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2-normal composition operators with linear fractional symbols on H^2

Rong Yang[a], Songxiao Li[b,*], Lian Hu[a]

[a] *Institute of Fundamental and Frontier Sciences, University of Electronic Science and Technology of China, 610054, Chengdu, Sichuan, P.R. China*
[b] *Department of Mathematics, Shantou University, 515063, Shantou, Guangdong, P.R. China*

Abstract. In this paper, some sufficient and necessary conditions for the composition operator C_φ to be 2-normal are investigated when the symbol φ is a linear fractional self-map of $\mathbb{D}$.

1. Introduction

Let H be a complex Hilbert space, $B(H)$ be the space of all bounded linear operators defined in H. An operator $T \in B(H)$ is called normal if it satisfies the condition $[T, T^*] = 0$, where $[T, T^*] = TT^* - T^*T$. An operator $T \in B(H)$ is subnormal if there is a Hilbert space K containing H and a normal operator M on K such that $MH \subset H$ and $T = M|H$. An operator T is called quasinormal if $[T, T^*T] = 0$. An operator $T \in B(H)$ is called p-hyponormal if $(T^*T)^p \geq (TT^*)^p$, where $0 < p \leq 1$. If $p = 1$, T is said to be hyponormal. An operator $T \in B(H)$ is called binormal when $[T^*T, TT^*] = 0$. An operator T is said to belong to Θ class if $[T^*T, T + T^*] = 0$. From [1, 9], we see that

$$\text{quasinormal} \subset \text{binormal}$$

$$\text{normal} \subset \text{quasinormal} \subset \text{subnormal} \subset \text{hyponormal}.$$

The operator T is said to be n-normal if T^* commutes with T^n, that is $[T^*, T^n] = 0$. When $n = 2$, the operator T is called 2-normal, that is, $[T^*, T^2] = 0$. It is clear that a normal operator is a 2-normal operator, but the converse is not true.

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$. Let $H(\mathbb{D})$ be the space of those analytic functions on $\mathbb{D}$. The Hardy space $H^2(\mathbb{D})$ is the space of all $f \in H(\mathbb{D})$ for which

$$\|f\|^2_{H^2(\mathbb{D})} = \sum_{n=0}^{\infty} |a_n|^2 < \infty,$$

where $\{a_n\}$ is the sequence of Maclaurin coefficients for f. The space H^2 is a reproducing kernel Hilbert space. In other word, for any $w \in \mathbb{D}$ and $f \in H^2$, there exists a unique function $K_w \in H^2$ such that

$$f(w) = \langle f, K_w \rangle.$$

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* Corresponding author: Songxiao Li
Email addresses: yangrong071428@163.com (Rong Yang), jyulsx@163.com (Songxiao Li), hl152808@163.com (Lian Hu)

It is well known that $K_w(z) = \frac{1}{1-\overline{w}z}$.

Let φ be an analytic self-map of $\mathbb{D}$. The composition operator C_φ with symbol φ is defined by

$$C_\varphi f = f \circ \varphi.$$

It is easy to see that $C_\varphi^* K_\alpha(z) = K_{\varphi(\alpha)}(z)$ for any $\alpha \in \mathbb{D}$. For $f \in L^\infty(\partial\mathbb{D})$ and $g \in H^2$, the Toeplitz operator T_f on H^2 is defined by $T_f(g) = P(fg)$, where P denotes the orthogonal projection of L^2 onto H^2. It is easy to check that

$$T_f^* K_\alpha = \overline{f(\alpha)} K_\alpha$$

for any $\alpha \in \mathbb{D}$ and $f \in H^\infty(\mathbb{D})$, the bounded analytic function space in $\mathbb{D}$.

H. Schwarz [11] showed that C_φ is normal if and only if $\varphi(z) = az$ with $|a| \leq 1$. S. Jung, Y. Kim and E. Ko [10] proved that C_φ is quasinormal if and only if C_φ is normal, where $\varphi(z) = \frac{az+b}{cz+d}$ is a linear fractional self-map of $\mathbb{D}$ with $\varphi(0) = 0$. Also they proved that, when $\varphi(z) = \frac{z}{uz+v}$ with $u \neq 0$ and $|v| \geq 1 + |u|$, C_φ is binormal if and only if C_φ is hyponormal, or C_φ is subnormal. Fatehi, Shaabani and Thompson studied hyponormal and quasinormal weighted composition operators on H^2 and the weighted Bergman space A^2_α in [8]. For more study on composition operators on H^2, see [2–11].

