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Global asymptotic stability for a classical controlled nonlinear periodic commensalism AG-ecosystem with distributed lags on time scales

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Abstract. Commensalism is a common phenomenon in nature. The Ayala-Gilpin (AG) dynamical system model is commonly used to describe the nonlinear interactions between species in ecosystems. Combining commensalism with AG-system models, the manuscript emphasizes on a classical controlled nonlinear periodic commensalism AG-ecosystem with distributed lags on time scales. In our model, the discrete and continuous cases are unified and generalized in the sense of time scale. Firstly, it is proved that a class of auxiliary functions have only two zeros in the real number field. Then, with the aid of these auxiliary functions, using the coincidence degree theory and inequality technique, we obtain some sufficient criteria for the existence of periodic solutions. Meanwhile, we prove that the periodic solution is globally asymptotically stable by applying Lyapunov stability theory. Finally, an example is numerically simulated with the help of MATLAB tools.

1. Introduction

The manuscript stresses on a nonlinear commensalism AG-ecosystem with distributed delays on time scales as follows

$$\begin{cases} \mathcal{X}^{\Delta}(\tau) = r_1(\tau) - a_{11}(\tau)[e^{\mathcal{X}(\tau)}]^{\vartheta_1} - \varphi_1(\tau)e^{-\mathcal{X}(\tau)}, \ \tau \in \mathbb{T}, \\ \mathcal{Y}^{\Delta}(\tau) = r_2(\tau) - a_{22}(\tau)[e^{\mathcal{Y}(\tau)}]^{\vartheta_2} + a_{21}(\tau)\int_{-\xi(\tau)}^{0} k(s)e^{\mathcal{X}(\tau+s)}\Delta s - \varphi_2(\tau)e^{-\mathcal{Y}(\tau)}, \ \tau \in \mathbb{T}, \end{cases} \tag{1}$$

where $\mathbb{T}$ is a time scale, Δ is the delta derivative on $\mathbb{T}$, $\mathcal{X}(\tau)$ is the population density of the host, $\mathcal{Y}(\tau)$ is the population density of cohabitants, $r_1(\tau) > 0$ and $r_2(\tau) > 0$ represent the inherent growth rates, $a_{11}(\tau) > 0$ and $a_{22}(\tau) > 0$ are the intraspecific competition rates, $a_{21}(\tau) > 0$ represents the population growth rate of cohabitants who obtain benefits from the host without harming them, $k(\tau)$ is a kernel function of distributed delays, $\xi(\tau) > 0$ is a distributed delayed function, $\varphi_1(\tau) > 0$ and $\varphi_2(\tau) > 0$ indicate human control such as fishing and catching, the constants $\vartheta_1 > 0$ and $\vartheta_2 > 0$ measure the nonlinear interferences within species.

In an ecosystem, commensalism is one of the interactions between different species. In commensalism, one species benefits from interactions such as shelter and food, while the other species is unaffected. The beneficiary species are called cohabitants, while the unaffected species are called hosts. Commensalism is a common phenomenon, for example, sharks and sucker fish, sea cucumber and imperial shrimp, livestock

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and cattle egret, hermit crab and gastropods. Therefore, it is of great value to use mathematical models to explore the dynamic behavior of commensalism ecosystems. In 2010, Zhao and Li [40] established a Lotka-Volterra (LV) model to investigate the multiplicity of solutions to a commensalism ecosystem with two species. In addition, the AG-ecosystem model was first proposed by Ayala, Gilpin and Eherenfeld [3] in 1973. When studying the competition relationship between fruit flies, they built the following nonlinear model

$$\begin{cases} \frac{d\mathcal{X}(\tau)}{d\tau} = r_1\mathcal{X}(\tau)\left[1 - \left(\frac{\mathcal{X}(\tau)}{K_1}\right)^{\vartheta_1} - a_{12}\frac{\mathcal{Y}(\tau)}{K_2}\right], \\ \frac{d\mathcal{Y}(\tau)}{d\tau} = r_2\mathcal{Y}(\tau)\left[1 - \left(\frac{\mathcal{Y}(\tau)}{K_2}\right)^{\vartheta_2} - a_{21}\frac{\mathcal{X}(\tau)}{K_1}\right], \end{cases} \tag{2}$$

where $r_1 > 0$ and $r_2 > 0$ are natural growth rates, $K_1 > 0$ and $K_2 > 0$ express the maximum number of species in the environment without competition. The nonlinear interferences within species are measured by constants $\vartheta_1 > 0$ and $\vartheta_2 > 0$. $a_{12} > 0$ and $a_{21} > 0$ are the measures of competition between species.

Taking $\vartheta_1 = \vartheta_2 = 1$ in (2), we get the following LV-system

$$\begin{cases} \frac{d\mathcal{X}(\tau)}{d\tau} = r_1\mathcal{X}(\tau)\left[1 - \frac{\mathcal{X}(\tau)}{K_1} - a_{12}\frac{\mathcal{Y}(\tau)}{K_2}\right], \\ \frac{d\mathcal{Y}(\tau)}{d\tau} = r_2\mathcal{Y}(\tau)\left[1 - \frac{\mathcal{Y}(\tau)}{K_2} - a_{21}\frac{\mathcal{X}(\tau)}{K_1}\right]. \end{cases} \tag{3}$$

So the AG-model extends the LV-model. In fact, the AG-model also includes other models. For example, (2) is the square root model, provided that $\vartheta_1 = \vartheta_2 = \frac{1}{2}$. Consequently, many scholars have conducted extensive and in-depth research on the AG-model. Some papers [6, 8, 15, 16, 19] dealt with the persistence, extinction and attraction. The others [11, 22–25, 41] probed the existence, multiplicity and stability of solutions. Furthermore, some scholars began to investigate the AG-type models concerned with some special effects such as time-delays [6, 15, 19, 22–25, 41], impulses [8, 26, 39] and randomness [1, 13, 14, 16, 21]. Moreover, there have been some papers [2, 6, 8, 14, 20–23, 41] involving in generalized or modified AG-type models.

As is well known, the number of cohabitants in a commensalism ecosystem does not increase instantaneously due to benefits from the host, and this process has a time delay. The number of various species in an ecosystem is usually influenced by environmental factors with periodic changes such as climate, weather, food, and mating. Additionally, the human behavior of catching, fishing, and logging has led to a decrease in the number of species in the ecosystem. Therefore, the model (1) describes these actual situations of the commensalism ecosystem.

