied in the boundary conditions. To determine the coefficients $\{c_j\}_{j=-N}^{N}$ in (20), v_m is substituted for (19) and then, 't' by collocation points $t=t_i=ih$, $i=-N,-N+1,...,N$ is replaced in which h is defined in (12), so the following nonlinear system of algebraic equations is obtained

$$\begin{aligned}\sum_{j=-N}^{N}c_j\Big[\frac{d^2}{dt^2}S_j(t_i)+\Big(\psi'(t_i)r(\psi(t_i))R\Big(c_i+\Lambda(t_i)\Big)-\frac{\psi''(t_i)}{\psi'(t_i)}\Big)\frac{d}{dt}S_j(t_i)\Big]\\ =(\psi'(t_i))^2G\Big(c_i+\Lambda(t_i),\psi(t_i)\Big), \qquad i=-N,\cdots,N,\end{aligned} \tag{21}$$

where G is considered in (19). By using Lemma 2.4 and considering that $\delta_{jk}^{(0)}=\delta_{kj}^{(0)}$, $\delta_{jk}^{(1)}=-\delta_{kj}^{(1)}$ and $\delta_{jk}^{(2)}=\delta_{kj}^{(2)}$, the nonlinear system (21) may be rewritten as below

$$\begin{aligned}\sum_{j=-N}^{N}c_j\Big[\frac{1}{h^2}\delta_{ij}^{(2)}+\frac{1}{h}\Big(\frac{\psi''(t_i)}{\psi'(t_i)}-R\Big(c_i+\Lambda(t_i)\Big)\psi'(t_i)r(\psi(t_i))\Big)\delta_{ij}^{(1)}\Big]\\ =(\psi'(t_i))^2G\Big(c_i+\Lambda(t_i),\psi(t_i)\Big), \qquad i=-N,\cdots,N.\end{aligned} \tag{22}$$

This nonlinear system can be solved by any suitable methods. The method chosen here is the Newton's method. To apply this method to solve the system of (22) we need to convert the system to $F(C)=0$ form. Also the matrix- vector form of the Jacobian matrix is required in each iteration. With a bit of sophisticated calculations and the use of introduced symbols, the matrix forms described are as follows:

$$F\Big(C\Big)=\begin{pmatrix}f_{-N}(c_{-N},...,c_N)\\ f_{-N+1}(c_{-N},...,c_N)\\ \vdots\\ f_N(c_{-N},...,c_N)\end{pmatrix}=\begin{pmatrix}0\\0\\ \vdots\\0\end{pmatrix}.$$

So the nonlinear system (22) can be rewritten as

$$\begin{aligned}f_k(c_{-N},...,c_N)=\sum_{j=-N}^{N}a_{kj}c_j+\sum_{j=-N}^{N}b_{kj}R\Big(c_k+\Lambda(t_i)\Big)c_j\\ -(\psi'(t_i))^2G\Big(c_i+\Lambda(t_i),\psi(t_i)\Big), \qquad k=-N,\cdots,N,\end{aligned} \tag{23}$$

where

$$a_{ij} = \frac{1}{h^2}\delta_{ij}^{(2)} + \frac{1}{h}\Big(\frac{\psi''(t_i)}{\psi'(t_i)}\Big)\delta_{ij}^{(1)}, \tag{24}$$

$$b_{ij} = -\frac{1}{h}\psi'(t_i)r(\psi(t_i))\delta_{ij}^{(1)}, \tag{25}$$

by using the notation of $I^{(l)}$ and "diag" in Lema 2.4, system (23) is summarized as

$$F(C) = A\ C\ + \Big(diag(R(c+\Lambda))B\Big)C - diag\Big((\psi'(t_i))^2\Big)H, \tag{26}$$

where

$$A = \frac{1}{h^2}I^{(2)} + \frac{1}{h}diag\Big(\frac{\psi''}{\psi'}\Big)I^{(1)}, \tag{27}$$

$$B = -\frac{1}{h}diag\Big(\psi' r(\psi)\Big)I^{(1)}, \tag{28}$$

$$R\Big(C+\Lambda\Big) = \begin{pmatrix} R\Big(c_{-N} + \Lambda(\psi(t_{-N}))\Big) \\ R\Big(c_{-N+1} + \Lambda(\psi(t_{-N+1}))\Big) \\ \vdots \\ R\Big(c_{N} + \Lambda(\psi(t_{N}))\Big) \end{pmatrix}, \tag{29}$$

$$H = \begin{pmatrix} G\Big(\psi(t_{-N}), c_{-N} + \Lambda(\psi(t_{-N}))\Big) \\ G\Big(\psi(t_{-N+1}), c_{-N+1} + \Lambda(\psi(t_{-N+1}))\Big) \\ \vdots \\ G\Big(\psi(t_{N}), c_{N} + \Lambda(\psi(t_{N}))\Big) \end{pmatrix}, \tag{30}$$