In this paper, we discuss 2-normal composition operators with linear fractional symbols on H^2. The necessary and sufficient conditions for C_φ to be 2-normal are given when $\varphi(z) = \frac{az+b}{cz+d}$ is a linear fractional self-map of $\mathbb{D}$. In particular, when $\varphi(z) = \frac{az+b}{cz+d}$ is a linear fractional self-map of $\mathbb{D}$, we prove that C_φ is 2-normal if and only if C_φ is normal when $b = 0$ or $c = 0$. We also give an example of a linear fractional self-map φ which induces a 2-normal operator C_φ but not a normal operator.

2. Auxiliary results

In this section, we state some lemmas which will be used in this paper.

Lemma 1. *[11] Let φ be an analytic self-map of $\mathbb{D}$. Then C_φ is normal if and only if $\varphi(z) = \alpha z$ with $|\alpha| \leq 1$.*

Lemma 2. *[3, Theorem 2] Let $\varphi(z) = \frac{az+b}{cz+d}$ be a linear fractional transformation mapping $\mathbb{D}$ into itself, where $ad - bc \neq 0$. Then $\sigma(z) = \frac{\bar{a}z-\bar{c}}{-\bar{b}z+\bar{d}}$ maps $\mathbb{D}$ into itself, $g(z) = \frac{1}{-\bar{b}z+\bar{d}}$ and $h(z) = cz + d$ are in H^∞, and*

$$C_\varphi^* = T_g C_\sigma T_h^*.$$

The map σ is called the Krein adjoint of φ. g and h are called the Cowen auxiliary functions.

Lemma 3. *Let $\varphi(z) = \frac{az+b}{cz+d}$ be a linear fractional self-map of $\mathbb{D}$. Then*

$$\begin{aligned} C_\varphi^* C_\varphi C_\varphi K_\alpha(z) =& \frac{c(a+d)}{c(a+d) - (a^2+bc)\overline{\alpha}} K_{\varphi(0)}(z) \\ &+ \left(\frac{bc+d^2}{bc+d^2 - b(a+d)\overline{\alpha}} - \frac{c(a+d)}{c(a+d) - (a^2+bc)\overline{\alpha}} \right) K_{\varphi(\sigma_2(\alpha))}(z) \end{aligned}$$

for any $\alpha \in \mathbb{D}$ with $c(a+d) \neq (a^2+bc)\overline{\alpha}$, where $\sigma(z) = \frac{\bar{a}z-\bar{c}}{-\bar{b}z+\bar{d}}$.

Proof. Let $\alpha \in \mathbb{D}$ with $c(a+d) \neq (a^2+bc)\overline{\alpha}$. Then

$$\begin{aligned} C_\varphi^* C_\varphi C_\varphi K_\alpha(z) =& C_\varphi^* K_\alpha(\varphi_2(z)) = C_\varphi^* \frac{1}{1 - \overline{\alpha}\varphi_2(z)} \\ =& C_\varphi^* \frac{1}{1 - \overline{\alpha}\frac{(a^2+bc)z+ab+bd}{(ac+cd)z+bc+d^2}} = C_\varphi^* \frac{Az+B}{Cz+D}, \end{aligned}$$

where

$$A = ac + cd, \qquad B = bc + d^2,$$
$$C = c(a+d) - \overline{\alpha}(a^2 + bc), \qquad D = (bc + d^2) - b(a+d)\overline{\alpha}.$$

From [10, Lemma 2.3] and $C \neq 0$, we have

$$C_\varphi^* \frac{Az + B}{Cz + D} = \frac{A}{C} K_{\varphi(0)}(z) + \left(\frac{B}{D} - \frac{A}{C}\right) K_{\varphi(-\frac{\overline{C}}{\overline{D}})}(z).$$

Since $\frac{A}{C} = \frac{c(a+d)}{c(a+d) - \overline{\alpha}(a^2+bc)}$,

$$\frac{B}{D} - \frac{A}{C} = \frac{bc + d^2}{bc + d^2 - b(a+d)\overline{\alpha}} - \frac{c(a+d)}{c(a+d) - (a^2+bc)\overline{\alpha}},$$

and

$$-\frac{\overline{C}}{\overline{D}} = \frac{\alpha(\overline{a}^2 + \overline{bc}) - (\overline{ac} + \overline{cd})}{(\overline{bc} + \overline{d}^2) - \alpha(\overline{ab} + \overline{bd})} = \sigma_2(\alpha),$$

we obtain

$$\begin{aligned} C_\varphi^* C_\varphi C_\varphi K_\alpha(z) = &\frac{c(a+d)}{c(a+d) - (a^2+bc)\overline{\alpha}} K_{\varphi(0)}(z) \\ &+ \left(\frac{bc + d^2}{bc + d^2 - b(a+d)\overline{\alpha}} - \frac{c(a+d)}{c(a+d) - (a^2+bc)\overline{\alpha}}\right) K_{\varphi(\sigma_2(\alpha))}(z). \end{aligned}$$

