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Inverse conformable Sturm–Liouville problems by three spectra with discontinuities and boundary conditions

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Abstract. In this manuscript, we consider the conformable fractional Sturm–Liouville problem (CFSLP) with finite numbers of transmission conditions at an interior point in $[0, \pi]$. Also, we study the uniqueness theorem for inverse second order of fractional differential operators by applying three spectra with a finite number of discontinuities at interior points. For this aim, we investigate the CFSLP in three intervals $[0, \pi]$, $[0, p]$, and $[p, \pi]$ such that $p \in (0, \pi)$ is an interior point.

1. Introduction

Sturm–Liouville problem is one of the most important problems in mathematics, physics and engineering. This problem arises in the modeling of many systems in vibration theory, quantum mechanics, hydrodynamics, and etc. [8, 16, 23].

There are two types of CFSLP: direct and inverse problems. In direct problems, the eigenvalues, eigenfunctions, and other properties are estimated from the known coefficients [14, 15, 25]. The inverse spectral problem can be considered as three aspects: existence, uniqueness, and reconstruction of coefficients with special property of eigenvalues and eigenfunctions, (see [5–7, 9, 13, 15, 17, 20, 21, 24] and the references therein).

The inverse three spectra problems to reconstruction of the potential function in the Sturm–Liouville problem firstly discussed by Pivovarchik in [17, 18] and Gesztesy and Simon in [7]. They proved if the three spectra are pairwise disjoint, then the potential q can be uniquely determined by the three spectra of the problems defined on three intervals $[0, 1]$, $[0, d]$ and $[d, 1]$ for some $d \in (0, 1)$. Also, in [7] they gave a violation example to demonstrate that the pairwise disjoint conditions are necessary. Recently, in [2–6, 19], the authors discussed the inverse three spectra problems in the several cases such as reconstruction of the potential function with different boundary and transmissions conditions and some uniqueness results.

The main purpose of this manuscript is to study the inverse CFSLP by using three spectra. One may consider the results of this paper as an extension of [5–7, 12, 17–19, 25] to the CFSLP. Furthermore, we introduce that a Weyl–Titchmarsh $\mathfrak{m}$-function uniquely determined the CFSLP. We also show that the Weyl–Titchmarsh function is a meromorphic Herglotz-Nevanlinna function.

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2. The conformable fractional Sturm-Liouville problem

In this section, before introducing the main results, we give some important contents of the conformable fractional (CF) derivative. In what follows, we always take $D_x^\alpha = D^\alpha$. In [11], Khalil and et al. defined the CF derivative as follows:

Definition 2.1. *For the function $h : [0, \infty) \to \mathbb{R}$, the CF derivative of order $\alpha \in (0, 1]$ defined by:*

$$D^\alpha h(x) = \lim_{\epsilon \to 0} \frac{h(x + \epsilon x^{1-\alpha}) - h(x)}{\epsilon},$$

for all $x > 0$, and

$$D^\alpha h(0) = \lim_{x \to 0^+} D^\alpha h(x).$$

If h is a differentiable function, then

$$D^\alpha h(x) = x^{1-\alpha} h'(x).$$

If $D^\alpha h(x_0)$ exists and finite, then the function h is α-differentiable at x_0.

Definition 2.2. *For the function $h : [0, \infty) \to \mathbb{R}$, the CF integral of order $\alpha \in (0, 1]$ defined by:*

$$J^\alpha h(x) = \int_0^x h(t) \mathrm{d}_\alpha t = \int_0^x t^{\alpha-1} h(t) \mathrm{d}t, \quad x > 0.$$

For CFSLP, we use some fundamental CF derivative relations as detailed in [1, 11, 21].
Let us consider the following three CFSLPs

$$\ell_0 y := -D^\alpha D^\alpha y + qy = \lambda y \tag{1}$$

with

$$\mathrm{B}_1(y) := D^\alpha y(0) + h\, y(0) = 0, \tag{2}$$
$$\mathrm{B}_2(y) := D^\alpha y(\pi) + H\, y(\pi) = 0, \tag{3}$$

subject to the following jump conditions

$$\begin{aligned} U_k(y) &:= y(p_k + 0) - b_k y(p_k - 0) = 0, \\ V_k(y) &:= D^\alpha y(p_k + 0) - c_k D^\alpha y(p_k - 0) - d_k y(p_k - 0) = 0, \end{aligned} \tag{4}$$