(31)

$$C = \Big(c_{-M}, c_{-M+1}, .., c_M\Big)^T. \tag{32}$$

Also Jacobian is calculated as follow:

$$J = A + B\ diag\Big(R(C+\Lambda)\Big) + diag\Big(B\ diag(\Upsilon)\ C\Big) - diag\Big((\psi'(t_i))^2\Big)diag(\Sigma), \tag{33}$$

where

$$\Upsilon = \frac{\partial}{\partial C} R\Big(C + \Lambda\Big) = \begin{pmatrix} \frac{\partial}{\partial c_{-N}} R\Big(c_{-N} + \Lambda(\psi(t_{-N}))\Big) \\ \frac{\partial}{\partial c_{-N+1}} R\Big(c_{-N+1} + \Lambda(\psi(t_{-N+1}))\Big) \\ \vdots \\ \frac{\partial}{\partial c_{N}} g\Big(c_{N} + \Lambda(\psi(t_{N}))\Big) \end{pmatrix}, \tag{34}$$

$$\Sigma = \frac{\partial}{\partial C} G\Big(\psi, C + \Lambda(\psi)\Big) = \begin{pmatrix} \frac{\partial}{\partial c_{-N}} G\Big(\psi(t_{-N}), c_{-N} + \Lambda(\psi(t_{-N}))\Big) \\ \frac{\partial}{\partial c_{-N+1}} G\Big(\psi(t_{-N+1}), c_{-N+1} + \Lambda(\psi(t_{-N+1}))\Big) \\ \vdots \\ \frac{\partial}{\partial c_{N}} G\Big(\psi(t_{N}), c_{N} + \Lambda(\psi(t_{N}))\Big) \end{pmatrix}, \tag{35}$$

now, by considering initial guess C_0, the Newton iteration is calculated as follows:

$$C_{j+1} = C_j - J^{-1}(C_j)\, F(C_j). \tag{36}$$

Here, C_j is the current iterate and C_{j+1} is the new iterate. A common numerical practices is to stop the Newton iteration wherever the distance between two successive iterates is less than a given tolerance , i.e., when $\|C_{j+1} - C_j\| \le \varepsilon$ or $\|F(C_j)\| \le \varepsilon$ where the Euclidean norm is used.

4. Convergence Analysis

This section deals with, the convergence analysis of the Sinc method for (18).

Lemma 4.1 Suppose w be the exact solution of nonlinear equation $L(w)+K(w) = 0$ in which $(Lw)(x) = w''(x) - \Big(\frac{\psi''(x)}{\psi'(x)}\Big)w'(x)$ is a linear operator and $(Kw)(x) = \Big(\psi(x)\psi'(x)R(w(x) + \Lambda(x))\Big)w'(x) - (\psi'(x))^2 G\Big(w(x) + \Lambda(x), \psi(t)\Big)$ is a nonlinear operator. Let $w \in H^1(D)$, $K'(x)$ and $K''(x)$ are well defined and bounded on the ball $B(w^0, d)$. Furthermore, let $(L+K'(w))^{-1}$ and $(L+K'(w^0))^{-1}(Lw^0+K(w^0))$ be bounded on $B(w^0, d)$, and $\|(Lw^0+K(w^0))\|_\infty \le M_0$, $\|(Lw^0 + K'(w^0))^{-1}\|_\infty \le M_1$, $\|K''(w^0)\|_\infty \le M_2$, $w \in B(w^0, d)$, if $\widetilde{h} = M_1 M_1^2 M_2 < 2$, and $d > M_1 M_2 \sum_{k=0}^{\infty} \Big(\frac{\widetilde{h}}{2}\Big)^{2k-1}$, then the sequence

$$w^{n+1} = w^n - (L + K'(w^n))^{-1}(Lw^n + K(w^n)), \tag{37}$$

is well defined, also $w^{n+1} \in H^1(D)$ for every positive integer n, and the sequence w^n converges to w^*, also

$$\|w^n - w^*\|_\infty \le M_1 M_0 \frac{(\widetilde{h}/2)^{2^n - 1}}{1 - (\widetilde{h}/2)^{2^n}}. \tag{38}$$