□

Lemma 4. *Let $\varphi(z) = \frac{az+b}{cz+d}$ be a linear fractional self-map of $\mathbb{D}$. Then*

$$\begin{aligned} C_\varphi C_\varphi C_\varphi^* K_\alpha(z) = &\frac{c(a+d)}{c(a+d) - (a^2+bc)\overline{\varphi(\alpha)}} \\ &+ \left(\frac{bc + d^2}{bc + d^2 - b(a+d)\overline{\varphi(\alpha)}} - \frac{c(a+d)}{c(a+d) - (a^2+bc)\overline{\varphi(\alpha)}}\right) K_{\sigma_2(\varphi(\alpha))}(z) \\ = &\frac{(\overline{c\alpha} + \overline{d})[(ac+cd)z + bc + d^2]}{(\overline{c\alpha} + \overline{d})[(ac+cd)z + bc + d^2] - (\overline{a\alpha} + \overline{b})[(a^2+bc)z + ab + bd]} \end{aligned}$$

for any $\alpha \in \mathbb{D}$ with $c(a+d) \neq (a^2+bc)\overline{\varphi(\alpha)}$, where $\sigma(z) = \frac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}}$.

Proof. For any $\alpha \in \mathbb{D}$ with $c(a+d) \neq (a^2+bc)\overline{\varphi(\alpha)}$, according to the proof of [10, Lemma 2.3], we have that

$$\begin{aligned} C_\varphi C_\varphi C_\varphi^* K_\alpha(z) = &\frac{c(a+d)}{c(a+d) - (a^2+bc)\overline{\varphi(\alpha)}} \\ &+ \left(\frac{bc + d^2}{bc + d^2 - b(a+d)\overline{\varphi(\alpha)}} - \frac{c(a+d)}{c(a+d) - (a^2+bc)\overline{\varphi(\alpha)}}\right) K_{\sigma_2(\varphi(\alpha))}(z). \end{aligned}$$

On the other hand,

$$\begin{aligned} C_\varphi C_\varphi C_\varphi^* K_\alpha(z) &= C_\varphi C_\varphi K_{\varphi(\alpha)}(z) = K_{\varphi(\alpha)}(\varphi_2(z)) = \frac{1}{1 - \overline{\varphi(\alpha)}\varphi_2(z)} \\ &= \frac{1}{1 - \frac{\overline{a\alpha} + \overline{b}}{\overline{c\alpha} + \overline{d}} \frac{(a^2+bc)z + ab + bd}{(ac+cd)z + bc + d^2}} \\ &= \frac{(\overline{c\alpha} + \overline{d})[(ac+cd)z + bc + d^2]}{(\overline{c\alpha} + \overline{d})[(ac+cd)z + bc + d^2] - (\overline{a\alpha} + \overline{b})[(a^2+bc)z + ab + bd]}. \end{aligned}$$

□

Lemma 5. *Let φ be an analytic self-map of $\mathbb{D}$. If C_φ is 2-normal, then $\overline{\varphi(0)}\varphi_2(0) = 0$.*

Proof. Note that

$$\left\langle C_\varphi^* C_\varphi C_\varphi K_0, K_0 \right\rangle = \langle K_0 \circ \varphi_2, K_0 \rangle = K_0(\varphi_2(0)) = \frac{1}{1 - \bar{0}\varphi_2(0)} = 1$$

and

$$\left\langle C_\varphi C_\varphi C_\varphi^* K_0, K_0 \right\rangle = \left\langle K_{\varphi(0)} \circ \varphi_2, K_0 \right\rangle = \frac{1}{1 - \overline{\varphi(0)}\varphi_2(0)}.$$

Since C_φ is 2-normal, we get the desired result. □

As an application of Lemma 5, we get the following simple example.

Example 1. *If $\varphi(z) = \frac{1}{2}iz + \frac{1}{4}$, then $\varphi_2(z) = -\frac{1}{4}z + \frac{1}{8}i + \frac{1}{4}$. Since $\overline{\varphi(0)}\varphi_2(0) \neq 0$, C_φ is not 2-normal by Lemma 5.*

3. Main results and proofs

3.1. Automorphism

Theorem 1. *Let φ be an automorphism of $\mathbb{D}$. Then the following statements are equivalent.*
(i) C_φ is 2-normal;
(ii) $\varphi(z) = -\lambda z$, $|\lambda| = 1$ or $\varphi(z) = \frac{z-a}{\bar{a}z-1}$ for $a \in \mathbb{D}$.

Proof. (*ii*) ⇒ (*i*). If $\varphi(z) = -\lambda z$, $|\lambda| = 1$, then C_φ is normal by Lemma 1 and hence C_φ is 2-normal. If $\varphi(z) = \frac{z-a}{\bar{a}z-1}$ for $a \in \mathbb{D}$, we note that $(\varphi \circ \varphi)(z) = z$, which implies that

$$C_\varphi^* C_\varphi C_\varphi = C_\varphi C_\varphi C_\varphi^*.$$

Thus, C_φ is 2-normal.