$$\ell_1 y := -D^\alpha D^\alpha y + q_1 y = \lambda y \tag{5}$$

with

$$\mathrm{B}_1(y) = 0, \;\; \mathrm{B}_3(y) := D^\alpha y(p) + H_1\, y(p) = 0, \tag{6}$$

subject to the following jump conditions

$$U_k(y) = 0, \;\; V_k(y) = 0, \text{ for } k = 1, 2, \ldots, p - 1, \tag{7}$$

and

$$\ell_2 y := -D^\alpha D^\alpha y + q_2 y = \lambda y \tag{8}$$

with

$$\mathrm{B}_4(y) := D^\alpha y(p) + H_2\, y(p) = 0,\ \mathrm{B}_2(y) = 0, \tag{9}$$

subject to the following jump conditions

$$U_k(y) = 0,\ \ V_k(y) = 0,\ \text{for } k = p+1, 2, \ldots, m-1, \tag{10}$$

where D^α is the CF derivative of order $0 < \alpha \leq 1$, $q(x) \in L^1_\alpha[0,\pi]$, $q_1 = q|_{[0,p)}$, and $q_2 = q|_{(p,\pi]}$ are real valued functions. Also, $h, H, H_1, H_2, b_k, c_k, d_k$, and p_k, $k = 1, 2, \ldots, m-1$ (with $m \geq 2$) are real numbers. The parameter λ is the spectral parameter. In this paper, we suppose that $b_k c_k > 0$, $p_0 = 0 < p_1 < p_2 < \ldots < p_{m-1} < p_m = \pi$. In this section, we assume that $p = p_s$ for $1 \leq s \leq m-1$. As well as, we use the notations $L_0 = L(q(x); h; H; p_k)$, $L_1 = L(q_1(x); h; H_1; p_k)$, $L_2 = L(q_2(x); H_2; H; p_k)$ for the problems (1)–(10). Using the jump conditions (4) in the transmission point $p = p_s$, $(1 \leq s \leq m-1)$, we must have $d_s = 0$ and

$$H_2 = \frac{c_s}{b_s} H_1,\ \text{for } H_1 \in (0, \infty). \tag{11}$$

To obtain a self-adjoint operator, we define the weighted inner product as follows

$$\langle f, g \rangle_{T_0} := \int_0^\pi f(t)\overline{g(t)} r_0(t) \mathrm{d}_\alpha t,$$

where $f, g \in L^2_\alpha((0,\pi); r_0)$ and

$$r_0(t) = \begin{cases} 1, & 0 \leq t < p_1, \\ \frac{1}{b_1 c_1}, & p_1 < t < p_2, \\ \vdots & \\ \frac{1}{b_1 c_1 \cdots b_{m-1} c_{m-1}}, & p_{m-1} < t \leq \pi. \end{cases}$$

Also, $r_1 = r_0(t)|_{[0,p)}$, and $r_2 = r_0|_{(p,\pi]}$. Note that $T_i := L^2_\alpha((0,\pi); r_i)$, $(i = 0, 1, 2)$, are the Hilbert spaces with the norms $\|f\|_{T_i} = \langle f, f \rangle^{1/2}_{T_i}$. Define the operators

$$A_i : T_i \to T_i, \quad i = 0, 1, 2 \tag{12}$$

with domains

$$\operatorname{dom}(A_0) = \left\{ f \in T_0 \,\middle|\, \begin{array}{c} f, D^\alpha f \in AC\Big(\cup_0^{m-1} (p_k, p_{k+1}) \Big),\ \ell_0 f \in L^2_\alpha(0,\pi) : \\ U_k(f) = V_k(f) = 0,\ k = 1, 2, \ldots, m-1 \end{array} \right\}, \tag{13}$$

$$\operatorname{dom}(A_1) = \left\{ f \in T_1 \,\middle|\, \begin{array}{c} f, D^\alpha f \in AC\Big(\cup_0^{s-1} (p_k, p_{k+1}) \Big),\ \ell_1 f \in L^2_\alpha(0,p) : \\ U_k(f) = V_k(f) = 0,\ k = 1, 2, \ldots, s-1 \end{array} \right\}, \tag{14}$$

and

$$\operatorname{dom}(A_2) = \left\{ f \in T_2 \,\middle|\, \begin{array}{c} f, D^\alpha f \in AC\Big(\cup_s^{m-1} (p_k, p_{k+1}) \Big),\ \ell_2 f \in L^2_\alpha(p,\pi) : \\ U_k(f) = V_k(f) = 0,\ k = s+1, s+2, \ldots, m-1 \end{array} \right\} \tag{15}$$