Proof
By using the Kantorovitch's theorem [10], the existence of the sequence $\{w^n\}_{n\geq 0}$ and the upper bound (38) could be proven. □
Theorem 4.2 Let us consider all conditions of Lemma 4.1, and let discrete equivalent of $K'(w), K''(w)$ and $(L+K'(w))^{-1}$ be well defined and bounded on the ball $\overline{B}(w^0, d)$, and the sequence y_m^n be the discrete equivalent of (37), then
a) sequence $\{y_m^n\}_{n\geq 0}$ converges to y^*, and $||y_m^n - y^*||_\infty$ has a upper bound as introduced in (38),
b) there exists a constant C_1 independent of N such that

$$||y_m^* - w^*||_\infty \leq C_1 N^2 \log(N) \exp\Big(-\frac{\pi d_1 \gamma N}{\log(\pi d_1 \gamma N/\beta)}\Big), \tag{39}$$

where C_1, d_1, γ, β are constant independent of N.
Proof
a) Let $\{y_m^n\}_{m\geq 0}$ be the discrete sequence by the Sinc-collocation method that defined by the discrete of (37), similarly by using Lemma 4.1 $\{y_m^n\}_{n\geq 0}$ exist and convergence to y_m^* and we have,

$$||y_m^n - y_m^*||_\infty \leq M_1 M_0 \frac{(\widetilde{h}/2)^{2^n-1}}{1-(\widetilde{h}/2)^{2^n}}, \tag{40}$$

where M_1, M_0, and $\widetilde{h}$ are defined in Lemma 4.1.
b) Let the sequence w^n be defined by using Lemma 4.1. We know that this sequence exists and converges to w^*; and,

$$||w^n - w^*||_\infty \leq M_1 M_0 \frac{(\widetilde{h}/2)^{2^n-1}}{1-(\widetilde{h}/2)^{2^n}}, \tag{41}$$

by considering bound $(L + K'(w))^{-1}$ on the ball $\overline{B}(w^0, d)$ and Theorem 3.2 [22], we have:

$$||y_m^n - w^n||_\infty \leq C_2 N^2 \log(N) \exp\Big(-\frac{\pi d_1 \gamma N}{\log(\pi d_1 \gamma N/\beta)}\Big), \tag{42}$$

where C_2, d_1, γ, β are constant independent of N.
Now consider the following inequality

$$||y_m^* - w^*||_\infty \leq ||y_m^* - y_m^n||_\infty + ||y_m^n - w^n||_\infty + ||w^n - w^*||_\infty. \tag{43}$$

By considering assumption in Lemma 4.1 and $\widetilde{h} < 2$, the following inequality can be made for n large enough

$$M_1 M_0 \frac{(\widetilde{h}/2)^{2^n-1}}{1-(\widetilde{h}/2)^{2^n}} \leq C_3 N^2 \log(N) \exp\Big(-\frac{\pi d_1 \gamma N}{\log(\pi d_1 \gamma N/\beta)}\Big), \tag{44}$$

By applying relations (40) - (44) we obtain

$$||y_m^* - w^*||_\infty \leq C_1 N^2 \log(N) \exp\Big(-\frac{\pi d_1 \gamma N}{\log(\pi d_1 \gamma N/\beta)}\Big), \tag{45}$$

□.

5. Numerical Results

To demonstrate the capability and efficiency of the our method, four different test examples are considered from open literature [3, 5, 7, 8, 15, 17, 25]. For all problems DE Sinc-collocation method (DE-SC) with the following step length is applied:

$$h = \frac{\log(\pi N/4)}{N}.$$

To compare the exact solution of the considered examples with the numerical solution obtained by Sinc-collocation method, the maximum absolute error (MAE) in 999 equally step points Ω calculated by

$$MAE = \max_{1\leq i\leq 999} |U_{exact\ solution}(x_i) - U_{Sinc-collocation}(x_i)|, \tag{46}$$

where

$$\Omega = \{x_1, x_2, ..., x_{999}\}, \tag{47}$$

$$x_i = a + \frac{b-a}{1000} i, \qquad i = 1, 2, 3, ..., 999. \tag{48}$$

Furthermore the error at special points, corresponding to the reports are determined, too. For solving arising nonlinear system of equations, Newton's method with stopping condition $\|C_{k+1} - C_k\| < 10^{-12}$ are applied. In all problem an initial of zero vector is used. All programs are coded by MATLAB software. The time is recorded on an intel CORE i5 PC at 2.50 GHz.