(*i*) ⇒ (*ii*). Assume that C_φ is 2-normal and $\varphi(z) = \frac{\lambda(z-a)}{\bar{a}z-1}$, where $a \in \mathbb{D}$ and $|\lambda| = 1$. We note that

$$\overline{\varphi(0)}\varphi_2(0) = \frac{|a|^2(\lambda - 1)}{\lambda|a|^2 - 1} = 0$$

from Lemma 5. Then $|a|^2(\lambda - 1) = 0$. Hence $|a| = 0$ or $\lambda = 1$.

If $a = 0$, then $\varphi(z) = -\lambda z$, $|\lambda| = 1$. If $\lambda = 1$, then $\varphi(z) = \frac{z-a}{\bar{a}z-1}$, as desired. The proof is complete. □

3.2. Linear fractional self-maps with $\varphi(0) = 0$

Theorem 2. *Let $\varphi(z) = \frac{az+b}{cz+d}$ be a linear fractional self-map of $\mathbb{D}$ and $\varphi(0) = 0$. Then C_φ is 2-normal if and only if C_φ is normal.*

Proof. Sufficiency. It is obvious.

Necessity. Assume that C_φ is 2-normal. Since $\varphi(0) = 0$, $a \neq 0$, and $\varphi(\mathbb{D}) \subset \mathbb{D}$, we can set

$$\varphi(z) = \frac{z}{mz + n},$$

where $m = \frac{c}{a}$, $n = \frac{d}{a}$ and $|n| \geq 1 + |m|$. If $m = 0$, then $\varphi(z) = \frac{z}{n}$. So C_φ is normal.

Now we assume that $m \neq 0$. Then $|n| > 1$. For Lemma 3 we obtain that

$$C_\varphi^* C_\varphi C_\varphi K_\alpha(z) = \frac{m(1+n)}{m(1+n) - \overline{\alpha}} - \frac{\overline{\alpha}}{m(1+n) - \overline{\alpha}} \frac{1}{1 - \frac{\overline{\alpha} - m(1+n)}{\overline{m}(\overline{\alpha} - m(1+n)) + |n|^2 n} z} \tag{1}$$

for $\overline{\alpha} \neq m(1+n)$. From Lemma 4 we get that

$$C_\varphi C_\varphi C_\varphi^* K_\alpha(z) = \frac{m(1+n)(\overline{m\alpha}+\overline{n})}{m(1+n)(\overline{m\alpha}+\overline{n})-\overline{\alpha}} - \frac{\overline{\alpha}}{m(1+n)(\overline{m\alpha}+\overline{n})-\overline{\alpha}} \frac{1}{1-\frac{\overline{\alpha}-m(1+n)(\overline{m\alpha}+\overline{n})}{(\overline{m\alpha}+\overline{n})n^2}z} \tag{2}$$

for $\overline{\alpha} \neq m(1+n)(\overline{m\alpha}+\overline{n})$. Since C_φ is 2-normal, by (1) and (2) we get that

$$\begin{aligned}&\frac{m(1+n)}{m(1+n)-\overline{\alpha}} - \frac{\overline{\alpha}}{m(1+n)-\overline{\alpha}} \frac{1}{1-\frac{\overline{\alpha}-m(1+n)}{\overline{m}(\overline{\alpha}-m(1+n))+|n|^2 n}z}\\ =&\frac{m(1+n)(\overline{m\alpha}+\overline{n})}{m(1+n)(\overline{m\alpha}+\overline{n})-\overline{\alpha}} - \frac{\overline{\alpha}}{m(1+n)(\overline{m\alpha}+\overline{n})-\overline{\alpha}} \frac{1}{1-\frac{\overline{\alpha}-m(1+n)(\overline{m\alpha}+\overline{n})}{(\overline{m\alpha}+\overline{n})n^2}z}\end{aligned}$$

for any $\alpha \in \mathbb{D}$ with $\overline{\alpha} \neq m(1+n)$ and $\overline{\alpha} \neq m(1+n)(\overline{m\alpha}+\overline{n})$. That is,

$$\begin{aligned}0 =&\frac{\overline{\alpha}m(1+n)(\overline{m\alpha}+\overline{n}-1)}{[m(1+n)-\overline{\alpha}][m(1+n)(\overline{m\alpha}+\overline{n})-\overline{\alpha}]} - \frac{\overline{\alpha}}{m(1+n)-\overline{\alpha}} \frac{1}{1-\frac{\overline{\alpha}-m(1+n)}{\overline{m}(\overline{\alpha}-m(1+n))+|n|^2 n}z}\\ &+ \frac{\overline{\alpha}}{m(1+n)(\overline{m\alpha}+\overline{n})-\overline{\alpha}} \frac{1}{1-\frac{\overline{\alpha}-m(1+n)(\overline{m\alpha}+\overline{n})}{(\overline{m\alpha}+\overline{n})n^2}z},\end{aligned}$$