In mathematical theory, the model (1) includes many types of functional differential equations. For example, if $\mathbb{T} = \mathbb{N}^+$, then the model (1) converts to a difference equation below

$$\begin{cases} \mathcal{X}(\tau+1) - \mathcal{X}(\tau) = r_1(\tau) - a_{11}(\tau)[e^{\mathcal{X}(\tau)}]^{\vartheta_1} - \varphi_1(\tau)e^{\mathcal{X}(\tau)}, \ \tau \in \mathbb{N}^+, \\ \mathcal{Y}(\tau+1) - \mathcal{Y}(\tau) = r_2(\tau) - a_{22}(\tau)[e^{\mathcal{Y}(\tau)}]^{\vartheta_2} + a_{21}(\tau)\sum\limits_{s=-\xi(\tau)}^{0} k(s)e^{\mathcal{X}(\tau+s)} - \varphi_2(\tau)e^{\mathcal{Y}(\tau)}\Big], \ \tau \in \mathbb{N}^+. \end{cases} \tag{4}$$

If $\mathbb{T} = \mathbb{R}$, set $x(\tau) = e^{\mathcal{X}(\tau)}$, $y(\tau) = e^{\mathcal{Y}(\tau)}$, then the model (1) changes into a differential equation below

$$\begin{cases} \frac{dx(\tau)}{d\tau} = x(\tau)\Big[r_1(\tau) - a_{11}(\tau)[x(\tau)]^{\vartheta_1}\Big] - \varphi_1(\tau), \ \tau \in \mathbb{R}, \\ \frac{dy(\tau)}{d\tau} = y(\tau)\Big[r_2(\tau) - a_{22}(\tau)[y(\tau)]^{\vartheta_2} + a_{21}(\tau)\int_{-\xi(\tau)}^{0} k(s)x(\tau+s)ds\Big] - \varphi_2(\tau), \ \tau \in \mathbb{R}. \end{cases} \tag{5}$$

Actually, a time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. To uniformly deal with discrete and continuous cases, the calculus on time scales was first proposed in Hilger's Ph.D. thesis [9] in 1988. Many scholars focused on and studied calculus and differential equation theory on time scales. For example, Srivastava and Tseng et al. [17, 18] studied some important inequalities on time scales. After more than 20 years of development, fruitful results have been achieved in the study of time scales and their differential equations. These achievements have been summarized and published as monographs. Specifically, readers can refer to the following two monographs [4, 5] to understand the theory of differential equations on time scales. In addition, there have been some research works involving ecosystems on time scales (see [26–28, 39]).

To the best our knowledge, there are few papers dealing with the solvability and stability of periodic commensalism AG-ecosystem on time scales. Therefore, the project of this manuscript is important and interesting. The remaining structure of this manuscript is as follows. Section 2 mainly includes the basic concepts and results of time scales, important assumptions and necessary propositions. In Section 3, sufficient conditions for the existence of periodic solutions are obtained. We shall show that the periodic solution is globally asymptotically stable by using Lyapunov functional in Section 4. A numerical example and simulation are provided in Section 5. Finally, Section 6 makes a brief summary and outlook.

2. Preliminaries

Thanks to [4, 5], we first retrospect some concepts and important conclusions of calculus on time scales in this portion.

A nonempty closed subset $\mathbb{T} \subset \mathbb{R}$ is called a time scale. Two jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ to forward and backward, and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(\tau) = \inf\{s \in \mathbb{T} : s > \tau\}, \quad \rho(\tau) = \sup\{s \in \mathbb{T} : s < \tau\} \quad \text{and} \quad \mu(\tau) = \sigma(\tau) - \tau, \ \text{ for all } \tau \in \mathbb{T}.$$

A point $\tau \in \mathbb{T}$ is named left-dense (right-dense) when $\tau > \inf \mathbb{T}$ and $\rho(\tau) = \tau$ ($\tau < \sup \mathbb{T}$ and $\sigma(\tau) = \tau$), left-scattered (right-scattered) when $\rho(\tau) < \tau$ ($\sigma(\tau) > \tau$). If $\mathbb{T}$ achieves a left-scattered maximum M, then $\mathbb{T}^k = \mathbb{T} \setminus \{M\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ reachs a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$.

Let $\mathbb{T}$ be a time scale, if there exists a constant $\omega > 0$ such that, for all $\tau \in \mathbb{T} \Rightarrow \tau + \omega \in \mathbb{T}$, then $\mathbb{T}$ is called an ω-periodic time scale. Obviously, if $\mathbb{T}$ is an ω-periodic time scale, then $\mathbb{T}$ is unbounded above.

Definition 2.1. *A function $u : \mathbb{T} \to \mathbb{R}$ is called regulated, provided that its right-side limits $u(\tau^+)$ and left-side limits $u(\tau^-)$ all exist (finite) for all $\tau \in \mathbb{T}$.*

Definition 2.2. *A function $u : \mathbb{T} \to \mathbb{R}$ is called rd-continuous, provided that it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $u : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.*

Definition 2.3. *Assume $u : \mathbb{T} \to \mathbb{R}$ and $\tau \in \mathbb{T}^k$. Then $u^\Delta(\tau)$ is defined to be the number (if exists) satisfying that, for given any $\varepsilon > 0$ there exists a neighborhood U of τ (i.e., $U = (\tau - \delta, \tau + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that*

$$|[u(\sigma(\tau)) - u(s)] - u^\Delta(\tau)[\sigma(\tau) - s]| < \varepsilon|\sigma(\tau) - s|,$$

for all $s \in U$. $u^\Delta(\tau)$ is called the delta (or Hilger) derivative of u at τ. The set of functions $u : \mathbb{T} \to \mathbb{R}$ that are Δ-differentiable and $u^\Delta(\tau)$ is rd−continuous, is denoted by $C^1_{rd}(\mathbb{T}, \mathbb{R})$.

According to the above definitions, one easily knows that u is Δ-differentiable $\Rightarrow$ u is continuous $\Rightarrow$ u is rd-continuous $\Rightarrow$ u is regulated.