respectively by

$$A_i f = \ell_i f \text{ with } f \in \operatorname{dom}(A_i),\ i = 0, 1, 2.$$

In this paper, the notation $AC\Big(\cup_0^{m-1} (p_k, p_{k+1}) \Big)$ denotes the set of all functions whose restriction to (p_k, p_{k+1}) is absolutely continuous for all $k = 0, 1, \ldots, m-1$. The function

$$W_\alpha(f, g) = r_0(t)\Big(f(t) D^\alpha g(t) - D^\alpha f(t) g(t) \Big) \tag{16}$$

is called the modified fractional Wronskian of f and g, where these functions are two solutions of $\ell_0 f = \lambda f$, $\ell_0 g = \lambda g$. The similar modified fractional Wronskian can be defined to L_1 and L_2. It is easy to see that the function W_α does not depend on t.

Lemma 2.3. *For $0 < \alpha \leq 1$, the operators A_i, $i = 0, 1, 2$, are self-adjoint on $L^2_\alpha((0, \pi); r_i)$.*

Proof. We prove this lemma for $i = 0$. After using α-integration by parts twice, we arrive to the following expression:

$$\left\langle \ell_0 f, g \right\rangle = W_\alpha\left(f, g\right)\Big|_{x=\pi} - W_\alpha\left(f, g\right)\Big|_{x=0} + \left\langle f, \ell_0 g \right\rangle. \tag{17}$$

So, from Eqs (2)–(4) we have:

$$W_\alpha\left(f, g\right)\Big|_{x=\pi} - W_\alpha\left(f, g\right)\Big|_{x=0} = 0$$

Then A_0 is self-adjoint operator on $L^2_\alpha((0, \pi); r_0)$. Similarly, the operators A_1 and A_2 are also self-adjoint. □

By applying Lemma 2.3, the eigenvalues of the problems A_i and hence of L_i are simple and real.
Since the associated Cauchy problem (1) with initial conditions $g(\nu \pm 0) = g_0$ and $g'(\nu \pm 0) = g_1$ (with $\nu \in (0, \pi)$)) has a unique solution.

Remark 2.4. *We will denote the restriction of any function g with $g \in \operatorname{dom}(A_i)$, by g_k, $1 \leq k \leq m$, to the subinterval (p_{k-1}, p_k). Also, we will set $g_k(p_{k-1}) = g(p_{k-1} + 0)$ and $g_i(p_k) = g(p_k - 0)$.*

Remark 2.5. *Without loss of generality of the problem* (1)–(4)*, by [20, Lemma 2.3], we can take $b_k c_k = 1$, for $k = 1, 2, \ldots, m$.*

3. Uniqueness result

In this section, we study the inverse CFSLP of the reconstruction of a boundary value problem L_0 from its spectral characteristics. For this purpose, we consider three boundary value problems L_i, $(i = 0, 1, 2)$, from three spectra $\{\lambda_n,\ \mu_n,\ \nu_n\}_{n\geq 0}$. To prove the uniqueness theorem, we use an adaptation of this technique, firstly it was discussed by F. Gesztesy and B. Simon in [7].

Consider the CFSLPs (1)–(4) on the interval $[0, \pi]$, CFSLPs (5)–(7) on subinterval $[0, p)$ and CFSLPs (8)–(10) $(p, \pi]$ which are imposed the boundary condition at p. Suppose that $v(x, \lambda)$ and $w(x, \lambda)$ are solutions of (1) with the initial conditions

$$\begin{aligned} &v(0, \lambda) = 1, \qquad D^\alpha v(0, \lambda) = -h, \\ &w(\pi, \lambda) = 1, \qquad D^\alpha w(\pi, \lambda) = -H, \end{aligned} \tag{18}$$

and the transmission conditions (4), respectively. The functions $v(x, \lambda)$, $D^\alpha v(x, \lambda)$, $w(x, \lambda)$, and $D^\alpha w(x, \lambda)$ for any fixed $x \in [0, \pi]$ are entire functions in λ of order $\frac{1}{2}$ [22]. The asymptotic form of solutions and characteristic function $\Delta(\lambda)$ are discussed as follows:

Theorem 3.1 ([21]). *Let $\lambda = \rho^2$ and $\rho := \sigma + i\tau$. The asymptotic forms of solutions $v(x, \lambda)$ and $D^\alpha v(x, \lambda)$ for CFSLP* (1)–(4) *as $|\lambda| \to \infty$, are the following forms:*

$$v(x, \lambda) = \begin{cases} \cos\left(\frac{\rho}{\alpha} x^\alpha\right) + O\left(\frac{1}{\rho} \exp\left(\frac{|\tau|}{\alpha} x^\alpha\right)\right), & 0 \leq x < p_1, \\ a_1 \cos\left(\frac{\rho}{\alpha} x^\alpha\right) + a_1' \cos\left(\frac{\rho}{\alpha}(x^\alpha - 2p_1^\alpha)\right) + O\left(\frac{1}{\rho} \exp\left(\frac{|\tau|}{\alpha} x^\alpha\right)\right), & p_1 < x < p_2, \\ a_1 a_2 \cos\rho\left(\frac{\rho}{\alpha} x^\alpha\right) + a_1' a_2 \cos\left(\frac{\rho}{\alpha}(x^\alpha - 2p_1^\alpha)\right) + a_1 a_2' \cos\left(\frac{\rho}{\alpha}(x^\alpha - 2p_2^\alpha)\right) & \\ \quad + a_1' a_2' \cos\left(\frac{\rho}{\alpha}(x^\alpha + 2p_1^\alpha - 2p_2^\alpha)\right) + O\left(\frac{1}{\rho} \exp\left(\frac{|\tau|}{\alpha} x^\alpha\right)\right), & p_2 < x < p_3, \\ \quad \vdots & \\ a_1 a_2 \ldots a_{m-1} \cos\left(\frac{\rho}{\alpha} x^\alpha\right) + & \\ \quad + a_1' a_2 \ldots a_{m-1} \cos\left(\frac{\rho}{\alpha}(x^\alpha - 2p_1^\alpha)\right) + \cdots & \\ \quad + a_1 a_2 \ldots a_{m-1}' \cos\left(\frac{\rho}{\alpha}(x^\alpha - 2p_{m-1}^\alpha)\right) + & \\ \quad + a_1' a_2' a_3 \ldots a_{m-1} \cos\left(\frac{\rho}{\alpha}(x^\alpha + 2p_1^\alpha - 2p_2^\alpha)\right) + \cdots & \\ \quad + a_1 \ldots a_i' \ldots a_j' \ldots a_{m-1} \cos\left(\frac{\rho}{\alpha}(x^\alpha + 2p_i^\alpha - 2p_j^\alpha)\right) & \\ \quad + a_1 \ldots a_i' \ldots a_j' \ldots a_k' \ldots a_{m-1} \cos\left(\frac{\rho}{\alpha}(x^\alpha - 2p_i^\alpha + 2p_j^\alpha - 2p_k^\alpha)\right) + \cdots & \\ \quad + a_1' a_2' \ldots a_{m-1}' \cos\left(\frac{\rho}{\alpha}(x^\alpha + 2(-1)^{m-1} p_1^\alpha + 2(-1)^{m-2} p_2^\alpha - 2p_m^\alpha)\right) & \\ \quad + O\left(\frac{1}{\rho} \exp\left(\frac{|\tau|}{\alpha} x^\alpha\right)\right), & p_{m-1} < x \leq \pi, \end{cases} \tag{19}$$