5.1. Example 1

In first example the homogenous case is considered as follow [25]

$$\begin{cases} u''(x) + \frac{1}{\sqrt{x}} u'(x) = f(x, u), & 0 < x < 1, \\ u(0) = 0, \quad u(1) = 0, \end{cases} \tag{49}$$

where the exact solution is given by $u(x) = (x - x^2)\sin(x)$, and

$$f(x, u) = -\frac{1}{x} u(x) + \sin(y^2(x)) + \exp(u^9(x)) - u^{11}(x) + \frac{1}{\sqrt{x}}\Big[(2\sqrt{x} + x - 4x\sqrt{x} - x^2)\cos(x) + (1 - \sqrt{x} - 2x - 2x\sqrt{x} + x^2\sqrt{x})\sin(x)\Big] - \sin\Big(((x - x^2)\sin(x))^2\Big) - \exp\Big(((x - x^2)\sin(x))^9\Big) + \Big((x - x^2)\sin(x)\Big)^{11}.$$

As you see this x=0 is a singularity. This problem has already been investigated by authors by using reproducing kernel method (RKM) [25] and DE-Sinc Galerkin method (DE-SGM) [15]. We applied the DE-Sinc collocation method (DE-SCM) to solve this problem. The results of comparing the approximation solution obtained from different methods with the exact solution are listed in Table 1. As can be seen, the our method is much more accurate than the method in [25] and almost as accurate as the DE-SGM. Also, the maximum absolute error (MAE) in the solution in set points Ω and runtime (per second) of methods are presented in Table 2 for different values of Ns for our method and the method in [15]. In Fig 1, the approximation solution of our method for several N has been shown. In this figure the exact solution is shown with red 'o' and approximation solution for N=1, N=2, N=3, N=4, are shown with blue triangular, '$*$', '$+$' and '$.$' respectively. As it is known, by increasing the value of N, the approximation solution converges towards the exact solution.

Table 1: Computing the absolute error in the solutions for example 1

x	RKM [25] $N = 120$	DE-SGM[15] $N = 20$	DE-SGM[15] $N = 30$	DE-SCM $N = 20$	DE-SCM $N = 30$
0.08	$2.51E-7$	$5.21E-11$	$1.51E-14$	$5.12E-11$	$4.35E-14$
0.16	$2.50E-7$	$1.29E-10$	$1.24E-14$	$1.52E-11$	$4.31E-14$
0.32	$2.25E-7$	$1.27E-10$	$1.37E-14$	$3.49E-12$	$4.32E-14$
0.48	$1.92E-7$	$3.14E-11$	$1.41E-14$	$1.21E-10$	$3.67E-14$
0.64	$1.57E-7$	$9.87E-11$	$1.65E-14$	$1.26E-10$	$3.16E-14$
0.80	$1.25E-7$	$4.11E-11$	$8.75E-15$	$5.74E-11$	$7.06E-15$
0.96	$9.61E-8$	$5.96E-11$	$1.41E-14$	$1.39E-11$	$2.18E-15$

Table 2: Comparing MAE and runtime for Example 1

N	DE-SGM[15]	DE-SGM RunTime	DE-SCM	DE-SCM RunTime
5	$4.40E-04$	0.6292s	$4.39E-04$	0.7898s
10	$1.03E-06$	0.6377s	$7.80E-07$	0.8489s
15	$9.63E-09$	0.6937s	$1.34E-08$	0.8655s
20	$1.40E-10$	0.7529s	$1.98E-10$	0.8707s
25	$2.11E-12$	0.7547s	$3.16E-12$	0.9174s
30	$3.43E-14$	0.8097s	$5.37E-14$	0.9461s