Lemma 2.4. *Let u be regulated, then there has a function F which is Δ-differentiable with region of differentiation D such that*

$$F^\Delta(\tau) = u(\tau), \ \ \forall\, \tau \in D.$$

Definition 2.5. *Assume $u : \mathbb{T} \to \mathbb{R}$ is a regulated function. Any function F as in Lemma 2.1 is called a Δ-antiderivative of u. The indefinite integral of a regulated function u is defined by*

$$\int u(\tau)\Delta\tau = F(\tau) + C,$$

where C is an arbitrary constant and F is a Δ-antiderivative of u. We define the definite integral by

$$\int_a^b u(s)\Delta s = F(b) - F(a), \ \ \forall\, a, b \in \mathbb{T}.$$

A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $u : \mathbb{T} \to \mathbb{R}$, provided that

$$F^{\Delta}(\tau) = u(\tau) \ \ \forall\, \tau \in \mathbb{T}^k.$$

Lemma 2.6. *If $a, b \in \mathbb{T}$, $\alpha, \beta \in \mathbb{R}$ and $u, v \in C_{rd}(\mathbb{T}, \mathbb{R})$, then*

(1) $\int_a^b [\alpha u(\tau) + \beta v(\tau)]\Delta\tau = \alpha \int_a^b u(\tau)\Delta\tau + \beta \int_a^b v(\tau)\Delta\tau$;

(2) $\forall\, a \le \tau < b,\ u(\tau) \ge 0 \Rightarrow \int_a^b u(t)\Delta\tau \ge 0$;

(3) $\forall\, \tau \in [a, b) \triangleq \{\tau \in \mathbb{T} : a \le \tau < b\},\ |u(\tau)| \le v(\tau) \Rightarrow |\int_a^b u(\tau)\Delta\tau| \le \int_a^b v(\tau)\Delta\tau$.

Lemma 2.7. *(Mawhin's coincidence theorem [7]) Let $\mathbb{X}$ and $\mathbb{Y}$ be two Banach spaces, $\Omega \subset \mathbb{X}$ be a nonempty bounded open set, $\mathscr{L} : \mathbb{X} \to \mathbb{Y}$ be a zero index Fredholm operator, an operator $\mathscr{N} : \mathbb{X} \times [0, 1] \to \mathbb{Y}$ be $\mathscr{L}$-compact on $\overline{\Omega} \times [0, 1]$, $\mathscr{Q} : \mathbb{Y} \to \mathbb{Y}$ be a projection operator, $\mathscr{J} : \mathbb{Y} \to \mathbb{Y}$ be a homotopy operator. Assume that the following conditions hold.*

(i) *every solution w of $\mathscr{L}w = \lambda\mathscr{N}(w, \lambda)$ satisfies $w \notin \partial\Omega \cap \mathrm{Dom}(\mathscr{L})$, $\forall\, \lambda \in (0, 1)$;*

(ii) *$\mathscr{Q}\mathscr{N}(w, 0)w \neq 0$, $\forall\, w \in \partial\Omega \cap \mathrm{Ker}(\mathscr{L})$;*

(iii) $\deg(\mathscr{J}\mathscr{Q}\mathscr{N}(w, 0), \Omega \cap \mathrm{Ker}(\mathscr{L}), 0) \neq 0$.

Then $\mathscr{L}w = \mathscr{N}(w, 1)$ exists at least one solution in $\overline{\Omega} \cap \mathrm{Dom}(\mathscr{L})$.

Lemma 2.8. *Let $a, b, c, \theta > 0$ be some constants, consider the function $h(z) = ae^{(1+\theta)z} - be^z + c$. Assume that $\theta a^{-\frac{1}{\theta}} \left(\frac{b}{1+\theta}\right)^{\frac{1+\theta}{\theta}} > c$, then the following assertions are true:*

(i) *$h(z)$ has a unique minimum point $z_0 = \frac{1}{\theta} \ln\left[\frac{b}{a(1+\theta)}\right]$ in $(-\infty, +\infty)$, and the minimum $h(z_0) = -\theta a^{-\frac{1}{\theta}} \left(\frac{b}{1+\theta}\right)^{\frac{1+\theta}{\theta}} + c < 0$.*

(ii) *$h(z)$ is strict decreasing in $(-\infty, z_0]$ and increasing in $[z_0, +\infty)$, respectively.*

(iii) *$h(z)$ has only two zeros z_1 and z_2 satisfying $-\infty < z_1 < z_0 < z_2 < +\infty$.*

Proof. Since $h'(z) = a(1 + \theta)e^{(1+\theta)z} - be^z$, $h''(z) = a(1 + \theta)^2 e^{(1+\theta)z} - be^z$, we follows from $h'(z) = 0$ that $z_0 = \frac{1}{\theta} \ln\left[\frac{b}{a(1+\theta)}\right]$ and $h''(z_0) = b\theta e^{z_0} > 0$. Combining hypothesis $\theta a^{-\frac{1}{\theta}} \left(\frac{b}{1+\theta}\right)^{\frac{1+\theta}{\theta}} > c$, we know that z_0 is a unique minimum point of $h(z)$ in $(-\infty, +\infty)$ and the minimum $h(z_0) = -\theta a^{-\frac{1}{\theta}} \left(\frac{b}{1+\theta}\right)^{\frac{1+\theta}{\theta}} + c < 0$. Moreover, $h(z)$ is strict decreasing in $(-\infty, z_0]$ and increasing in $[z_0, +\infty)$, respectively. Thus the assertions (i) and (ii) hold. In addition,

$$\lim_{z\to-\infty} h(z) = c > 0,\ \lim_{z\to+\infty} h(z) = \lim_{z\to+\infty} e^z(ae^{\theta z} - b + c) = +\infty > 0.$$

It follows from existence theorem of zeros that there only exist two real numbers z_1, z_2 such that $-\infty < z_1 < z_0 < z_2 < +\infty$ and $h(z_1) = h(z_2) = 0$. The proof is completed. □

For simplicity, we use the following symbols.

$$\kappa = \min\{[0, +\infty) \cap \mathbb{T}\},\ \ I_\omega = [\kappa, \kappa + \omega] \cap \mathbb{T},\ \ \overline{u} = \sup_{\tau\in I_\omega} u(\tau),$$

$$\underline{u} = \inf_{\tau\in I_\omega} u(\tau),\ \ \hat{u} = \frac{1}{\omega} \int_{I_\omega} u(s)\Delta s = \frac{1}{\omega} \int_\kappa^{\kappa+\omega} u(s)\Delta s,$$

where $u \in C_{rd}(\mathbb{T}, \mathbb{R})$ satisfies $u(\tau + \omega) = u(\tau)$ for all $\tau \in \mathbb{T}$. In the whole paper, we need the following assumptions.