$$D^\alpha v(x,\lambda)=\begin{cases}\rho\left[-\sin\left(\frac{\rho}{\alpha}x^\alpha\right)\right]+O\left(\exp\left(\frac{|\tau|}{\alpha}x^\alpha\right)\right), & 0\le x<p_1,\\ \rho\left[-a_1\sin\left(\frac{\rho}{\alpha}x^\alpha\right)-a_1'\sin\left(\frac{\rho}{\alpha}(x^\alpha-2p_1^\alpha)\right)\right]+O\left(\exp\left(\frac{|\tau|}{\alpha}x^\alpha\right)\right), & p_1<x<p_2,\\ \rho\left[-a_1a_2\sin\rho\left(\frac{\rho}{\alpha}x^\alpha\right)-a_1'a_2\sin\left(\frac{\rho}{\alpha}(x^\alpha-2p_1^\alpha)\right)-\alpha_1\alpha_2'\sin\left(\frac{\rho}{\alpha}(x^\alpha-2p_2^\alpha)\right)\right. & \\ \quad\left.-a_1'a_2'\sin\left(\frac{\rho}{\alpha}(x^\alpha+2p_1^\alpha-2p_2^\alpha)\right)\right]+O\left(\exp\left(\frac{|\tau|}{\alpha}x^\alpha\right)\right), & p_2<x<p_3,\\ \quad\vdots & \\ \rho\left[-a_1a_2\ldots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}x^\alpha\right)\right. & \\ \quad -a_1'a_2\ldots a_{m-1}\sin\left(\frac{\rho}{\alpha}(x^\alpha-2p_1^\alpha)\right)+\cdots & \\ \quad -a_1a_2\ldots a_{m-1}'\sin\left(\frac{\rho}{\alpha}(x^\alpha-2p_{m-1}^\alpha)\right)+ & \\ \quad -a_1'a_2'a_3\ldots a_{m-1}\sin\left(\frac{\rho}{\alpha}(x^\alpha+2p_1^\alpha-2p_2^\alpha)\right)+\cdots & \\ \quad -a_1\ldots a_i'\ldots a_j'\ldots a_{m-1}\sin\left(\frac{\rho}{\alpha}(x^\alpha+2p_i^\alpha-2p_j^\alpha)\right) & \\ \quad -a_1\ldots a_i'\ldots a_j'\ldots a_k'\ldots a_{m-1}\sin\left(\frac{\rho}{\alpha}(x^\alpha-2p_i^\alpha+2p_j^\alpha-2p_k^\alpha)\right)+\cdots & \\ \quad \left.-a_1'a_2'\ldots a_{m-1}'\sin\left(\frac{\rho}{\alpha}(x^\alpha+2(-1)^{m-1}p_1^\alpha+2(-1)^{m-2}p_2^\alpha-2p_m^\alpha)\right)\right] & \\ \quad +O\left(\exp\left(\frac{|\tau|}{\alpha}x^\alpha\right)\right), & p_{m-1}<x\le\pi,\end{cases}\tag{20}$$

where

$$a_k=\frac{1}{2}(b_k+c_k),\qquad a_k'=\frac{1}{2}(b_k-c_k),\tag{21}$$

for $k=1,2,\ldots,m-1$.

From Theorem 3.1 and Definition 2.1 we get that

$$\begin{aligned}|v(x,\lambda)|&=O\left(\exp\left(\frac{|\tau|}{\alpha}x^\alpha\right)\right),\\ |D^\alpha v(x,\lambda)|=|x^{1-\alpha}v'(x,\lambda)|&=O\left(|\rho|\exp\left(\frac{|\tau|}{\alpha}x^\alpha\right)\right),\ 0\le x\le\pi.\end{aligned}\tag{22}$$

By changing x to $\pi-x$ and using the jump conditions (4) and Definition 2.1, we obtain the asymptotic forms of $w(x,\lambda)$ and $D^\alpha w(x,\lambda)$. Specially,

$$\begin{aligned}|w(x,\lambda)|&=O\left(\exp\left(\frac{|\tau|}{\alpha}(\pi-x)^\alpha\right)\right),\\ |D^\alpha w(x,\lambda)|=|x^{1-\alpha}w'(x,\lambda)|&=O\left(|\rho|\exp\left(\frac{|\tau|}{\alpha}(\pi-x)^\alpha\right)\right),\quad 0\le x\le\pi.\end{aligned}\tag{23}$$

Moreover, from Eqs. (2) and Remark 2.4 we set

$$\begin{aligned}\Delta(\lambda):&=W_\alpha(v(\lambda),w(\lambda))\\ &=B_1(w(\lambda))\\ &=-r(\pi)B_2(v(\lambda))\\ &=r(p)\left(c_sv(p,\lambda)D^\alpha w(p,\lambda)-b_sD^\alpha v(p,\lambda)w(p,\lambda)\right).\end{aligned}\tag{24}$$

From Eq. (24) the characteristic function $\Delta(\lambda)$ is the composition of the solutions and from [10] it is known that each solutions are entire function of order $\frac{1}{2}$. Consequently $\Delta(\lambda)$ is an entire function of order $\frac{1}{2}$ whose roots λ_n coincide with the eigenvalues of L. The asymptotic form of characteristic function satisfies