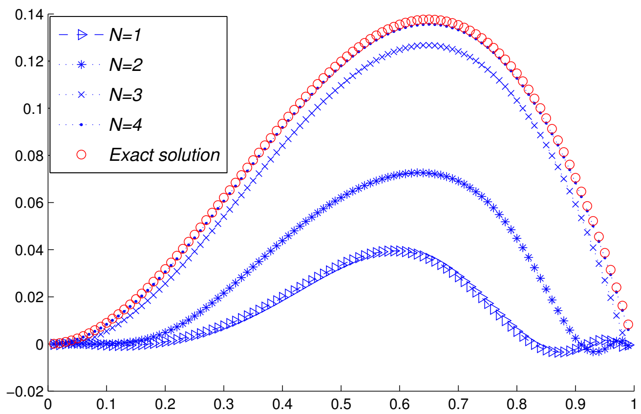

Figure 1: Graph of approximation solution for N=1, 2, 3, 4 and exact solution for Example 1.

Table 3: Computing the absolute error in the solutions for Example 2

	DE-SGM[15]			DE-SCM		
x	$N = 10$	$N = 20$	$N = 40$	$N = 10$	$N = 20$	$N = 40$
0.010	$6.28E-05$	$5.60E-09$	$7.19E-11$	$4.74E-06$	$1.01E-09$	$2.49E-11$
0.120	$2.77E-04$	$1.93E-07$	$2.46E-09$	$1.63E-04$	$3.48E-08$	$8.49E-10$
0.270	$3.92E-04$	$2.70E-07$	$3.43E-09$	$2.28E-04$	$4.86E-08$	$1.18E-09$
0.378	$3.82E-04$	$2.62E-07$	$3.33E-09$	$2.21E-04$	$4.72E-08$	$1.14E-09$
0.500	$3.32E-04$	$2.27E-07$	$2.89E-09$	$1.92E-04$	$4.09E-08$	$9.96E-10$
0.621	$2.62E-04$	$1.79E-07$	$2.28E-09$	$1.51E-04$	$3.22E-08$	$7.85E-10$
0.729	$1.91E-04$	$1.30E-07$	$1.66E-09$	$1.10E-04$	$2.35E-08$	$5.73E-10$
0.879	$8.69E-05$	$5.89E-08$	$7.52E-10$	$5.02E-05$	$1.06E-08$	$2.59E-10$
0.970	$2.18E-05$	$1.47E-08$	$1.87E-10$	$1.27E-07$	$2.64E-09$	$6.44E-11$
0.999	$1.11E-06$	$4.98E-10$	$6.23E-12$	$7.87E-07$	$8.81E-11$	$2.14E-12$

Table 4: Comparing MAE and runtime for Example 2

N	DE-SGM [15]	DE-SGM RunTime	DE-SCM	DE-SCM RunTime
5	$2.7456E-02$	0.6765s	$2.3586E-02$	0.2673s
10	$4.0838E-04$	.....	$2.2907E-04$	0.3140s
15	$1.2738E-05$	0.6779s	$5.4814E-06$	0.3189s
20	$2.7140E-07$	.....	$4.8837E-08$	0.3518s
30	$1.4731E-07$	.....	$1.7183E-10$	0.3898s

5.2. *Example 2*

Consider the homogeneous nonlinear singular BVP as :

$$\begin{cases} u''(x) + \frac{1}{x}u^3(x)u'(x) = f(x,u), \qquad 0 < x < 1, \\ u(0) = 0, \quad u(1) = 0, \end{cases} \tag{50}$$

where

$$f(x,u) = -\frac{1}{x^2}\sin(u(x)) - 2 + (1-x)^3x^2(1-2x) + \frac{1}{x^2}\sin(x - x^2),$$

and the exact solution is $u(x) = x - x^2$.

The proposed method is developed for solving this problem and the results are summarized in Table 3 and 4. Table 3 show the comparison between DE-SGM [15] and DE-SCM for different values of N. Also Table 4 show the MAE and runtime of the methods. These tables show that the DE-SGM and DE-SCM are almost equally accurate.