(H_1) Assume that $0 < r_1(\tau)$, $r_2(\tau)$, $\xi(\tau)$, $\varphi_1(\tau)$, $\varphi_2(\tau)$, $a_{11}(\tau)$, $a_{22}(\tau)$, $a_{21}(\tau)$, $k(\tau) \in C_{rd}(\mathbb{T}, \mathbb{R})$ are all ω-periodic, and $\int_{-\overline{\xi}}^0 k(s)\Delta s < \infty$, where $\overline{\xi} = \sup_{\tau\in I_\omega} \xi(\tau)$.

3. Existence of periodic solution

This section focuses on the existence of periodic solution for system (1) by applying Lemma 2.7. To this end, set $\mathbb{X} = \mathbb{Y} = W_1 \oplus W_2$, where $W_1 = \{w(\tau) \equiv (C_1, C_2)^T \in \mathbb{R}^2\}$,

$$W_2 = \Big\{w(\tau) = (w_1(\tau), w_2(\tau))^T : w_j(\tau) \in C_{rd}(\mathbb{T}, \mathbb{R}), w_j(\tau + \omega) = w_j(\tau), j = 1, 2\Big\},$$

A norm $\|\cdot\|$ is defined by

$$\|w\| = \max_{1\le j\le 2} \sup_{\tau\in I_\omega} |w_j(\tau)|, \ \forall w = (w_1, w_2)^T \in \mathbb{X} = \mathbb{Y}. \tag{6}$$

Using similar methods in Ref. [39], we can easily get Lemmas 3.1-3.4. Therefore, their proof is omitted.

Lemma 3.1. *Under the norm $\|\cdot\|$ defined as* (6)*, $\mathbb{X} = \mathbb{Y}$ is a Banach space.*

Lemma 3.2. *For all $w(\tau) = (w_1(\tau), w_2(\tau))^T \in \mathbb{X}$, define $\mathscr{L} : \mathbb{X} \to \mathbb{Y}$ as $\mathscr{L}w(\tau) = w^\Delta(\tau) = (w_1^\Delta(\tau), w_2^\Delta(\tau))^T$, then $\mathscr{L}$ is a zero index Fredholm operator.*

Lemma 3.3. *For all $w = (\mathcal{X}, \mathcal{Y})^T \in \mathbb{X} = \mathbb{Y}$, define an operator $\mathscr{N}(w, \lambda) : \mathbb{X} \times [0, 1] \to \mathbb{Y}$ and two projection operators $\mathscr{P} : \mathbb{X} \to \mathbb{Y}$, $\mathscr{Q} : \mathbb{Y} \to \mathbb{Y}$ as follows:*

$$\mathscr{N}(w, \lambda) = \begin{pmatrix} r_1(\tau) - a_{11}(\tau)e^{\vartheta_1\mathcal{X}(\tau)} - \varphi_1(\tau)e^{-\mathcal{X}(\tau)} \\ r_2(\tau) - a_{22}(\tau)e^{\vartheta_2\mathcal{Y}(\tau)} + \lambda a_{21}(\tau)\int_{-\xi(\tau)}^0 k(s)e^{\mathcal{X}(\tau+s)}ds - \varphi_2(\tau)e^{-\mathcal{Y}(\tau)} \end{pmatrix},$$

$$\mathscr{P}w = \mathscr{Q}w = (\hat{\mathcal{X}}, \hat{\mathcal{Y}})^T = \left(\frac{1}{\omega}\int_\kappa^{\kappa+\omega} \mathcal{X}(\tau)\Delta\tau, \ \frac{1}{\omega}\int_\kappa^{\kappa+\omega} \mathcal{Y}(\tau)\Delta\tau\right)^T,$$

$$\mathrm{Ker}(\mathscr{L}) = \{w = (\mathcal{X}, \mathcal{Y}) \in \mathbb{X} : (\mathcal{X}, \mathcal{Y}) = (C_1, C_2), \tau \in \mathbb{T}\},$$

$$\mathrm{Im}(\mathscr{L}) = \{w = (\mathcal{X}, \mathcal{Y}) \in \mathbb{Y} : (\hat{\mathcal{X}}, \hat{\mathcal{Y}}) = (0, 0)\},$$

$$(\mathscr{L}^{-1}|_{\mathscr{P}})(w(\tau)) = \begin{pmatrix} \int_\kappa^\tau \mathcal{X}(s)\Delta s - \frac{1}{\omega}\int_\kappa^{\kappa+\omega}\int_\kappa^\tau \mathcal{X}(s)\Delta s\Delta\tau \\ \int_\kappa^\tau \mathcal{Y}(s)\Delta s - \frac{1}{\omega}\int_\kappa^{\kappa+\omega}\int_\kappa^\tau \mathcal{Y}(s)\Delta s\Delta\tau \end{pmatrix},$$

Then, on $\overline{\Omega} \times [0, 1]$, $\mathscr{N}(w, \lambda)$ is $\mathscr{L}$-compact.

Lemma 3.4. *Let $\mathbb{T}$ be an ω-periodic time scale. Suppose $\psi : \mathbb{T} \to \mathbb{R}$ be an ω-periodic function which is rd-continuous, then*

$$0 \le \sup_{s\in I_\omega} \psi(s) - \inf_{s\in I_\omega} \psi(s) \le \frac{1}{2}\int_\kappa^{\kappa+\omega} |\psi^\Delta(s)|\Delta s.$$

(H_2) Suppose that the following inequalities are fulfilled.

$$\vartheta_1(\underline{a_{11}}e^{-\omega\vartheta_1\overline{r_1}})^{-\frac{1}{\vartheta_1}}\left(\frac{\overline{r_1}}{1+\vartheta_1}\right)^{\frac{1+\vartheta_1}{\vartheta_1}} > \underline{\varphi_1},$$

$$\vartheta_2(\underline{a_{22}}e^{-\omega\vartheta_2\overline{r_2}})^{-\frac{1}{\vartheta_2}}\left(\frac{\overline{r_2} + \overline{a_{21}}e^{L^+}\int_{-\overline{\xi}}^0 k(s)\Delta s}{1+\vartheta_2}\right)^{\frac{1+\vartheta_2}{\vartheta_2}} > \underline{\varphi_2}.$$