$$\begin{aligned}\Delta(\lambda) =&\rho r(\pi)\Big[a_1a_2\dots a_{m-1}\sin\Big(\frac{\rho}{\alpha}\pi^\alpha\Big)+a_1'a_2\dots a_{m-1}\sin\Big(\frac{\rho}{\alpha}(\pi^\alpha-2p_1^\alpha)\Big)+\cdots\\
&+a_1a_2\dots a_{m-1}'\sin\Big(\frac{\rho}{\alpha}(\pi^\alpha-2p_{m-1}^\alpha)\Big)+a_1'a_2'a_3\dots a_{m-1}\sin\Big(\frac{\rho}{\alpha}(\pi^\alpha+2p_1^\alpha-2p_2^\alpha)\Big)\\
&+\cdots+a_1\dots a_i'\dots a_j'\dots a_{m-1}\sin\Big(\frac{\rho}{\alpha}(\pi^\alpha+2p_i^\alpha-2p_j^\alpha)\Big)\\
&+a_1\dots a_i'\dots a_j'\dots a_k'\dots a_{m-1}\sin\Big(\frac{\rho}{\alpha}(\pi^\alpha-2p_i^\alpha+2p_j^\alpha-2p_k^\alpha)\Big)+\cdots\\
&+a_1'a_2'\dots a_{m-1}'\sin\Big(\frac{\rho}{\alpha}(\pi^\alpha+2(-1)^{m-1}d_1^\alpha+2(-1)^{m-2}p_2^\alpha-2p_m^\alpha)\Big)\Big]\\
&+O\Big(\exp\Big(\frac{|\tau|}{\alpha}\pi^\alpha\Big)\Big).\end{aligned}\tag{25}$$

We need to the following lemma on asymptotic, poles and residues determining a meromorphic Herglotz–Nevanlinna function, see Theorem 2.3 in [7].

Lemma 3.2. *Suppose that the functions $h_1(z)$ and $h_2(z)$ are two meromorphic Herglotz–Nevanlinna functions with the same sets of poles and residues. If*

$$h_1(it)-h_2(it)\to 0,\quad \textit{as } t\to\infty,$$

then $h_1=h_2$.

Define the Weyl–Titchmarsh $\mathfrak{m}$-functions

$$\mathfrak{m}_-(\lambda)=-\frac{D^\alpha v(p,\lambda)}{v(p,\lambda)},\qquad \mathfrak{m}_+(\lambda)=\frac{D^\alpha w(p,\lambda)}{w(p,\lambda)}.\tag{26}$$

As a consequence of theorem ([7, Thms. 2.1 and 2.2]) we obtain:

Lemma 3.3. *The functions $\mathfrak{m}_-(\lambda)$ and $\mathfrak{m}_+(\lambda)$ are the Herglotz–Nevanlinna functions, (i.e. it maps the upper half plane to the upper half plane).*

Proof. Suppose that the functions v and $\bar{v}$ are solutions of $\ell_1 v=\lambda v$ and $\overline{\ell_1 v}=\ell_1\bar{v}=\bar{\lambda}\bar{v}$. It is easy to see that

$$(\lambda-\bar{\lambda})\int_0^x v(t)\bar{v}(t)r_1(t)\mathrm{d}_\alpha t=W_\alpha(v,\bar{v})(x)-W_\alpha(v,\bar{v})(0).$$

From definition of $\mathfrak{m}_-(\lambda)$ in the point $x=p$ and the condition (18), we get

$$\mathrm{Im}(\lambda)\|v\|_{T_1}^2=\mathrm{Im}(\mathfrak{m}_-(\lambda))|v(t)|^2.$$

Then the function $\mathfrak{m}_-(\lambda)$ is Herglotz–Nevanlinna function. Similarly the function $\mathfrak{m}_+(\lambda)$ is also Herglotz–Nevanlinna function. □

Lemma 3.4. *For any $\varepsilon>0$, if $\varepsilon<\arg\lambda<2\pi-\varepsilon$, then $\mathfrak{m}_-(\lambda)$ and $\mathfrak{m}_+(\lambda)$ have the following asymptotic behavior*

$$\mathfrak{m}_+(\lambda)=i\sqrt{\lambda}+o(\sqrt{\lambda}),\quad \mathfrak{m}_-(\lambda)=i\sqrt{\lambda}+o(\sqrt{\lambda}),\quad \textit{as } \lambda\to\infty.\tag{27}$$

Specially, when $\lambda\to-\infty$, we have

$$\mathfrak{m}_+(\lambda)=-\sqrt{|\lambda|}+o(\sqrt{|\lambda|}),\quad \mathfrak{m}_-(\lambda)=-\sqrt{|\lambda|}+o(\sqrt{|\lambda|})\quad \textit{as } \lambda\to-\infty.\tag{28}$$

Proof. Using the asymptotic forms of $v(x,\lambda)$ and $D^\alpha v(x,\lambda)$ in (19) and (20) and similar asymptotic forms for $w(x,\lambda)$ and $D^\alpha w(x,\lambda)$. It is easy to check that the asymptotic forms of $\mathfrak{m}_-(\lambda)$ and $\mathfrak{m}_+(\lambda)$ are satisfying in (27)–(28). □

