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A stable numerical scheme for pricing American put options under irrational behavior with rationality parameter

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Abstract. This study investigates the irrational behavior of American put options holders that results in exercising options at non-optimal times. Investors usually react to market information and consequently market movements. These emotional reactions lead to exercising options strategy at a time that might not be optimal. In this situation, we consider irrational behavior in the option pricing problem. For this, we used the proposed intensity-based models with stochastic intensity parameters. Under these models, the option pricing problem leads to a nonlinear parabolic partial differential equation (PDE) with an additional term to the PDE of the American option under rational strategy (classical American option with optimal exercise strategy) due to the intensity functions of models. In this paper, we are interested in finding a stable solution for the resulting PDE using a finite element method. For this, we show the stability of the proposed finite element method by proving some theoretical results. Our numerical experiments demonstrate the accuracy and efficiency of the proposed method to obtain fast solutions for the pricing problem of American put options under irrational behavior.

1. Introduction

During the last three decades, financial products have been used as a significant tool for hedging and risk management in modern financial markets. Among these products, financial derivatives, particularly American-style options, are more attractive due to the option holder's right to exercise the option based on his (or her) choice at any time until its expiration date and get the payoff. Thus, the pricing problem of an American option is formulated as an optimal stopping time problem [7] where the stopping time is optimal for the option holder to exercise and receive the maximum exercise value. However, in real financial markets, experimental data show that a large number of irrational behaviors lead to irrational exercises [3, 12, 13]. Various reasons may cause irrational behavior of option holders, such as emotional reactions to market movements, incorrect information, or using imperfect input data for models [12]. In addition, sometimes holding American options as a hedging strategy can lead to exercise at a time that

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might not be optimal and is known as an irrational exercise strategy. Due to the irrational behavior and the irrational exercise strategy, the pricing problem of American-style options results in overvalued option prices, so we cannot consider the classical model of American options under a rational exercise strategy. Thus, irrational exercise behavior is needed to take into account through other alternative option pricing models.

Recently in [13], under the Black-Scholes model a penalty approach is proposed for the valuation of American options to capture irrational behavior. For this, the authors assumed the exercise time as the first jump time of a point process with a stochastic intensity parameter called a rationality parameter. On the other hand, if the option holder decides to exercise the option at the non-optimal time (early or late exercise), his (or her) profit at each exercise time can be measured as the difference between the payoff and the value of the option, so the dependence of the exercise intensity in terms of the profitability can be described by the rationality parameter. The authors also provided probabilistic proof of the existence of a solution to the pricing problem and show that under the proposed model the American option prices converge to the corresponding American options under the rational strategy when the rational parameter tends to infinity.

To obtain the fair value of American put options under irrational strategy with the PDE approach, a finite difference method has been applied by authors in [2]. As far as we know, no research paper has been done on this issue with other accurate and efficient numerical methods to compare the results. This motivated us to propose a finite element method to obtain more accurate and fast solutions for the pricing problem of American put options under the irrational strategy. In this way, our main contribution in this paper is finding a stable, fast, and accurate solution for the pricing problem of American put options under irrational behavior. For this purpose, we first use a variable transformation technique to transform the pricing problem of the American put option under the irrational strategy into a nonlinear parabolic equation with constant coefficients in an infinite domain. Then the truncated problem over a finite domain is written in a variational form. A finite element method is applied to solve the variational problem for option price on a truncated domain. We then study the stability of the finite element method by proving some theoretical results. Our numerical experiments demonstrated the accuracy and efficiency of the proposed method to obtain fast solutions for the pricing problem of American put options under irrational behavior.

The paper is organized as follows: In section 2 we consider the intensity-based models to obtain an appropriate PDE with boundary conditions for the American put option under the irrational behavior of its holder, and use a transformation technique to take some numerical advantages of working on PDEs with constant coefficients. In section 3, we first write the variational form of the truncated problem and apply a finite element method. In section 4, a theorem is proven to illustrate the stability of the applied finite element method. Some numerical examples are examined in section 5 to show the accurate and fast results of applying the proposed method for American put option prices under the irrational strategy. Finally, the conclusion is presented in section 6.

2. American Put Options with Rationality Parameter

Let's fix a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ and assume that the dynamics of an underlying asset price $\{S_t, t \geq 0\}$ follows a geometric Brownian motion (GBM) under a risk-neutral probability measure $\mathbb{Q}$, which are captured by the following stochastic differential equation (SDE)

$$dS_t = rS_t dt + \sigma S_t dW_t,$$

where, r denotes the constant risk-free interest rate, $\sigma > 0$ is the constant instantaneous volatility of the asset and W_t represents the Wiener process (standard Brownian motion).

Now, if we consider an American put option on the underlying asset S_t, with the strike price \$$E$, maturity time T and the exercise value which is given by the payoff function $(E - S_t)^+ = \max(E - S_t, 0)$ at time $t < T$, then the value of the put option, $P(t, S_t)$, can be characterized as the solution to the following optimal stopping time problem [8]:

$$P(t, S_t) = \sup_{\tau\in\mathscr{S}} \mathbb{E}_{\mathbb{Q}}\Big[e^{-r(\tau-t)}\Big(E - S_\tau\Big)^+ |\mathcal{F}_t\Big],$$

where, $\mathscr{S}$ is the set of the stopping times taking values in $[t, T]$ and $\mathbb{E}_{\mathbb{Q}}$ denotes the conditional expectation under the risk-neutral probability measure $\mathbb{Q}$ and filtration $\mathcal{F}_t = \sigma(\{S_s, \quad 0 \le s \le t\})$.