which gives that

$$\begin{aligned}0 =&\overline{\alpha}m(1+n)(\overline{m\alpha}+\overline{n}-1)\left(1-\frac{\overline{\alpha}-m(1+n)}{\overline{m}(\overline{\alpha}-m(1+n))+|n|^2 n}z\right)\left(1-\frac{\overline{\alpha}-m(1+n)(\overline{m\alpha}+\overline{n})}{(\overline{m\alpha}+\overline{n})n^2}z\right)\\ &-\overline{\alpha}[m(1+n)(\overline{m\alpha}+\overline{n})-\overline{\alpha}]\left(1-\frac{\overline{\alpha}-m(1+n)(\overline{m\alpha}+\overline{n})}{(\overline{m\alpha}+\overline{n})n^2}z\right)\\ &+\overline{\alpha}[m(1+n)-\overline{\alpha}]\left(1-\frac{\overline{\alpha}-m(1+n)}{\overline{m}(\overline{\alpha}-m(1+n))+|n|^2 n}z\right)\end{aligned}$$

for any $\alpha \in \mathbb{D}$ with $\overline{\alpha} \neq m(1+n)$ and $\overline{\alpha} \neq m(1+n)(\overline{m\alpha}+\overline{n})$. Multiply this by $(\overline{m\alpha}+\overline{n})$, we get

$$\begin{aligned}0 =&\overline{\alpha}m(1+n)(\overline{m\alpha}+\overline{n}-1)\left(1-\frac{\overline{\alpha}-m(1+n)}{\overline{m}(\overline{\alpha}-m(1+n))+|n|^2 n}z\right)\\ &\cdot\left(\overline{m\alpha}+\overline{n}-\frac{\overline{\alpha}-m(1+n)(\overline{m\alpha}+\overline{n})}{n^2}z\right)\\ &-\overline{\alpha}[m(1+n)(\overline{m\alpha}+\overline{n})-\overline{\alpha}]\left(\overline{m\alpha}+\overline{n}-\frac{\overline{\alpha}-m(1+n)(\overline{m\alpha}+\overline{n})}{n^2}z\right)\\ &+\overline{\alpha}[m(1+n)-\overline{\alpha}]\left(1-\frac{\overline{\alpha}-m(1+n)}{\overline{m}(\overline{\alpha}-m(1+n))+|n|^2 n}z\right)(\overline{m\alpha}+\overline{n})\end{aligned} \tag{3}$$

for any $\alpha \in \mathbb{D}$ with $\overline{\alpha} \neq m(1+n)$ and $\overline{\alpha} \neq m(1+n)(\overline{m\alpha}+\overline{n})$. Since (3) holds for any $z \in \mathbb{D}$, the coefficient of z^2 in (3) must be zero. This implies that

$$\overline{\alpha}m(1+n)(\overline{m\alpha}+\overline{n}-1)\frac{\overline{\alpha}-m(1+n)}{\overline{m}(\overline{\alpha}-m(1+n))+|n|^2 n}\frac{\overline{\alpha}-m(1+n)(\overline{m\alpha}+\overline{n})}{n^2}=0.$$

Since $m \neq 0$, from the last equality we obtain that

$$\overline{\alpha}(1+n)(\overline{m\alpha}+\overline{n}-1)=0$$

for any $\alpha \in \mathbb{D}$ with $\overline{\alpha} \neq m(1+n)$ and $\overline{\alpha} \neq m(1+n)(\overline{m\alpha}+\overline{n})$. After a calculation, we get

$$\overline{m}(1+n)\overline{\alpha}^2+(1+n)(\overline{n}-1)\overline{\alpha}=0$$

for any $\alpha \in \mathbb{D}$ with $\overline{\alpha} \neq m(1+n)$ and $\overline{\alpha} \neq m(1+n)(\overline{m\alpha}+\overline{n})$, which is a contradiction. So $m=0$ and $\varphi(z)=\frac{z}{n}, |n| \geq 1$. Therefore C_φ is normal. The proof is complete. □

3.3. Linear fractional self-maps with $c = 0$

Lemma 6. *[9] If $\varphi(z) = \frac{az+b}{cz+d}$ is a linear fractional self-map into $\mathbb{D}$ and $c = 0$, then*

$$C_\varphi = C_{\tilde{\sigma}}^* T_{\tilde{g}}^*,$$

where $\tilde{\sigma}(z) = \frac{\bar{a}z}{-\bar{b}z+\bar{d}}$ and $\tilde{g}(z) = \frac{\bar{d}}{-\bar{b}z+\bar{d}}$.