Theorem 3.5. *If the conditions* (H_1) *and* (H_2) *hold, then model* (1) *exists at least an* ω*-periodic solution* $(\widetilde{\mathcal{X}}(\tau), \widetilde{\mathcal{Y}}(\tau))^T$ *on time scale* $\mathbb{T}$ *satisfying* $L^- - \omega\overline{r_1} < \widetilde{\mathcal{X}}(\tau) < L^+$ *and* $M^- - \omega\overline{r_2} < \widetilde{\mathcal{Y}}(\tau) < M^+$*, where* L^-, L^+ *are two roots of* $f(\mathcal{Z}) = 0$*,* M^-, M^+ *are two roots of* $g(\mathcal{Z}) = 0$*, and* $f(\mathcal{Z})$*,* $g(Z)$ *are given by*

$$
\begin{aligned}
f(\mathcal{Z}) =& (\underline{a_{11}} e^{-\omega\vartheta_1\overline{r_1}}) e^{(1+\vartheta_1)\mathcal{Z}} - \overline{r_1} e^{\mathcal{Z}} + \underline{\varphi_1},\\
g(\mathcal{Z}) =& (\underline{a_{22}} e^{-\omega\vartheta_2\overline{r_2}}) e^{(1+\vartheta_2)\mathcal{Z}} - \left[\overline{r_2} + \overline{a_{21}} e^{L^+} \int_{-\overline{\xi}}^{0} k(s)\Delta s\right] e^{\mathcal{Z}} + \underline{\varphi_2} = 0.
\end{aligned}
$$

Proof. To apply Lemma 2.7 showing that model (1) has an ω-periodic solution, we define $\mathbb{X}$ and $\mathbb{Y}$ as Lemma 3.1 and $\mathscr{L}, \mathscr{N}, \mathscr{P}, \mathscr{Q}$ as Lemmas 3.2-3.3.

Now we find the existence region $\Omega \subset \mathbb{X}$ of solution. Assume that an operator equation $\mathscr{L}w = \lambda\mathscr{N}(w, \lambda)$ has an ω-periodic solution $w = (\mathcal{X}, \mathcal{Y})^T \in \mathbb{X}$, then we have

$$
\begin{cases}
\mathcal{X}^{\Delta}(\tau) = \lambda\Big[r_1(\tau) - a_{11}(\tau)e^{\vartheta_1\mathcal{X}(\tau)} - \varphi_1(\tau)e^{-\mathcal{X}(\tau)}\Big],\\
\mathcal{Y}^{\Delta}(\tau) = \lambda\Big[r_2(\tau) - a_{22}(\tau)e^{\vartheta_2\mathcal{Y}(\tau)} + \lambda a_{21}(\tau)\int_{-\xi(\tau)}^{0} k(s)e^{\mathcal{X}(\tau+s)}\Delta s - \varphi_2(\tau)e^{-\mathcal{Y}(\tau)}\Big].
\end{cases} \tag{7}
$$

We integrate at both ends of (7) to obtain

$$
\begin{cases}
0 = \int_{\kappa}^{\kappa+\omega}\Big[r_1(\tau) - a_{11}(\tau)e^{\vartheta_1\mathcal{X}(\tau)} - \varphi_1(\tau)e^{-\mathcal{X}(\tau)}\Big]\Delta\tau,\\
0 = \int_{\kappa}^{\kappa+\omega}\Big[r_2(\tau) - a_{22}(\tau)e^{\vartheta_2\mathcal{Y}(\tau)} + \lambda a_{21}(\tau)\int_{-\xi(\tau)}^{0} k(s)e^{\mathcal{X}(\tau+s)}\Delta s - \varphi_2(\tau)e^{-\mathcal{Y}(\tau)}\Big]\Delta\tau.
\end{cases} \tag{8}
$$

Since $\mathcal{X}(\tau)$ and $\mathcal{Y}(\tau)$ are all ω-periodic, there exist μ_1, μ_2, ν_1 and $\nu_2 \in I_\omega$ such that $\mathcal{X}(\mu_1) = \overline{\mathcal{X}}$, $\mathcal{X}(\mu_2) = \underline{\mathcal{X}}$, $\mathcal{Y}(\nu_1) = \overline{\mathcal{Y}}$, $\mathcal{Y}(\nu_2) = \underline{\mathcal{Y}}$. From the first equation of (7), and (8), we get

$$
\int_{\kappa}^{\kappa+\omega} |\mathcal{X}^{\Delta}(\tau)|\Delta\tau < 2\omega\overline{r_1}. \tag{9}
$$

In light of the first equation of (8), and (9) together with Lemma 3.4, we get

$$
\begin{aligned}
\omega\overline{r_1} \geq& \int_{\kappa}^{\kappa+\omega} r_1(s)\Delta s = \int_{\kappa}^{\kappa+\omega} a_{11}(\tau)e^{\vartheta_1\mathcal{X}(\tau)}\Delta\tau + \int_{\kappa}^{\kappa+\omega} \varphi_1(\tau)e^{-\mathcal{X}(\tau)}\Delta\tau\\
>& \omega\underline{a_{11}}e^{\vartheta_1\mathcal{X}(\mu_2)} + \omega\underline{\varphi_1}e^{-\mathcal{X}(\mu_1)} \geq \omega\underline{a_{11}}e^{\vartheta_1[\mathcal{X}(\mu_1)-\omega\overline{r_1}]} + \omega\underline{\varphi_1}e^{-\mathcal{X}(\mu_1)},
\end{aligned}
$$

which implies that

$$
(\underline{a_{11}}e^{-\omega\vartheta_1\overline{r_1}})e^{(1+\vartheta_1)\mathcal{X}(\mu_1)} - \overline{r_1}e^{\mathcal{X}(\mu_1)} + \underline{\varphi_1} < 0. \tag{10}
$$

By Lemma 2.8, we know that $f(\mathcal{Z})$ has unique minimum point L_0 and minimum $f(L_0)$

$$
L_0 = \frac{1}{\vartheta_1}\ln\left[\frac{\overline{r_1}}{(\underline{a_{11}}e^{-\omega\vartheta_1\overline{r_1}})(1+\vartheta_1)}\right],\ f(L_0) = -\vartheta_1(\underline{a_{11}}e^{-\omega\vartheta_1\overline{r_1}})^{-\frac{1}{\vartheta_1}}\left(\frac{\overline{r_1}}{1+\vartheta_1}\right)^{\frac{1+\vartheta_1}{\vartheta_1}} + \underline{\varphi_1},
$$

in $(-\infty, +\infty)$ such that $f'(L_0) = 0$. From (H_2) and Lemma 2.8, we conclude that $f(L_0) < 0$, and there exist only two constants L^- and L^+ such that

$$
L^- < L_0 < L^+,\ f(L^{\pm}) = 0. \tag{11}
$$

From (11) and Lemma 2.8, we find the solution to inequality (10) as

$$
L^- < \mathcal{X}(\mu_1) < L^+. \tag{12}
$$

In view of Lemma 3.4 and (12), we obtain

$$L^- - \omega\overline{r_1} < \mathcal{X}(\mu_2) \le \mathcal{X}(\mu_1) < L^+. \tag{13}$$