It is shown that there exists an optimal stopping time τ^* and corresponding rational boundary exercise price S_{τ^*}, for which the supremum is attained, (It is known as an optimal strategy) and the option should be exercised for $S \ge S_{\tau^*}$. Under the optimal strategy, we assume that the option investor is rational and his decision occurs at rational exercise time. Thus, we derive the option price as follows:

$$P(t, S_t) = \mathbb{E}_{\mathbb{Q}}[e^{-r(\tau^*-t)}\Big(E - S_{\tau^*}\Big)^+ |\mathcal{F}_t].$$

However, irrational behavior as a reaction to real market movements can lead to an exercising option strategy (at a time τ up to the time of the contract) which might not be an optimal exercising time and consequently a non-optimal strategy. Under the irrational exercising time strategy, the pricing problem of the American style option results in an overvalued price. Thus, to study the irrational behavior of option holders and its impact on the option pricing problem, the authors in [13] introduced a rationality parameter $\lambda > 0$ associated with an intensity function $f^\lambda : [-E, E] \longrightarrow [0, \infty]$ which denotes the differences between the American option payoff and its value under the corresponding exercise strategy τ which is not necessarily optimal time:

$$f^\lambda\Big[(E - S_t)^+ - P(t, S_t; \tau)\Big], \quad 0 \le t \le T,$$

where $P(t, S_t; \tau)$ is American put option price under irrational exercise strategy τ,

$$P(t, S_t; \tau) = \mathbb{E}_{\mathbb{Q}}\Big[e^{-r(\tau-t)}\Big(E - S_\tau\Big)^+ |\mathcal{F}_t\Big].$$

Theorem 2 in [13], states that the value of the American put option under irrational exercise strategy τ converges to the value of corresponding American put option $P(t, S_t)$ with rational exercise strategy τ^* when λ tends to infinity:

$$\lim_{\lambda\to\infty} P(t, S_t; \tau) = P(t, S_t), \quad 0 \le t \le T.$$

For more details, see Theorem 2 in [13].

Now to obtain the American put option price under the irrational exercise strategy τ, we assume that τ is the first jump time of a point process with stochastic intensity $\mu_t = \alpha(t, S_t)$, which α is a positive deterministic measurable function [13]. In this paper, α is defined as follows [2]:

$$\alpha(t, S_t) = f^\lambda\Big[(E - S_t)^+ - P(t, S_t; \tau)\Big].$$

To consider a family of intensity functions f^λ, the parameter λ must satisfy the condition of rationality parameter according to Theorem 2 in [13]. The two following intensity functions have been defined in [13]:

$$f_1^\lambda(x) = \begin{cases} \lambda & x \ge 0, \\ 0 & x < 0, \end{cases} \tag{1}$$

and

$$f_2^\lambda(x) = \lambda e^{\lambda^2 x}. \tag{2}$$

Under the first family of intensity functions, the option holder certainly does not exercise when it is not profitable. Thus, the non-optimal behavior of option holders is to exercise too late. However, the second family of functions [13] shows that the option holder is affected by profitability.

Recently, more functions are proposed to use as intensity functions, which we mention the two followings:

$$f_3^\lambda(x) = \frac{2\lambda}{1 + e^{-\lambda^2 x}}, \tag{3}$$

$$f_4^\lambda(x) = \lambda(1 + \frac{2}{\pi}\arctan \lambda^2 x). \tag{4}$$

Now let define

$$\mathcal{G}_t = \sigma(\{1_{\{\tau\leq s\}}, \quad 0 \leq s \leq t\}),$$

and apply the following Lemma from [9] to price American put options under the irrational strategy τ.

Lemma 2.1. *Under the assumption*

$$\mathbb{E}_{\mathbb{Q}}\Big[\int_t^T \Big(\alpha(u,S_u)e^{-r(u-t)-\int_t^\tau \alpha(u,S_u)du}\Big(E-S_u\Big)^+\Big)du\Big] < \infty,$$

we have

$$\mathbb{E}_{\mathbb{Q}}\Big[1_{\{\tau\geq T\}}|\mathcal{F}_T \vee \mathcal{G}_t\Big] = 1_{\{\tau>t\}}e^{-\int_t^T \alpha(u,S_u)du},$$

thus,

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\Big[e^{-r(\tau-t)}(E-S_\tau)^+1_{\{\tau\geq T\}}|\mathcal{F}_T \vee \mathcal{G}_t\Big]\\ &\qquad = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_t^T \alpha(u,S_u)du}\Big(E-S_T\Big)^+|\mathcal{F}_t\Big],\end{aligned} \tag{5}$$

and

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\Big[e^{-r(\tau-t)}\Big(E-S_\tau\Big)^+1_{\{\tau<T\}}|\mathcal{F}_t \vee \mathcal{G}_t\Big]\\ &\quad = \int_t^T e^{-r(u-t)}\Big(\mathbb{E}_{\mathbb{Q}}\Big[\alpha(u,S_u)e^{-\int_t^\tau \alpha(u,S_u)du}\Big(E-S_u\Big)^+|\mathcal{F}_t\Big]\Big)du.\end{aligned} \tag{6}$$

Proof. See Proposition 3.1 in ref. [9]. □

Under an irrational exercise strategy τ, the American put option price at each time t is given by the expression

$$\begin{aligned}P(t,S_t;\tau) &= \mathbb{E}_{\mathbb{Q}}\Big[e^{-r(\tau-t)}\Big(E-S_\tau\Big)^+|\mathcal{F}_t\Big]\\ &= \mathbb{E}_{\mathbb{Q}}\Big[e^{-r(\tau-t)}\Big(E-S_\tau\Big)^+1_{\{\tau\geq T\}}|\mathcal{F}_T \vee \mathcal{G}_t\Big]\\ &+ \mathbb{E}_{\mathbb{Q}}\Big[e^{-r(\tau-t)}\Big(E-S_\tau\Big)^+1_{\{\tau<T\}}|\mathcal{F}_t \vee \mathcal{G}_t\Big].\end{aligned} \tag{7}$$

Using the Lemma (2.1), expression (7) can be rewritten as

$$\begin{aligned}P(t,S_t;\tau) &= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_t^T \alpha(u,S_u)du}\Big(E-S_T\Big)^+|\mathcal{F}_t\Big]\\ &+ \int_t^T e^{-r(u-t)}\mathbb{E}_{\mathbb{Q}}\Big[\alpha(u,S_u)e^{-\int_t^\tau \alpha(u,S_u)du}\Big(E-S_u\Big)^+|\mathcal{F}_t\Big]du.\end{aligned} \tag{8}$$

In the rest of this paper, to emphasize the significance of the intensity parameter on option price, we denote the value of the American put option under irrational strategy τ and the corresponding intensity parameter λ by $P(t,S_t;\lambda)$.