Theorem 3. *Let $\varphi(z) = \frac{az+b}{cz+d}$ be a linear fractional self-map into $\mathbb{D}$ with $c = 0$. Then C_φ is 2-normal if and only if C_φ is normal.*

Proof. Sufficiency. It is obvious.

Necessity. Suppose that C_φ is 2-normal, that is,

$$C_\varphi^* C_\varphi C_\varphi K_\alpha(z) = C_\varphi C_\varphi C_\varphi^* K_\alpha(z)$$

for any $\alpha, z \in \mathbb{D}$. Since $c = 0$, we set $\varphi(z) = sz + t$, where $s = \frac{a}{d}$, $t = \frac{b}{d}$ and $|s| + |t| \le 1$. Put

$$\sigma(z) = \frac{\bar{s}z}{1 - \bar{t}z}, g(z) = \frac{1}{1 - \bar{t}z}.$$

According to the proof of [9, Theorem 2.4], by Lemma 6 we obtain that

$$\begin{aligned} C_\varphi^* C_\varphi C_\varphi K_\alpha(z) &= C_\varphi^* C_\sigma^* T_g^* C_\sigma^* T_g^* K_\alpha(z) = \overline{g(\alpha) g(\sigma(\alpha))} K_{\varphi(\sigma_2(\alpha))}(z) \\ &= \frac{1}{1 - t(s+1)\overline{\alpha} - \left[\bar{t} + (|s|^2 s - |t|^2 s - |t|^2)\,\overline{\alpha}\right] z}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} C_\varphi C_\varphi C_\varphi^* K_\alpha(z) &= C_\sigma^* T_g^* C_\sigma^* T_g^* K_{\varphi(\alpha)}(z) = \overline{g(\varphi(\alpha)) g(\sigma(\varphi(\alpha)))} K_{\sigma_2(\varphi(\alpha))}(z) \\ &= \frac{1}{1 - t\overline{\varphi(\alpha)}} \frac{1}{1 - t\overline{\sigma(\varphi(\alpha))}} \frac{1}{1 - \overline{\sigma_2(\varphi(\alpha))} z} \\ &= \frac{1}{1 - t\overline{\varphi(\alpha)}} \frac{1}{1 - t\overline{\sigma(\varphi(\alpha))}} \frac{1}{1 - \frac{s\overline{\sigma(\varphi(\alpha))}}{1 - t\overline{\sigma(\varphi(\alpha))}} z} \\ &= \frac{1}{1 - t\overline{\varphi(\alpha)}} \frac{1}{1 - t\overline{\sigma(\varphi(\alpha))} - s\overline{\sigma(\varphi(\alpha))} z} \\ &= \frac{1}{1 - t\overline{\varphi(\alpha)} - st\overline{\varphi(\alpha)} - s^2 \overline{\varphi(\alpha)} z} \\ &= \frac{1}{1 - ts\overline{\alpha} - |t|^2 - t|s|^2 \overline{\alpha} - s|t|^2 - s|s|^2 \overline{\alpha} z - s^2 \bar{t} z}. \end{aligned}$$

Since C_φ is 2-normal, we get

$$\begin{aligned} &\frac{1}{1 - t(s+1)\overline{\alpha} - \left[\bar{t} + (|s|^2 s - |t|^2 s - |t|^2)\,\overline{\alpha}\right] z} \\ =&\frac{1}{1 - ts\overline{\alpha} - |t|^2 - t|s|^2 \overline{\alpha} - s|t|^2 - s|s|^2 \overline{\alpha} z - s^2 \bar{t} z} \end{aligned} \tag{4}$$

for any $\alpha, z \in \mathbb{D}$. Taking $\alpha = 0$ in (4), we obtain

$$\frac{1}{1 - |t|^2 - s|t|^2 - s^2 \bar{t} z} = \frac{1}{1 - \bar{t} z} \tag{5}$$

for any $z \in \mathbb{D}$. So we have $t = 0$ or $s = -1$. If $s = -1$, we know that $t = 0$ since $|s| + |t| \le 1$. Therefore, by Lemma 1 we get the desired result. The proof is complete. □

3.4. Linear fractional self-maps with $\varphi(0) \neq 0$ and $a = 0$

Lemma 7. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a constant function. Then C_φ is 2-normal if and only if φ is zero on $\mathbb{D}$.*