Similarly, it follows from the second equation of (7) and (13) that

$$\begin{aligned}&\omega\overline{r_2} + \omega\overline{a_{21}}e^{L^+}\int_{-\overline{\xi}}^{0} k(s)\Delta s > \int_{\kappa}^{\kappa+\omega} r_2(\tau)\Delta\tau + \lambda\int_{\kappa}^{\kappa+\omega} a_{21}(\tau)\Big[\int_{-\xi(\tau)}^{0} k(s)e^{\mathcal{X}(\tau+s)}\Delta s\Big]\Delta\tau\\ &= \int_{\kappa}^{\kappa+\omega} a_{22}(\tau)e^{\vartheta_2\mathcal{Y}(\tau)}\Delta\tau + \int_{\kappa}^{\kappa+\omega}\varphi_2(\tau)e^{-\mathcal{Y}(\tau)}\Delta\tau > \omega\underline{a_{22}}e^{\vartheta_2[\mathcal{Y}(\nu_1)-\omega\overline{r_2}]} + \omega\underline{\varphi_2}e^{-\mathcal{Y}(\nu_1)},\end{aligned}$$

which indicates that

$$(\underline{a_{22}}e^{-\omega\vartheta_2\overline{r_2}})e^{(1+\vartheta_2)\mathcal{Y}(\nu_1)} - \left[\overline{r_2} + \overline{a_{21}}e^{L^+}\int_{-\overline{\xi}}^{0} k(s)\Delta s\right]e^{\mathcal{Y}(\nu_1)} + \underline{\varphi_2} < 0. \tag{14}$$

By Lemma 2.8, we conclude that $g(\mathcal{Z})$ has unique minimum point M_0 and minimum $g(M_0)$

$$M_0 = \frac{1}{\vartheta_2}\ln\left[\frac{\overline{r_2} + \overline{a_{21}}e^{L^+}\int_{-\overline{\xi}}^{0} k(s)\Delta s}{(\underline{a_{22}}e^{-\omega\vartheta_2\overline{r_2}})(1+\vartheta_2)}\right],$$

$$g(M_0) = -\vartheta_2(\underline{a_{22}}e^{-\omega\vartheta_2\overline{r_2}})^{-\frac{1}{\vartheta_2}}\left(\frac{\overline{r_2} + \overline{a_{21}}e^{L^+}\int_{-\overline{\xi}}^{0} k(s)\Delta s}{1+\vartheta_2}\right)^{\frac{1+\vartheta_2}{\vartheta_2}} + \underline{\varphi_2},$$

in $(-\infty,+\infty)$ such that $g'(M_0) = 0$. We derive from (H_2) and Lemma 2.8 that $g(M_0) < 0$ and there exist two constants M^- and M^+ such that

$$M^- < M_0 < M^+,\ \ g(M^\pm) = 0. \tag{15}$$

Due to (15) and Lemma 2.8, the inequality (14) is solved by

$$M^- < \mathcal{Y}(\nu_1) < M^+. \tag{16}$$

From Lemma 3.4 and (16), we have

$$M^- - \omega\overline{r_2} < \mathcal{Y}(\nu_2) \le \mathcal{Y}(\nu_1) < M^+. \tag{17}$$

According to (13) and (17), we choose

$$\Omega = \Big\{(\mathcal{X}(\tau),\mathcal{Y}(\tau))^T \in \mathbb{X} : L^- - \omega\overline{r_1} < \mathcal{X}(\tau) < L^+, M^- - \omega\overline{r_2} < \mathcal{Y}(\tau) < M^+\Big\}.$$

Obviously, $\Omega \subset \mathbb{X}$ is a bounded open subset satisfying the condition (i) in Lemma 2.7.

Next, it is necessary to verify that condition (ii) of Lemma 2.7 is true, namely, when $w \in \partial\Omega \cap \mathrm{Ker}(\mathscr{L}) = \partial\Omega \cap \mathbb{R}^2, \mathscr{Q}\mathscr{N}(w,0) \ne (0,0)$. If it is not true, then when $w \in \partial\Omega \cap \mathrm{Ker}(\mathscr{L}) = \partial\Omega \cap \mathbb{R}^2$, constant vector $w^* = (u^*, v^*)$ with $w \in \partial\Omega$ satisfies

$$\begin{cases} \int_{\kappa}^{\kappa+\omega}\Big[r_1(\tau) - a_{11}(\tau)e^{\vartheta_1 u^*} - \varphi_1(\tau)e^{-u^*}\Big]\Delta\tau = 0,\\ \int_{\kappa}^{\kappa+\omega}\Big[r_2(\tau) - a_{22}(\tau)e^{\vartheta_2 u^*} + a_{21}(\tau)e^{u^*}\int_{-\xi(\tau)}^{0} k(s)\Delta s - \varphi_2(\tau)e^{-v^*}\Big]\Delta\tau = 0.\end{cases} \tag{18}$$

Similar to (8)-(17), we derive from (18) that $w^* = (u^*, v^*) \in \Omega \cap \mathbb{R}^2$, which contradicts with $w^* = (u^*, v^*) \in \partial\Omega \cap \mathbb{R}^2$. Thus, condition (ii) in Lemma 2.7 is true.

Take $\mathscr{J} = \mathscr{I}$ is the identity mapping, by direct calculation, one has

$$\deg\Big\{\mathscr{J}\mathscr{Q}\mathscr{N}(w,0), \Omega \cap \mathrm{Ker}(\mathscr{J}), (0,0)^T\Big\} \ne 0.$$

Thus, all the assumptions of Lemma 2.7 are true. Consequently, the model (1) exists at least an ω-periodic solutions $(\widetilde{\mathcal{X}}(\tau), \widetilde{\mathcal{Y}}(\tau))^T$. The proof is completed. □

4. Global asymptotic stability

In this section, we centralize on the global asymptotic stability of model (1). To this end, the following definition is necessary.