Now consistent with Feynman-Kac Theorem [9] and applying Ito's Lemma, we obtain the subsequent nonlinear Black-Scholes equation for option price $P(t,S_t;\lambda)$ corresponding to price (8) under exercise strategy τ, intensity parameter λ, time to maturity $t \in (0,T]$ and S_t in the unbounded domain $(0,\infty)$ [2]:

$$\begin{aligned}\frac{\partial P}{\partial t}(t,S_t;\lambda) &= \frac{1}{2}\sigma^2S^2\frac{\partial^2 P}{\partial S^2}(t,S_t;\lambda) + rS\frac{\partial P}{\partial S}(t,S_t;\lambda) - rP(t,S_t;\lambda)\\ &\quad + \Big((E-S_t)^+ - P(t,S_t;\lambda)\Big)f^\lambda\Big((E-S_t)^+ - P(t,S_t;\lambda)\Big),\end{aligned} \tag{9}$$

with the initial condition

$$P(0, S; \lambda) = (E - S)^+, \tag{10}$$

and boundary conditions:

$$\lim_{S\to\infty} P(t, S; \lambda) = 0, \tag{11}$$

$$\frac{\partial P}{\partial t}(t, 0; \lambda) = -rP(t, 0; \lambda) + \Big(E - P(t, 0; \lambda)\Big) f^{\lambda}\Big((E - P(t, 0; \lambda)\Big). \tag{12}$$

Thus, based on the intensity parameter λ and the corresponding intensity function f^{λ}, the value of the American put option can be governed by equation (9) with initial and boundary conditions (10)-(12).

To value American put options under the irrational strategy, we need to solve the pricing problem (9)-(12) under the intensity functions (1)-(4) with a numerical approach. To take some numerical advantages of working on PDEs with constant coefficients, we first use the following variable transform [5]:

$$S = Ee^x, \quad P(T - t, S; \lambda) = Ee^{-\alpha x - \beta t} u(t, x; \lambda),$$

where α and β are constants to be determined.
By simple calculation, the pricing problem (9)-(12) of American put options under irrational exercise strategy corresponding to intensity parameter $\lambda > 0$ is transformed to the following PDE for $(t, x) \in ([0, T] \times \mathbb{R})$ with constant coefficients:

$$\frac{\partial u}{\partial t}(t, x; \lambda) - \gamma \frac{\partial^2 u}{\partial x^2}(t, x; \lambda) + \nu \frac{\partial u}{\partial x}(t, x; \lambda) + \varepsilon u(t, x; \lambda) + \Gamma(t, x) f^{\lambda}\Big(E\Gamma(t, x)\Big) = 0, \tag{13}$$

with the initial condition

$$u(0, x; \lambda) = g(0, x), \tag{14}$$

and boundary conditions:

$$\lim_{x\to+\infty} u(t, x; \lambda) = \lim_{x\to+\infty} g(t, x), \tag{15}$$

$$\lim_{x\to-\infty} \frac{\partial u}{\partial t}(t, x; \lambda) + \varepsilon u(t, x; \lambda) + \Big(1 - u(t, x; \lambda)\Big) f^{\lambda}\Big(E\Big(1 - u(t, x; \lambda)\Big)\Big) = 0, \tag{16}$$

where the coefficients γ, ν and ε are defined as

$$\gamma = \frac{\sigma^2}{2}, \qquad \nu = \gamma(1 + 2\alpha) - r, \qquad \varepsilon = r + \alpha r - \gamma\alpha(1 + \alpha) - \beta,$$

and the functions $g(t, x)$ and $\Gamma(t, x)$ are defined as follows:

$$g(t, x) = e^{\alpha x + \beta t}(1 - e^x)^+, \qquad \Gamma(t, x) = g(t, x) - u(t, x; \lambda).$$

Now to find the American put option prices with a numerical approach, we solve the approximation pricing problem on a bounded domain. For this, we truncate the problem over a bounded interval $\Omega = (X_1, X_2)$ for large negative number X_1 and positive number X_2. Thus, problem (13)-(16) for $\lambda > 0$ turns to the following truncated problem on bounded domain $(t, x) \in ([0, T] \times \Omega)$:

$$\frac{\partial u}{\partial t}(t, x; \lambda) - \gamma \frac{\partial^2 u}{\partial x^2}(t, x; \lambda) + \nu \frac{\partial u}{\partial x}(t, x; \lambda) + \varepsilon u(t, x; \lambda) + \Gamma(t, x) f^{\lambda}\Big(E\Gamma(t, x)\Big) = 0, \tag{17}$$

$$u(0, x; \lambda) = g(0, x), \tag{18}$$

$$u(t, X_2; \lambda) = g(t, X_2), \tag{19}$$

$$\frac{\partial u}{\partial t}(t, X_1; \lambda) + \varepsilon u(t, X_1; \lambda) + \Big(1 - u(t, X_1; \lambda)\Big) f^{\lambda}\Big(E(1 - u(t, X_1; \lambda))\Big) = 0. \tag{20}$$