Proof. Let $\varphi(z) \equiv c$ for some $c \in \mathbb{D}$. Then

$$C_\varphi^* C_\varphi C_\varphi K_\alpha(z) = C_\varphi^* C_\varphi K_\alpha(\varphi(z)) = \frac{1}{1-\overline{\alpha}c} C_\varphi^* C_\varphi K_0(z) = \frac{1}{1-\overline{\alpha}c} K_{\varphi(0)}(z) = \frac{1}{1-\overline{\alpha}c}\frac{1}{1-\overline{c}z},$$

and

$$C_\varphi C_\varphi C_\varphi^* K_\alpha(z) = K_{\varphi(\alpha)}(\varphi_2(z)) = \frac{1}{1-\overline{\varphi(\alpha)}\varphi_2(z)} = \frac{1}{1-|c|^2}$$

for any $\alpha, z \in \mathbb{D}$. Hence C_φ is 2-normal if and only if $c = 0$. □

Theorem 4. *Let $\varphi(z) = \frac{az+b}{cz+d}$ be a linear fractional self-map of $\mathbb{D}$ with $\varphi(0) \neq 0$ and $a = 0$. Then C_φ is not 2-normal.*

Proof. Since $\varphi(0) \neq 0$ and $a = 0$, we can set $\varphi(z) = \frac{1}{uz+v}$ where $u = \frac{c}{b}$ and $v = \frac{d}{b}$. If $u = 0$, then $\varphi(z) = \frac{1}{v} \neq 0$ and so C_φ is not 2-normal from Lemma 7. Suppose C_φ is 2-normal and $u \neq 0$ and $v \neq 0$. Form Lemma 3, we get

$$C_\varphi^* C_\varphi C_\varphi K_\alpha(z) = \frac{uv}{uv - u\overline{\alpha}} K_{\varphi(0)}(z) + \left(\frac{u+v^2}{u+v^2-v\overline{\alpha}} - \frac{uv}{uv-u\overline{\alpha}}\right) K_{\varphi(\sigma_2(\alpha))}(z) \tag{6}$$

for any $z \in \mathbb{D}$ and $\alpha \in \mathbb{D}$ with $uv \neq u\overline{\alpha}$. From Lemma 4, we have

$$C_\varphi C_\varphi C_\varphi^* K_\alpha(z) = \frac{(\overline{u\alpha}+\overline{v})(uvz+u+v^2)}{(\overline{u\alpha}+\overline{v})(uvz+u+v^2) - (uz+v)} \tag{7}$$

for any $z \in \mathbb{D}$ and $\alpha \in \mathbb{D}$. In particular, taking $\alpha = 0$ in (6) and (7), we get

$$C_\varphi^* C_\varphi C_\varphi K_0(z) = \frac{\overline{v}}{\overline{v} - z} \tag{8}$$

and

$$C_\varphi C_\varphi C_\varphi^* K_0(z) = \frac{\overline{v}(uvz+u+v^2)}{\overline{v}(uvz+u+v^2) - (uz+v)}. \tag{9}$$

Since C_φ is 2-normal, (8) and (9) are equal, that is,

$$\frac{\overline{v}}{\overline{v} - z} = \frac{\overline{v}(uvz+u+v^2)}{\overline{v}(uvz+u+v^2) - (uz+v)} \tag{10}$$

for any $z \in \mathbb{D}$. After a calculation, we obtain that

$$u|v|^2 z^2 + |v|^2 vz - |v|^2 = 0$$

for any $z \in \mathbb{D}$. Since $v \neq 0$, dividing both sides by $|v|^2$, we get

$$uz^2 + vz - 1 = 0 \tag{11}$$

for any $z \in \mathbb{D}$, which is a contradiction. Hence C_φ is not 2-normal. □

3.5. Linear fractional self-maps with $\varphi(0) \neq 0$, $a \neq 0$ and $c \neq 0$

Lemma 8. *Let $\varphi(z) = \frac{az+b}{cz+1}$ be a linear fractional self-map of $\mathbb{D}$ with $\varphi(0) \neq 0$, $a \neq 0$ and $c \neq 0$. If C_φ is 2-normal, then*

$$(a^2 + bc)\overline{\varphi(0)} \neq c(a+1).$$

Proof. If $(a^2 + bc)\overline{\varphi(0)} = c(a+1)$, then we obtain that $a^2\overline{b} = (a+1-|b|^2)c$. Since $a \neq 0$ and $\varphi(0) = b \neq 0$, it must be hold that $a+1-|b|^2 \neq 0$, and so $c = \frac{a^2\overline{b}}{a+1-|b|^2}$. Therefore,

$$\varphi(z) = \frac{az+b}{\frac{a^2\overline{b}}{a+1-|b|^2}z+1}.$$

After a calculation,

$$\overline{\varphi(0)}\varphi_2(0) = \frac{(a+1-|b|^2)|b|^2}{1+(a-1)|b|^2}.$$

Since $a+1-|b|^2 \neq 0$ and $b \neq 0$, $\overline{\varphi(0)}\varphi_2(0) \neq 0$. Thus C_φ is not 2-normal by Lemma 5. □