Definition 4.1. *[12] For each $\tau \in \mathbb{T}$, let U be a neighborhood of τ. Then, for $V \in C_{rd}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^+)$, define the Dini derivative $D^+V^\Delta(\tau, x(\tau))$, to mean that, given $\epsilon > 0$, there exists a right neighborhood $U_\epsilon \cap U$ of τ such that*

$$\frac{V(\sigma(\tau), x(\sigma(\tau))) - V(s, x(s))}{\sigma(\tau) - s} < D^+V^\Delta(\tau, x(\tau)) + \epsilon, \ \forall\, s \in U_\epsilon, s > \tau.$$

If τ is right-scattered and $V(\tau, x(\tau))$ is continuous at τ, this reduces to

$$D^+V^\Delta(\tau, x(\tau)) = \frac{V(\sigma(\tau), x(\sigma(\tau))) - V(\tau, x(\tau))}{\sigma(\tau) - \tau}.$$

In view of Theorem 3.5, one concludes that system (1) exists at least an ω-periodic solutions $(\widetilde{\mathcal{X}}(\tau), \widetilde{\mathcal{Y}}(\tau))^T \in \Omega$. Let $u(\tau) = e^{\mathcal{X}(\tau)}$, $v(\tau) = e^{\mathcal{Y}(\tau)}$, then $\mathcal{X}^\Delta(\tau) = (\ln u(\tau))^\Delta$ and $\mathcal{Y}^\Delta(\tau) = (\ln v(\tau))^\Delta$. Thus the system (1) changes into

$$\begin{cases} (\ln u(\tau))^\Delta = r_1(\tau) - a_{11}(\tau)[u(\tau)]^{\vartheta_1} - \frac{\varphi_1(\tau)}{u(\tau)}, \ \tau \in \mathbb{T}, \\ (\ln v(\tau))^\Delta = r_2(\tau) - a_{22}(\tau)[v(\tau)]^{\vartheta_2} + a_{21}(\tau)\int_{-\xi(\tau)}^{0} k(s)u(\tau+s)\Delta s - \frac{\varphi_2(\tau)}{v(\tau)}, \ \tau \in \mathbb{T}. \end{cases} \tag{19}$$

System (19) has at least an ω-periodic positive solution $(\widetilde{u}(\tau), \widetilde{v}(\tau))^T \in \widetilde{\Omega}$, where

$$\widetilde{\Omega} = \Big\{(u(\tau), v(\tau))^T : e^{L^- - \omega\overline{r_1}} < u(\tau) < e^{L^+}, e^{M^- - \omega\overline{r_2}} < v(\tau) < e^{M^+}\Big\}.$$

Let ρ and ϑ be some positive constants satisfying $0 < \rho < \min\{e^{L^- - \omega\overline{r_1}}, e^{M^- - \omega\overline{r_2}}\}$ and $\vartheta \geq \max\{1, \vartheta_1, \vartheta_2\}$. We further assume that

(H_3) $-\rho^{\frac{\vartheta_1}{\vartheta}}\underline{a_{11}} + \rho^{-\frac{1}{\vartheta}}\overline{\varphi_1} + \rho^{\frac{1}{\vartheta}}\overline{a_{21}}\int_{-\overline{\xi}}^{0} k(s)\Delta s < 0$ and $-\rho^{\frac{\vartheta_2}{\vartheta}}\underline{a_{22}} + \rho^{-\frac{1}{\vartheta}}\overline{\varphi_2} < 0$.

Let $u(\tau) = (\rho X(\tau))^{\frac{1}{\vartheta}}$ and $v(\tau) = (\rho Y(\tau))^{\frac{1}{\vartheta}}$, then

$$(\ln u(\tau))^\Delta = \left[\frac{1}{\vartheta}\ln(\rho X(\tau))\right]^\Delta = \frac{1}{\vartheta}\Big[\ln\rho + \ln X(\tau)\Big]^\Delta = \frac{1}{\vartheta}(\ln X(\tau))^\Delta,$$

and

$$(\ln v(\tau))^\Delta = \left[\frac{1}{\vartheta}\ln(\rho Y(\tau))\right]^\Delta = \frac{1}{\vartheta}\Big[\ln\rho + \ln Y(\tau)\Big]^\Delta = \frac{1}{\vartheta}(\ln Y(\tau))^\Delta.$$

Therefore, system (19) becomes

$$\begin{cases} (\ln X(\tau))^\Delta = \vartheta\Big[r_1(\tau) - \rho^{\frac{\vartheta_1}{\vartheta}}a_{11}(\tau)X^{\frac{\vartheta_1}{\vartheta}}(\tau) - \rho^{-\frac{1}{\vartheta}}\varphi_1(\tau)X^{-\frac{1}{\vartheta}}(\tau)\Big], \\ (\ln Y(\tau))^\Delta = \vartheta\Big[r_2(\tau) - \rho^{\frac{\vartheta_2}{\vartheta}}a_{22}(\tau)Y^{\frac{\vartheta_2}{\vartheta}}(\tau) + \rho^{\frac{1}{\vartheta}}a_{21}(\tau)\int_{-\xi(\tau)}^{0} k(s)X^{\frac{1}{\vartheta}}(\tau+s)\Delta s - \rho^{-\frac{1}{\vartheta}}\varphi_2(\tau)Y^{-\frac{1}{\vartheta}}(\tau)\Big]. \end{cases} \tag{20}$$

Obviously, there has a positive ω-periodic function $(\widetilde{X}(\tau), \widetilde{Y}(\tau))^T = (\frac{1}{\rho}\widetilde{u}^\vartheta(\tau), \frac{1}{\rho}\widetilde{v}^\vartheta(\tau))^T \in \widetilde{\Omega}'$ satisfying system (20), where

$$\widetilde{\Omega}' = \Big\{(X(\tau), Y(\tau))^T : \frac{1}{\rho}e^{\vartheta(L^- - \omega\overline{r_1})} < X(\tau) < \frac{1}{\rho}e^{\vartheta L^+}, \frac{1}{\rho}e^{\vartheta(M^- - \omega\overline{r_2})} < Y(\tau) < \frac{1}{\rho}e^{\vartheta M^+}\Big\}.$$