3. Finite Element Method

In this section, we propose a finite element method to solve the approximation problem (17)-(20). Let define the space $H_0^1(\Omega)$ and the admissible space V as follows [6]:

$$H_0^1(\Omega) = \{v : \quad v \in L^2(\Omega), \quad v_x \in L^2(\Omega), \quad v|_{\partial\Omega} = 0\},$$

and

$$V = \{v : \quad v \in L^2(0,T;H_0^1(\Omega)), \quad v_t \in L^2(0,T;H^{-1}(\Omega))\},$$

where $L^2(\Omega)$ is the space of square-integrable functions on Ω and $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$.
We denote the inner product of $L^2(\Omega)$ by $\langle .,. \rangle$ and define the following bilinear forms for $u \in V$ and $v \in H_0^1(\Omega)$:

$$a\langle u,v\rangle = \gamma\langle\frac{\partial u}{\partial x},\frac{\partial v}{\partial x}\rangle + \nu\langle\frac{\partial u}{\partial x},v\rangle + \varepsilon\langle u,v\rangle, \qquad b\langle u,v\rangle = \langle Ru,v\rangle,$$

where

$$Ru = \Gamma(t,x)f^{\lambda}\Big(E\Gamma(t,x)\Big).$$

Then the variational form for problem (17)-(20) is as follows:
Find $u \in V$ such that for $v \in H_0^1(\Omega)$ and $0 \le t \le T$

$$\langle\frac{\partial u}{\partial t},v\rangle + a\langle u,v\rangle + b\langle u,v\rangle = 0, \tag{21}$$

$$u(t,X_2;\lambda) = g(t,X_2), \tag{22}$$

$$\frac{\partial u}{\partial t}(t,X_1;\lambda) + \varepsilon u(t,X_1;\lambda) + \Big(1-u(t,X_1;\lambda)\Big)f^{\lambda}\Big(E(1-u(t,X_1;\lambda))\Big) = 0. \tag{23}$$

Now to obtain a discrete solution for the variational problem (21)-(23), we need to define a finite subspace $V_h \subset V$ of piecewise linear functions with basic functions

$$\varphi_j(x_i) = \delta_{i,j},$$

for any $j = 1,2,\ldots,N$ and $i = 0,1,2,\ldots,N$ where $\delta_{i,j}$ is the Kronecker delta.
Under the spatial partition $\Delta_x : X_1 = x_0 < x_1 < \cdots < x_N = X_2$ and the time partition $\Delta_t : 0 = t_0 < t_1 < \cdots < t_M = T$, we show the discrete solutions of the variational problem (21)-(23) by $u_i^j = u(t_j,x_i)$ for any $i = 0,1,\ldots,N$ and $j = 0,1,\ldots,M$.
Therefore, for $u_h^m \in V_h$ with $h = \max_{1\le j\le N}(x_j - x_{j-1})$, there are constant coefficients u_j^m such that

$$u_h^m(x) = \sum_{j=1}^{N} u_j^m \varphi_j(x).$$

Now the finite element approximation to the variational problem (21)-(23) is as follows [5]:
For $m = 1,2,\ldots,M$ find $u_h^m \in V_h$ such that

$$\langle\delta_t u_h^m,v\rangle + \mathcal{L}\langle u_h^{m-\theta},v\rangle = 0, \tag{24}$$

$$u_N^m = g(t_m,x_N), \tag{25}$$

$$\delta_t u_0^m + \varepsilon u_0^m + \Big(1-u_0^m\Big)f^{\lambda}\Big(E(1-u_0^m)\Big) = 0, \tag{26}$$

where linear operator $\mathcal{L}$ is defined as

$$\mathcal{L}\langle u_h^{m-\theta},v\rangle = a\langle u_h^{m-\theta},v\rangle + b\langle u_h^{m-\theta},v\rangle,$$

and

$$\delta_t u_h^m = \frac{u_h^m - u_h^{m-1}}{\Delta t_m}, \qquad \Delta t_m = t_m - t_{m-1},$$

$$u_h^{m-\theta} = (1-\theta)u_h^m + \theta u_h^{m-1},$$

for $\theta \in [0,1]$.

By definition $\alpha_j = E\Big((1-e^x)^+ - e^{-\alpha x-\beta t} u_j^{m-\theta}\varphi_j(x)\Big)$ and simple calculation, we also obtain following integrals for $i, j = 1, 2, \ldots, N$

$$b\langle u_h^{m-\theta}, \varphi_i(x)\rangle = \int_{X_1}^{X_2} g(t,x) f^{\lambda}(\alpha_j)\varphi_i(x)dx - \int_{X_1}^{X_2} u_h^{m-\theta} f^{\lambda}(\alpha_j)\varphi_i(x)dx.$$

Then, the scheme (24)-(26) can be presented in the following vector form:

$$AU^m + (\gamma B + \nu C + \mu A + D)U^{m-\theta} = L, \tag{27}$$

$$u_N^m = g(t_m, x_N), \tag{28}$$

$$u_0^m = u_0^{m-1} - \varepsilon(t_m - t_{m-1})u_0^m - \Big(1 - u_0^m\Big) f^{\lambda}\Big(E(1 - u_0^m)\Big), \tag{29}$$

where

$$A = (\varphi_j, \varphi_i)_{N\times N} = \begin{cases} \frac{2h}{3}, & i = j, \\ \frac{h}{6}, & |i-j| = 1, \\ 0, & otherwise, \end{cases}$$

$$B = (\varphi_j', \varphi_i')_{N\times N} = \begin{cases} \frac{2}{h}, & i = j, \\ \frac{-1}{h}, & |i-j| = 1, \\ 0, & otherwise, \end{cases}$$

$$C = (\varphi_j', \varphi_i)_{N\times N} = \begin{cases} -\frac{1}{2}, & i = j = 1, \\ \frac{1}{2}, & i = j = N, \\ \frac{1}{2}, & |i-j| = 1, \\ 0, & otherwise, \end{cases}$$