Table 5: Computing the absolute error in the solutions for Example 3

x	HPM[3]	NSHM[8]	TSDM[17]	VIM [7]	DE-SGM[15] N=30	DE-SCM N=30
1.10	$0.15E-10$	$0.5E-05$	$7.16E-09$	....	$6.15E-17$	$6.93E-17$
1.20	$0.30E-10$	$0.9E-05$	$1.17E-08$	$3.0E-06$	$1.21E-16$	$3.29E-17$
1.30	$0.45E-10$	$0.1E-05$	$1.19E-08$	....	$1.63E-16$	$5.20E-18$
1.40	$0.50E-10$	$0.1E-05$	$1.18E-08$	$3.0E-07$	$1.56E-16$	$3.12E-17$
1.50	$0.75E-10$	$0.1E-05$	$1.09E-08$	....	$1.87E-16$	$6.93E-18$
1.60	$0.96E-10$	$0.1E-05$	$9.33E-09$	$1.3E-06$	$1.90E-17$	$1.09E-16$
1.70	$0.13E-09$	$0.8E-05$	$7.37E-09$	......	$1.09E-16$	$1.04E-17$
1.80	$0.20E-09$	$0.6E-05$	$5.12E-09$	$3.0E-06$	$1.23E-16$	$6.93E-18$
1.90	$0.26E-09$	$0.3E-05$	$2.66E-09$	....	$1.25E-16$	$1.30E-17$

Table 6: Comparing MAE and runtime for Example 3

N	DE-SGM[15]	DE-SGM RunTime	DE-SCM	DE-SCM RunTime
5	$4.48E-05$	0.6503s	$4.28E-05$	0.7641s
10	$3.29E-08$	0.6540s	$3.67E-08$	0.8251s
15	$9.24E-11$	0.7103s	$7.03E-11$	0.8566s
20	$3.66E-13$	0.7267s	$5.10E-13$	0.8661s
25	$4.52E-15$	0.7992s	$6.14E-15$	0.8849s
30	$2.41E-16$	0.8072s	$2.02E-16$	0.9256s

5.3. Example 3

Consider the non-homogeneous BVP [3], [7], [8], [17] :

$$\begin{cases} u''(x) + u(x)u'(x) = u^3(x), & 1 < x < 2, \\ u(1) = \frac{1}{2}, \quad u(2) = \frac{1}{3}, \end{cases} \tag{51}$$

with the exact solution $u(x) = \frac{1}{1+x}$.

This problem has been considered by Homotopy perturbation method (HPM) [3], the nonlinear shooting method(NSHM) [8], the two-step direct method (TSDM) [17], the DE-SGM [15], and recently by different version of the variational iteration method (VIM) [7]. DE-SCM is applied form solving this problem. The numerical results is tabulated in Table 5 and 6. In Table 5 the absolute error, obtained by DE-SCM, at special points $x = 1.1, 1.2, ..., 1.9$ comparing with results of other methods reported by authors are presented. The runtime and MAE at points Ω for DE-SGM and DE-SCM for different values of N are shown in Table 6. The results of Tables reveal that DE-SCM is accurate and its error is decreases rapidly with increasing N.

Table 7: Computing the absolute error in the solutions for Example 4

	SE-SGM[5]	DE-SGM[15]		DE-SCM	
x	$N = 40$	$N = 20$	$N = 30$	$N = 20$	$N = 30$
0.077701	$6.0E-8$	$9.25E-11$	$3.09E-15$	$9.54E-11$	$4.21E-16$
0.12058	$2.0E-7$	$1.26E-10$	$5.87E-15$	$7.14E-11$	$1.08E-15$
0.27022	$1.5E-7$	$8.23E-11$	$1.09E-16$	$4.46E-11$	$1.41E-15$
0.37830	$2.0E-8$	$2.95E-11$	$3.47E-15$	$3.77E-11$	$2.56E-15$
0.5	$1.4E-7$	$7.68E-11$	$5.12E-15$	$3.36E-11$	$3.04E-15$
0.62169	$2.0E-8$	$9.17E-12$	$1.87E-16$	$3.13E-11$	$3.19E-15$
0.72977	$1.6E-7$	$8.47E-11$	$8.45E-15$	$2.97E-11$	$4.34E-15$
0.87941	$2.0E-7$	$1.21E-10$	$3.68E-15$	$3.01E-11$	$5.52E-15$
0.97012	$2.3E-7$	$2.58E-11$	$1.28E-14$	$4.47E-11$	$3.40E-15$