Suppose that the eigenvalues of the CFSLPs (5)–(7) and CFSLPs (6)–(10) are denoted by $\{\mu_n\}_{n=1}^\infty$ and $\{\nu_n\}_{n=1}^\infty$, respectively. In this part, we express the main uniqueness theorem for the of problems (1)–(10). For the uniqueness theorem we need using the similar operators $\tilde{L}_i$, with operators L_i but with different coefficients $\tilde{q}(x)$, $\tilde{h}$, $\tilde{H}$, $\tilde{H}_1$, $\tilde{b}_k$, $\tilde{c}_k$, $\tilde{d}_k$, $\tilde{p}_k$. Given a function

$$f(\lambda) := \begin{cases} -\frac{\Delta(\lambda)}{r(p)v(p,\lambda)w(p,\lambda)}, & H_1 = \infty, \\ -\frac{\Delta(\lambda)}{r(p)[D^\alpha v(p,\lambda)+H_1 v(p,\lambda)][D^\alpha w(p,\lambda)+H_2 w(p,\lambda)]}, & H_1 \neq \infty. \end{cases} \tag{29}$$

It is easy to check that $f(\lambda)$ is a meromorphic function and the set of poles of $f(\lambda)$ is all values of $\{\mu_n\}_{n=1}^\infty \cup \{\nu_n\}_{n=1}^\infty$. Using Eq. (24) and $H_2 = \frac{c_s}{b_s} H_1$, we have

$$\begin{aligned} f(\lambda) &= \begin{cases} -c_s \frac{D^\alpha w(p,\lambda)}{w(p,\lambda)} + b_s \frac{D^\alpha v(p,\lambda)}{v(p,\lambda)}, & H_1 = \infty, \\ -c_s \frac{v(p,\lambda)}{D^\alpha v(p,\lambda)+H_1 v(p,\lambda)} + b_s \frac{w(p,\lambda)}{D^\alpha w(p,\lambda)+H_2 w(p,\lambda)}, & H_1 \neq \infty, \end{cases} \\ &:= \mathcal{M}_+(\lambda) + \mathcal{M}_-(\lambda), \end{aligned} \tag{30}$$

where from (26)

$$\mathcal{M}_+(\lambda) = \begin{cases} -c_s \mathfrak{m}_+(\lambda), & H_2 = \infty, \\ \frac{b_s}{H_2 + \mathfrak{m}_+(\lambda)}, & H_2 \in \mathbb{R}, \end{cases} \qquad \mathcal{M}_-(\lambda) = \begin{cases} -b_s \mathfrak{m}_-(\lambda), & H_1 = \infty, \\ \frac{c_s}{\mathfrak{m}_-(\lambda) - H_1}, & H_1 \in \mathbb{R}. \end{cases} \tag{31}$$

Lemma 3.5. *Fixed $H_1 \in \mathbb{R} \cup \{\infty\}$. For any $\varepsilon > 0$, if $\varepsilon < \arg\lambda < 2\pi - \varepsilon$, then $\mathcal{M}_-(\lambda)$ and $\mathcal{M}_+(\lambda)$ have the following asymptotic behavior*

$$\mathcal{M}_-(\lambda) = \begin{cases} i b_s \sqrt{\lambda} + o(\sqrt{\lambda}), & H_1 = \infty, \\ \frac{i c_s}{\sqrt{\lambda}} + o\left(\frac{1}{\sqrt{\lambda}}\right), & H_1 \in \mathbb{R}, \end{cases} \tag{32}$$

and

$$\mathcal{M}_+(\lambda) = \begin{cases} i c_s \sqrt{\lambda} + o(\sqrt{\lambda}), & H_2 = \infty, \\ \frac{i b_s}{\sqrt{\lambda}} + o\left(\frac{1}{\sqrt{\lambda}}\right), & H_2 \in \mathbb{R}. \end{cases} \tag{33}$$

Theorem 3.6. *If $\lambda_n = \tilde{\lambda}_n$, $\mu_n = \tilde{\mu}_n$, and $\nu_n = \tilde{\nu}_n$ for $n \geq 0$, and $r(x) = \tilde{r}(x)$, $h = \tilde{h}$, and $H = \tilde{H}$, and if $\{\mu_n\}_{n=1}^{+\infty}$ and $\{\nu_n\}_{n=1}^{+\infty}$ are pairwise disjoint, then $L = \tilde{L}$.*