$$D = \begin{bmatrix} \int_\Omega \varphi_1(x)\varphi_1(x) f^{\lambda}(\alpha_1)dx & \ldots & \int_\Omega \varphi_1(x)\varphi_N(x) f^{\lambda}(\alpha_N)dx \\ \int_\Omega \varphi_2(x)\varphi_1(x) f^{\lambda}(\alpha_1)dx & \ldots & \int_\Omega \varphi_2(x)\varphi_N(x) f^{\lambda}(\alpha_N)dx \\ \vdots & \vdots & \vdots \\ \int_\Omega \varphi_N(x)\varphi_1(x) f^{\lambda}(\alpha_1)dx & \ldots & \int_\Omega \varphi_N(x)\varphi_N(x) f^{\lambda}(\alpha_N)dx \end{bmatrix},$$

$$L = \begin{bmatrix} \int_\Omega g(t,x)\varphi_1(x) f^{\lambda}(\alpha_1)dx \\ \int_\Omega g(t,x)\varphi_2(x) f^{\lambda}(\alpha_2)dx \\ \vdots \\ \int_\Omega g(t,x)\varphi_N(x) f^{\lambda}(\alpha_N)dx \end{bmatrix},$$

and

$$U^m = (u_1^m, u_2^m, \ldots, u_N^m)^T.$$

It can be seen that the above matrices are three-diagonal and definite positive. So we use the Thomas algorithm to find the solutions to the equation.

4. Stability Analysis

In this section, we study the stability of the proposed finite element method which is used in the previous section. For this purpose, we apply the approach in [1, 5].

Let substitute $v = u_h^{m-\theta}$ in (24), therefore

$$\langle \delta_t u_h^m, u_h^{m-\theta} \rangle + \mathcal{L}\langle u_h^{m-\theta}, u_h^{m-\theta} \rangle = 0. \tag{30}$$

We use the following equation for the first term in (30) (see Appendix A.)

$$\langle \delta_t u_h^m, u_h^{m-\theta} \rangle = \frac{1}{2\Delta t_m}\left[\|u_h^m\|^2 - \|u_h^{m-1}\|^2 + (1-2\theta)\|u_h^m - u_h^{m-1}\|^2 \right]. \tag{31}$$

From the inner product definition, we also have

$$\langle u_{hx}^{m-\theta}, u_{hx}^{m-\theta} \rangle = \|u_{hx}^{m-\theta}\|^2, \qquad \langle u_h^{m-\theta}, u_h^{m-\theta} \rangle = \|u_h^{m-\theta}\|^2. \tag{32}$$

Applying integration by parts, we obtain

$$\langle u_{hx}^{m-\theta}, u_h^{m-\theta} \rangle = \frac{1}{2}\left[|u_h^{m-\theta}(X_2)|^2 - |u_h^{m-\theta}(X_1)|^2 \right]. \tag{33}$$

Thus, we have the following theorem about the stability of the proposed finite element method.

Theorem 4.1. *Assume that the constant parameters α and β are determined such that $\varepsilon \geq 0$ and $\nu \geq 0$. For $u_h^m \in H_0^1 \cap H^2$, the system* (24)-(26) *is stable when $\theta = \frac{1}{2}$ and $\theta = 0$, and we have*

$$\|u_h^M\| \leq \|u_h^0\| + C,$$

where C is a positive constant independent of time.

Proof. According to the results (31)-(33) the equation (30) turns to

$$\frac{1}{2\Delta t_m}\left[\|u_h^m\|^2 - \|u_h^{m-1}\|^2 + (1-2\theta)\|u_h^m - u_h^{m-1}\|^2 \right] + \gamma\|u_{hx}^{m-\theta}\|^2 + \varepsilon\|u_h^{m-\theta}\|^2 + \frac{\nu}{2}\left[|u_h^{m-\theta}(X_2)|^2 - |u_h^{m-\theta}(X_1)|^2 \right] + \langle R u_h^{m-\theta}, u_h^{m-\theta} \rangle = 0 \tag{34}$$

It is obvious that $\gamma \geq 0$ and for $\theta = 0$ or $\frac{1}{2}$, we also have $1 - 2\theta \geq 0$. On the other hand, according to the discussion in [2], we know that the term $R u_h^{m-\theta}$ in (34) is non-negative. Therefore, the equation (34) turns to

$$\|u_h^m\|^2 - \|u_h^{m-1}\|^2 + 2\varepsilon\Delta t_m\|u_h^{m-\theta}\|^2 + \nu\Delta t_m\left[|u_h^{m-\theta}(X_2)|^2 - |u_h^{m-\theta}(X_1)|^2 \right] \leq 0 \tag{35}$$

Note that $u_h^{m-\theta}(X_2) = g^{m-\theta}(X_2) = e^{\alpha X_2 + \beta(m-\theta)}(1 - e^{X_2})^+$ is bounded and positive.

Now assuming that the constants α and β are chosen so that $\nu \geq 0$ and $\varepsilon \geq 0$, we get

$$\|u_h^m\|^2 - \|u_h^{m-1}\|^2 \leq \nu\Delta t_m |u_h^{m-\theta}(X_1)|^2. \tag{36}$$

Then by summing (36) for time steps $m = 1, \ldots, M$ we have the desired result,

$$\|u_h^M\|^2 \leq \|u_h^0\|^2 + C,$$

where C is a positive constant independent of time steps. □

5. Numerical Implementations

In this section, we illustrate some numerical examples to show the performance of the finite element method to solve the pricing problem of American put options under irrational behavior. Our numerical implementations verify the accuracy and stability of our computational method compared to some recent results.