Theorem 5. *Let $\varphi(z) = \frac{az+b}{cz+1}$ be a linear fractional self-map with $\varphi(0) \neq 0$, $a \neq -1$ and $c \neq 0$, then C_φ is not 2-normal.*

Proof. Assume that C_φ is 2-normal. By Lemma 3, we get

$$\begin{aligned} C_\varphi^* C_\varphi C_\varphi K_\alpha(z) =& \frac{c(a+1)}{c(a+1)-(a^2+bc)\overline{\alpha}} K_{\varphi(0)}(z) \\ &+ \left(\frac{bc+1}{bc+1-b(a+1)\overline{\alpha}} - \frac{c(a+1)}{c(a+1)-(a^2+bc)\overline{\alpha}} \right) K_{\varphi(\sigma_2(\alpha))}(z) \end{aligned} \tag{12}$$

for any $\alpha \in \mathbb{D}$ with $c(a+1) \neq (a^2+bc)\overline{\alpha}$.

From Lemma 4, we see

$$\begin{aligned} C_\varphi C_\varphi C_\varphi^* K_\alpha(z) =& \frac{c(a+1)}{c(a+1)-(a^2+bc)\,\overline{\varphi(\alpha)}} \\ &+ \left(\frac{bc+1}{bc+1-b(a+1)\overline{\varphi(\alpha)}} - \frac{c(a+1)}{c(a+1)-(a^2+bc)\,\overline{\varphi(\alpha)}} \right) K_{\sigma_2(\varphi(\alpha))}(z) \end{aligned} \tag{13}$$

for any $\alpha \in \mathbb{D}$ with $c(a+1) \neq (a^2+bc)\overline{\varphi(\alpha)}$. Since $c(a+1) \neq 0$ and $(a^2+bc)\overline{\varphi(0)} \neq c(a+1)$ from Lemma 8, we can take $\alpha = 0$ in (12) and (13). Then

$$C_\varphi^* C_\varphi C_\varphi K_0(z) = \frac{1}{1-\overline{b}z} \tag{14}$$

and

$$\begin{aligned} & C_\varphi C_\varphi C_\varphi^* K_0(z) \\ =& \frac{c(a+1)}{c(a+1)-(a^2+bc)\overline{\overline{b}}} + \left(\frac{bc+1}{bc+1-b(a+1)\overline{\overline{b}}} - \frac{c(a+1)}{c(a+1)-(a^2+bc)\overline{\overline{b}}} \right) K_{\sigma_2(\varphi(0))}(z) \\ =& \frac{c(a+1)}{c(a+1)-(a^2+bc)\overline{b}} - \frac{\overline{b}[(bc+1)(a^2+bc)-bc(a+1)^2]}{[bc+1-|b|^2(a+1)][c(a+1)-(a^2+bc)\overline{b}]} \frac{1}{1-\frac{(a^2+bc)\overline{b}-(a+1)c}{-|b|^2(a+1)+bc+1}z}. \end{aligned} \tag{15}$$

Let

$$A = bc+1-|b|^2(a+1), \quad B = c(a+1)-(a^2+bc)\overline{b}.$$

Since C_φ is 2-normal, (14) and (15) are equal, i,e.,

$$\frac{1}{1-\overline{b}z} = \frac{c(a+1)}{B} - \frac{\overline{b}[(bc+1)(a^2+bc)-bc(a+1)^2]}{AB}\frac{A}{A+Bz}$$

for any $z \in \mathbb{D}$. Hence,

$$\frac{\overline{b}[(bc+1)(a^2+bc)-bc(a+1)^2]A}{AB(A+Bz)} = \frac{c(a+1)(1-\overline{b}z)-B}{B(1-\overline{b}z)}$$

for any $z \in \mathbb{D}$. Calculating the above equation and noting that the coefficients of z^2 must be zero, we obtain that $AB^2c(a+1)\overline{b} = 0$. However, $b, c(a+1), A$ and B are all nonzero, which gives a contradiction. Hence C_φ is not 2-normal. □

In the end of this paper, we give an example of a linear fractional self-map φ which induces a 2-normal operator C_φ but not a normal operator.

Example 2. *Let $\varphi(z) = \frac{-z+b}{cz+1}$ be a linear fractional self-map of $\mathbb{D}$ with $\varphi(0) \neq 0$ and $c \neq 0$. Since*

$$\varphi_2(z) = \varphi(\varphi(z)) = z, \ C_\varphi^2 = C_{\varphi_2} = I$$

and so

$$C_\varphi^* C_\varphi C_\varphi = C_\varphi C_\varphi C_\varphi^*.$$

Hence C_φ is a 2-normal operator. However, according to Lemma 1 we see that C_φ is not a normal operator.

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Conflicts of Interest The authors declare that they have no conflicts of interest.

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