Proof. From Lemma 3.3, $\mathfrak{m}_-(\lambda)$ and $\mathfrak{m}_+(\lambda)$ are Herglotz–Nevanlinna functions. Therefore, it is easy to check that the function $\mathcal{M}_+(\lambda)$ and $\mathcal{M}_-(\lambda)$ are Herglotz–Nevanlinna functions. The functions $\tilde{\mathfrak{m}}_-(\lambda)$, $\tilde{\mathfrak{m}}_+(\lambda)$, $\tilde{\mathcal{M}}_-(\lambda)$, $\tilde{\mathcal{M}}_+(\lambda)$, and $\tilde{f}(\lambda)$ defined by analogous manner by replacing L to $\tilde{L}$. Define the function

$$G(\lambda) := \frac{f(\lambda)}{\tilde{f}(\lambda)}.$$

Since the functions $f(\lambda)$ and $\tilde{f}(\lambda)$ have the same zeros and poles, thus $G(\lambda)$ is an entire function. Applying the Lemmas 3.4 and 3.5, we have

$$G(\lambda) = \frac{f(\lambda)}{\tilde{f}(\lambda)} = 1 + o(1)$$

for any $\varepsilon > 0$ in the sector of $\varepsilon \leq \arg\lambda \leq 2\pi - \varepsilon$. Using the Liouville's theorem, we obtain

$$G(\lambda) = 1$$

then

$$f(\lambda) = \tilde{f}(\lambda).$$

From (29) and (31), the poles of $\mathcal{M}_-(\lambda)$ and $\mathcal{M}_+(\lambda)$ are exactly the same $\{\mu_n\}_{n=1}^{\infty}$ and $\{\nu_n\}_{n=1}^{\infty}$, respectively. Then we get

$$\operatorname*{Res}_{\lambda=\mu_n} \mathcal{M}_-(\lambda) = \operatorname*{Res}_{\lambda=\mu_n} f(\lambda) \text{ and } \operatorname*{Res}_{\lambda=\nu_n} \mathcal{M}_+(\lambda) = \operatorname*{Res}_{\lambda=\nu_n} f(\lambda), \text{ for } n = 1, 2, 3, \ldots.$$

Which means that

$$\operatorname*{Res}_{\lambda=\mu_n} \mathcal{M}_-(\lambda) = \operatorname*{Res}_{\lambda=\mu_n} \tilde{\mathcal{M}}_-(\lambda) \text{ and } \operatorname*{Res}_{\lambda=\nu_n} \mathcal{M}_+(\lambda) = \operatorname*{Res}_{\lambda=\nu_n} \tilde{\mathcal{M}}_+(\lambda), \text{ for } n = 1, 2, 3, \ldots.$$

From Lemmas 3.2 and 3.5, we get

$$\mathcal{M}_-(\lambda) = \tilde{\mathcal{M}}_-(\lambda) \text{ and } \mathcal{M}_+(\lambda) = \tilde{\mathcal{M}}_+(\lambda).$$

Applying the Borg's theorem [13] for the $\mathcal{M}$-Weyl-Titchmarsh functions $\mathcal{M}_+(\lambda)$ and $\mathcal{M}_-(\lambda)$, we get

$$L = \tilde{L}.$$

□

Assuming $b_s = c_s = 1$ in Eq. (11) we have $H_1 = H_2$. From this assumptions, the main result (Theorem 3.6) can be extended to the case $p \in (p_s - 1, p_{s+1})$.

Corollary 3.7. *Let $\lambda_n = \tilde{\lambda}_n$, $\mu_n = \tilde{\mu}_n$, and $\nu_n = \tilde{\nu}_n$ for $n \geq 0$, and $r(x) = \tilde{r}(x)$, $h = \tilde{h}$, $H = \tilde{H}$, $b_s = 1$, and $c_s = 1$, and if $\{\mu_n\}_{n=1}^{+\infty}$ and $\{\nu_n\}_{n=1}^{+\infty}$ are pairwise disjoint, then $L = \tilde{L}$.*

Let $b_i = c_i = 1$, $d_i = 0$ for $i = 1, 2, \ldots, m - 1$ in Eqs. (4), then our CFSLP changes to the continuous case equation.

Corollary 3.8. *If $\lambda_n = \tilde{\lambda}_n$, $\mu_n = \tilde{\mu}_n$, and $\nu_n = \tilde{\nu}_n$ for $n \geq 0$, $h = \tilde{h}$, $H = \tilde{H}$, and if $\{\mu_n\}_{n=1}^{+\infty}$ and $\{\nu_n\}_{n=1}^{+\infty}$ are pairwise disjoint, then $L = \tilde{L}$.*

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