Example 1: In this example, we consider an American put option with the possibility of the irrational exercise of its holder with parameters as follows:

$$r = 0.05, \qquad \sigma = 0.2, \qquad T = 3, \qquad S = E = 100.$$

By applying the proposed finite element scheme, the American put option price with irrational strategy, for the different values of the rational parameter λ under the intensity functions (1)-(4), and considering $M = 10000$ (the number of time steps) and $N = 600$ (the number of spatial steps) on the finite interval $(X_1, X_2) = (-3, 3)$ are illustrated in Table 1. Our numerical results verify that the American put option prices under irrational strategy tend to the classical American put option prices as λ is increasing. In addition, it can be seen that under the intensity function (2), for smaller values of λ, the option price converges to the corresponding American put option price.

To investigate whether the proposed approach still meets the desired expectations under the different values of volatility, we implemented Example 1 for volatilities $\sigma = 0.25$ and $\sigma = 0.3$. This implies that the proposed method is robust and can handle different variations in volatility without significantly affecting the results. The evidence supporting this claim can be found in both Table 2 and Table 3, which show that the results obtained using the proposed approach remain consistent even when different values of volatility are used.

To show the accuracy of our algorithm, we also compared our results with other numerical techniques, like the Penalty method [4] and the Tree method [11].

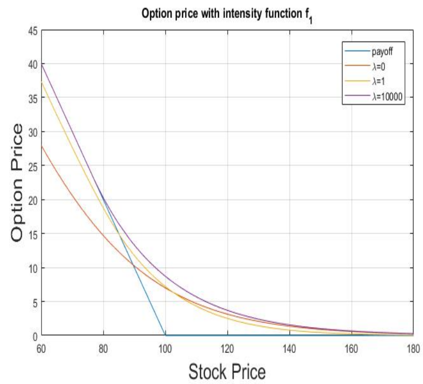

Figure 1: Numerical option prices with the intensity function belonging to family (1) under different values of the rational parameter λ by applying the finite element method for $M = 10000$ and $N = 600$.

Additionally, in Fig. 1 for the family of the intensity function (1) and different values of the rational parameter λ the American put option price is shown. Note that the case $\lambda = 0$ corresponds to the European option, while $\lambda = 1$ corresponds to a near-zero rationality parameter which can be understood as a case with a large irrational exercise and the value of the American put option is below the exercise value for small values of the asset. This situation may be caused by additional circumstances that prevent the owner from exercising, although the option price is below the exercise prices. For $\lambda = 10000$ the irrational case tends to be the rational one, corresponding to the American put option pricing problem.

To compare the speed of convergence for each of the intensity functions (1)-(4), Figures 2-3 are displaying the convergence rate of option prices under irrational behavior to American put option prices regard to the rational parameter λ for intensity functions f_1, f_2, f_3 and f_4, respectively. Finally, to compare the accuracy

Table 1: The speed of convergence American put option prices under irrational strategy to the values of the classical American put options with increasing λ for intensity functions (1)-(4) and $\sigma = 0.2$.

λ	f_1	f_2	f_3	f_4
0	6.9948	6.9948	6.9948	6.9948
1	7.9543	8.0815	7.7269	7.2048
10	8.6022	8.6978	8.6512	8.5291
100	8.6987	8.7105	8.7043	8.6914
1000	8.7090	8.7105	8.7096	8.7083
10000	8.7102	8.7105	8.7108	8.7107
Tree	8.7106	8.7106	8.7106	8.7106
Penalty	8.7100	8.7100	8.7100	8.7100

Table 2: The speed of convergence American put option prices under irrational strategy to the values of the classical American put options with increasing λ for intensity functions (1)-(4) and $\sigma = 0.25$.

λ	f_1	f_2	f_3	f_4
0	9.9124	9.9124	9.9124	9.9124
1	10.9065	11.1152	10.7848	10.0850
10	11.5927	11.6949	11.6458	11.5193
100	11.6952	11.6949	11.7012	11.6880
1000	11.7061	11.6949	11.7067	11.7054
10000	11.7073	11.6949	11.7084	11.7083

Table 3: The speed of convergence American put option prices under irrational strategy to the values of the classical American put options with increasing λ for intensity functions (1)-(4) and $\sigma = 0.3$.

λ	f_1	f_2	f_3	f_4
0	12.8760	12.8760	12.8760	12.8760
1	13.8994	14.1674	13.8489	13.0358
10	14.6197	14.7278	14.6762	14.5472
100	14.7276	14.7278	14.7339	14.7206
1000	14.7390	14.7278	14.7397	14.7384
10000	14.7404	14.7278	14.7423	14.7421

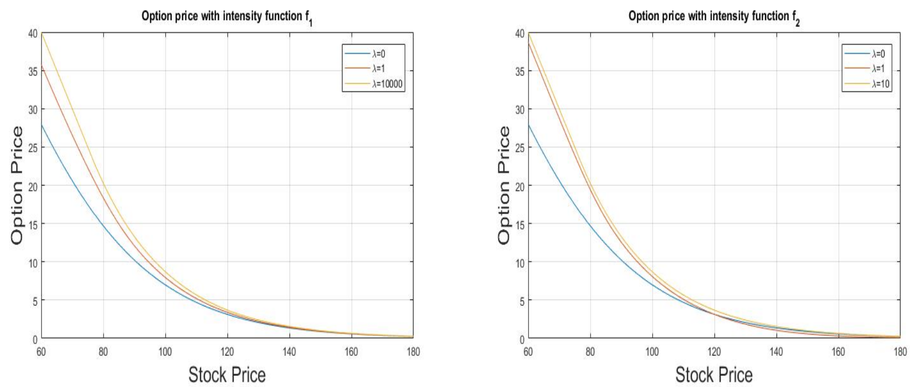

Figure 2: The speed of convergence American put option price under irrational behavior to corresponding American put option price for intensity functions f_1 and f_2 by applying the finite element method for $M = 10000$ and $N = 600$.