5.4. Example 4

In the last example the following nonlinear singular BVP is considered [5]

$$\begin{cases} u''(x) + u(x)u'(x) = f(x,u), \qquad 0 < x < 1, \\ u(0) = 0, \quad u(1) = 0, \end{cases} \tag{52}$$

where $f(x,u) = -u^3(x) + \frac{1}{x} + x\log(x)(1+\log(x)) + (x\log(x))^3$, with the exact solution $u(x) = x\log(x)$. This problem has been selected to compare the results of our method with Sinc Galerkin method that used single exponential transformation. In fact the purpose is to investigate the effect of selected the appropriate transformation in the Sinc method. This problem was considered by El-Gamel et al. [5] and solved by Sinc Galerkin method based on single transformation (SE-SGM). To compare the methods, the result of comparing the approximate solutions and the exact solution of the problem is reported in the points used by El-Gamel. Also the maximum relative error (MRE) defined by follow, are calculate and listed in Table 7.

$$MRE = \frac{|U_{exact\ solution} - U_{Sinc-method}|}{|U_{exact\ solution}|}.$$

Furthermore the MAE defined in (46) for the methods are summarized in Table 8. The reported results for SE-SGM obtained for $N = 40$, it means the arising nonlinear system has 81 algebraic equations, instead in DE-SCM and DE-SGM $N = 20, 30$ is selected, so the 41, 61 nonlinear algebraic equations are created, respectively. The results show that DE-SCM and DE-SGM are more accurate and faster than SE-SGM presented in [5].

Figure 2 demonstrated the approximation solution obtained by DE-SCM for different values of N. It also shows how the approximation solution converges to exact solution.

6. Conclusions

A collocation method was developed to solve a class of singular or regular strongly nonlinear two-point BVPs by an efficient basic Sinc functions coupled with double exponential transformation. By applying this method, the strongly nonlinear equation was discritized into strongly nonlinear system of algebraic equations. Newton's iteration method was successfully used to solve the created nonlinear

Table 8: Comparing MAE and runtime for Example 4

M	DE-SGM[15]	DE-SGM RunTime	DE-SCM	DE-SCM RunTime
5	$4.15E-03$	0.6031s	$4.03E-03$	0.7620s
10	$6.19E-06$	0.6551s	$5.87E-06$	0.8331s
15	$1.07E-08$	0.6696s	$1.06E-08$	0.8508s
20	$3.38E-11$	0.7712s	$2.03E-11$	0.8711s
25	$1.19E-13$	0.8028s	$4.02E-14$	0.9282s
30	$2.22E-15$	0.8089s	$1.33E-15$	0.9401s

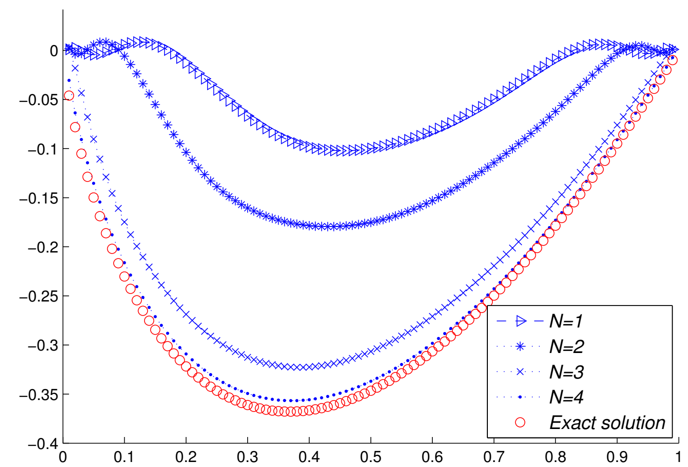

Figure 2: Graph of approximation solution with $N = 1, 2, 3, 4$ and exact solution for Example 4.

system of equations. For this purpose, the matrix-vector form of system and Jacobian was calculated. The convergence analysis was studied and it is clear that the method has an exponential convergence. In order to illustrate the capabilities of the method, it was used to solve several kinds of problems from the open literature. The obtained numerical results from the DE Sinc-collocation method and existing methods were compared. As is clear in the tables, the DE Sinc-collocation and the DE Sinc-Galerkin have almost same convergence rate and runtime. But discretization by the Sinc-collocation method is so simpler than that by the Sinc-Galerkin method. The results reveal the higher capability of the DE-Sinc methods versus the other existing methods in the open literature for solving such strongly nonlinear problems.

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