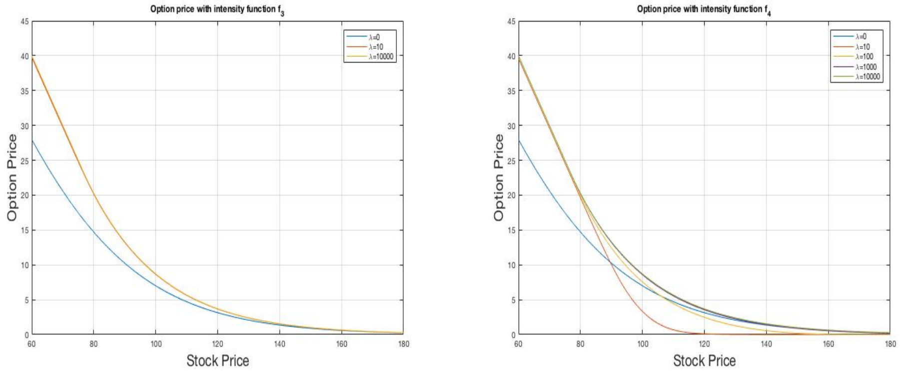

Figure 3: The speed of convergence American put option price under irrational behavior to corresponding American put option price for intensity functions f_3 and f_4 by applying the finite element method for $M = 10000$ and $N = 600$.

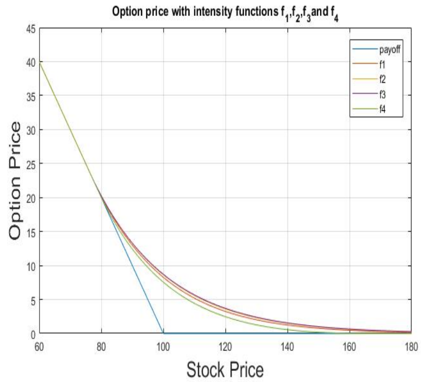

Figure 4: Comparing American put option prices for different intensity functions f_1, f_2, f_3 and f_4.

of each intensity function, we display the graphs of intensity functions f_1, f_2, f_3, and f_4 in Figure 4. **Example 2:** As we mentioned in example 1, our numerical results show that under the intensity function (2), for smaller values of λ, the option prices are close to the corresponding American put option price. To see that this feature is satisfied for the American options problem independent of the pricing problem parameters, in this example we consider an American put option with expiry $T = 1$ year, strike price $E = 9$, interest rate $r = 0.02$, and volatility $\sigma = 0.2$. Table 4 illustrates our results obtaining option prices for intensity functions (1)-(4) and different values of intensity parameter λ. It can be seen that American put option prices can be obtained for large enough values of λ. We also see that under the intensity function (2), for smaller values of $\lambda = 80$, the option price converges to the corresponding American put option price. Figures 5-6 show the results for the speed of convergence under the intensity functions (1)-(4).

Since exact solutions for the pricing problem of American options are unknown, we employed the double mesh technique to calculate the errors between two numerical solutions u_h and $u_{\frac{h}{2}}$ corresponding to the mesh steps h and $\frac{h}{2}$. Refining the grid points, enable us to assess the convergence rate of the proposed method under L^2-norm in the absence of an exact solution [5, 10].

Figure 7 depicts the American option price error curve under the L^2-norm for intensity functions f_1,f_2,f_3 and f_4, with a spatial step length of h. The results indicate that the method achieves second-order convergence under L^2-norm, consistent with our expectations.

Table 4: American put Option prices for different values of intensity parameter λ and intensity functions (1)-(4) by applying the finite element method for $M = 10000$ and $N = 600$.

λ	f_1	f_2	f_3	f_4
1	0.5050	0.5094	0.4950	0.4997
10	0.6350	0.6347	0.6325	0.5889
80	0.6380	0.6398	0.6391	0.6264
100	0.6390	0.6400	0.6397	0.6345
1000	0.6395	0.6400	0.6400	0.6394
10000	0.6400	0.6400	0.6401	0.6400

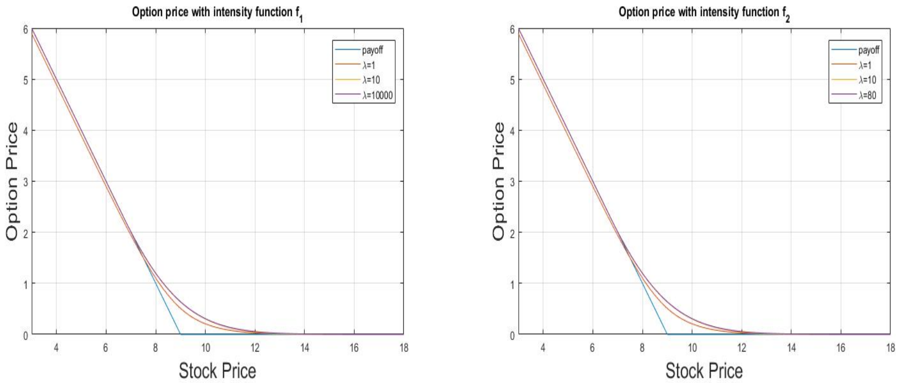

Figure 5: The speed of convergence American put option price under irrational behavior to corresponding American put option price for intensity functions f_1 and f_2 by applying the finite element method for $M = 10000$ and $N = 600$.

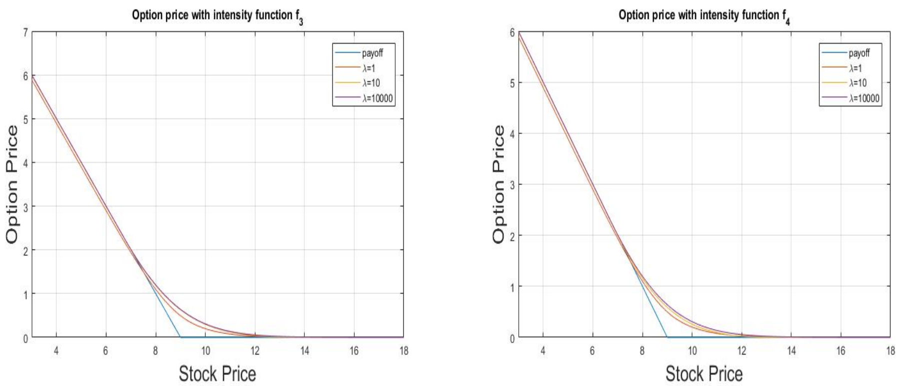

Figure 6: The speed of convergence American put option price under irrational behavior to corresponding American put option price for intensity functions f_3 and f_4 by applying the finite element method for $M = 10000$ and $N = 600$.

6. Conclusion

In this paper, we studied the irrational behavior of the American put option holder to market movements that lead to an exercising option strategy at a time that might not be an optimal time and is known as an irrational exercising time strategy. Under this situation, the pricing problem of American-style options results in overvalued option prices such that different alternative models need to be considered to incorporate possible irrational exercises. In this study, irrational exercise strategy was considered through families of intensity functions depending on rational parameters. Under these models, we obtained the resulting

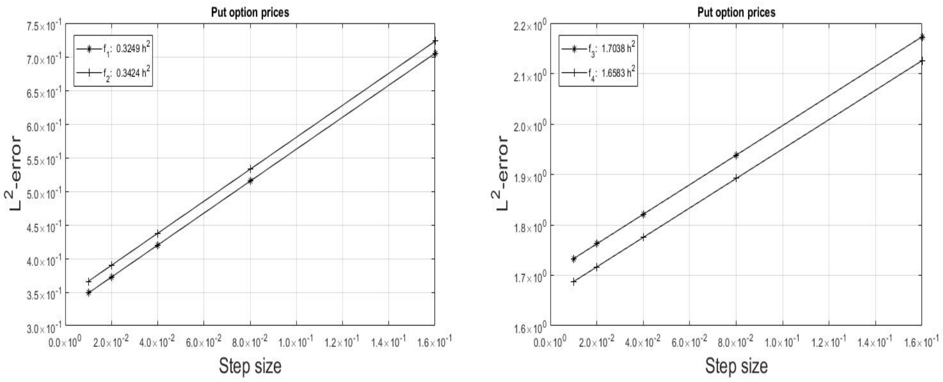

Figure 7: The convergence rates for the American put option prices under the L^2-norm.

nonlinear PDE with appropriate boundary conditions and intensity functions for American options. By using a variable transformation technique, the problem was transformed into a nonlinear parabolic equation with constant coefficients in an infinite domain. Then a finite element method was applied to solve the resulting variational problem for option price. Under some appropriate assumptions, we analyzed the stability of the proposed numerical method. Our numerical experiments also demonstrated the accuracy and efficiency of the proposed method to obtain fast solutions for the pricing problem of American put options under irrational behavior.

Appendix A

In this appendix, we try to prove the equality (31) for general values of θ. For this, from the definitions (27) and (28) and inner product property we have

$$\langle \delta_t u_h^m, u_h^{m-\theta} \rangle = \frac{1}{\Delta t_m}\Big[\Big\langle u_h^m, (1-\theta)u_h^m \rangle + \langle u_h^m, \theta u_h^{m-1} \rangle - \langle u_h^{m-1}, (1-\theta)u_h^m \rangle - \langle u_h^{m-1}, \theta u_h^{m-1} \rangle \Big]. \tag{37}$$

By adding term $\frac{1}{2}\langle u_h^{m-1}, u_h^{m-1} \rangle$, (37) turns to

$$\frac{1}{\Delta t_m}\Big[\frac{1}{2}\langle u_h^m, (1-\theta)u_h^m \rangle + \frac{1}{2}\langle u_h^m, \theta u_h^{m-1} \rangle + \frac{1}{2}\langle u_h^m, (1-\theta)u_h^m \rangle + \frac{1}{2}\langle u_h^m, \theta u_h^{m-1} \rangle - \frac{1}{2}\langle u_h^{m-1}, (1-\theta)u_h^m \rangle - \frac{1}{2}\langle u_h^{m-1}, \theta u_h^{m-1} \rangle - \frac{1}{2}\langle u_h^{m-1}, (1-\theta)u_h^m \rangle - \frac{1}{2}\langle u_h^{m-1}, \theta u_h^{m-1} \rangle + \frac{1}{2}\langle u_h^{m-1}, u_h^{m-1} \rangle - \frac{1}{2}\langle u_h^{m-1}, u_h^{m-1} \rangle \Big]. \tag{38}$$

Now by simple calculations on (38), we obtain the desired equality (31) as follows:

$$\frac{1}{2\Delta t_m}\bigg[\langle u_h^m, u_h^m\rangle - \langle u_h^{m-1}, u_h^{m-1}\rangle + (1-2\theta)\langle u_h^m, u_h^m\rangle + (1-2\theta)\langle u_h^{m-1}, u_h^{m-1}\rangle$$

$$- (1-2\theta)\langle u_h^m, u_h^{m-1}\rangle + (1-2\theta)\langle u_h^{m-1}, u_h^m\rangle\bigg]$$

$$= \frac{1}{2\Delta t_m}\bigg[\langle u_h^m, u_h^m\rangle - \langle u_h^{m-1}, u_h^{m-1}\rangle + (1-2\theta)\langle u_h^m - u_h^{m-1}, u_h^m - u_h^{m-1}\rangle\bigg]$$

$$= \frac{1}{2\Delta t_m}\bigg[\|u_h^m\|^2 - \|u_h^{m-1}\|^2 + (1-2\theta)\|u_h^m - u_h^{m-1}\|^2\bigg].$$

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