According to Theorem 3.5 and $\widetilde{\Omega}'$, we get

$$1 < \frac{1}{\rho}e^{\vartheta(L^- - \omega\overline{r_1})} < \widetilde{X}(\tau) < \frac{1}{\rho}e^{\vartheta L^+}, \ 1 < \frac{1}{\rho}e^{\vartheta(M^- - \omega\overline{r_2})} < \widetilde{Y}(\tau) < \frac{1}{\rho}e^{\vartheta M^+}. \tag{21}$$

Theorem 4.2. *If* (H_1)-(H_3) *are ture, then a unique* ω*-periodic solution* $(\widetilde{\mathcal{X}}(\tau), \widetilde{\mathcal{Y}}(\tau))^T$ *of* (1) *is globally asymptotically stable.*

Proof. Assume that the ω-periodic solution $(\widetilde{\mathcal{X}}(\tau), \widetilde{\mathcal{Y}}(\tau))^T$ of (1) is globally asymptotically stable, then $(\widetilde{\mathcal{X}}(\tau), \widetilde{\mathcal{Y}}(\tau))^T$ is attractive, that is, for any solution $(\mathcal{X}(\tau), \mathcal{Y}(\tau))^T$ of (1), we have $\lim\limits_{\tau\to+\infty}[\mathcal{X}(\tau) - \widetilde{\mathcal{X}}(\tau)] = 0$, $\lim\limits_{\tau\to+\infty}[\mathcal{Y}(\tau) - \widetilde{\mathcal{Y}}(\tau)] = 0$. If the system (1) has another ω-periodic solution $(\mathcal{X}^*(\tau), \mathcal{Y}^*(\tau))^T \in \Omega$ with $(\mathcal{X}^*(\tau), \mathcal{Y}^*(\tau))^T \neq (\widetilde{\mathcal{X}}(\tau), \widetilde{\mathcal{Y}}(\tau))^T$, without loss of generality, assume that $\mathcal{X}^*(\tau) \neq \widetilde{\mathcal{X}}(\tau)$, then we obtain $0 < |\widetilde{\mathcal{X}}(\tau) - \mathcal{X}^*(\tau)| \leq |\widetilde{\mathcal{X}}(\tau) - \mathcal{X}(\tau)| + |\mathcal{X}(\tau) - \mathcal{X}^*(\tau)| \to 0$, as $\tau \to +\infty$, which is a clear contradiction. Thus, we prove that the ω-periodic solution $(\widetilde{\mathcal{X}}(\tau), \widetilde{\mathcal{Y}}(\tau))^T$ of (1) is unique provided that $(\widetilde{\mathcal{X}}(\tau), \widetilde{\mathcal{Y}}(\tau))^T$ is globally asymptotically stable. In addition, since the global asymptotical stability of ω-periodic solution $(\widetilde{\mathcal{X}}(\tau), \widetilde{\mathcal{Y}}(\tau))^T \in \Omega$ of (1) and $(\widetilde{X}(\tau), \widetilde{Y}(\tau))^T$ of (20) is equivalent, it suffices to prove that the ω-periodic solution $(\widetilde{X}(\tau), \widetilde{Y}(\tau))^T$ of (20) is globally asymptotically stable. In fact, by (H_1) and (H_2), one knows from Theorem 3.5 that $(\widetilde{X}(\tau), \widetilde{Y}(\tau))^T$ is an ω-periodic positive solution of (20). For any positive solution $(X(\tau), Y(\tau))^T$ of (20), we construct the following Lyapunov functional $V(\tau) = V_1(\tau) + V_2(\tau)$, where

$$V_1(\tau) = |\ln X(\tau) - \ln \widetilde{X}(\tau)| + |\ln Y(\tau) - \ln \widetilde{Y}(\tau)|, \tag{22}$$

$$V_2(\tau) = \vartheta \rho^{\frac{1}{\vartheta}} \overline{a_{21}} \int_{-\overline{\xi}}^{0} k(s) \Big[\int_{\tau+s}^{\tau} \Big| X^{\frac{1}{\vartheta}}(\zeta) - \widetilde{X}^{\frac{1}{\vartheta}}(\zeta) \Big| \Delta\zeta \Big] \Delta s. \tag{23}$$

Apparently, $V(0) < +\infty$ and $V(\tau) \geq V_1(\tau)$. By (21), we calculate the Δ-derivation along (20) to obtain

$$\begin{aligned}
&D^+(|\ln X(\tau) - \ln \widetilde{X}(\tau)|)^{\Delta} \leq -\vartheta \rho^{\frac{\vartheta_1}{\vartheta}} \underline{a_{11}} \Big| X(\tau) - \widetilde{X}(\tau) \Big| + \vartheta \rho^{\frac{-1}{\vartheta}} \overline{\varphi_1} \Big| X^{\frac{-1}{\vartheta}}(\tau) - \widetilde{X}^{\frac{-1}{\vartheta}}(\tau) \Big| \\
=& -\vartheta \rho^{\frac{\vartheta_1}{\vartheta}} \underline{a_{11}} \Big| X(\tau) - \widetilde{X}(\tau) \Big| + \vartheta \rho^{\frac{-1}{\vartheta}} \overline{\varphi_1} \Big| X^{\frac{1}{\vartheta}}(\tau) - \widetilde{X}^{\frac{1}{\vartheta}}(\tau) \Big| X^{\frac{-1}{\vartheta}}(\tau) \widetilde{X}^{\frac{-1}{\vartheta}}(\tau) \\
\leq& -\vartheta \rho^{\frac{\vartheta_1}{\vartheta}} \underline{a_{11}} \Big| X(\tau) - \widetilde{X}(\tau) \Big| + \vartheta \rho^{\frac{-1}{\vartheta}} \overline{\varphi_1} \Big| X^{\frac{1}{\vartheta}}(\tau) - \widetilde{X}^{\frac{1}{\vartheta}}(\tau) \Big|,
\end{aligned} \tag{24}$$

$$\begin{aligned}
&D^+(|\ln Y(\tau) - \ln \widetilde{Y}(\tau)|)^{\Delta} \leq -\vartheta \rho^{\frac{\vartheta_2}{\vartheta}} \underline{a_{22}} \Big| Y(\tau) - \widetilde{Y}(\tau) \Big| \\
&+ \vartheta \rho^{\frac{1}{\vartheta}} \overline{a_{21}} \int_{-\xi}^{0} k(s) \Big| X^{\frac{1}{\vartheta}}(\tau+s) - \widetilde{X}^{\frac{1}{\vartheta}}(\tau+s) \Big| \Delta s + \vartheta \rho^{\frac{-1}{\vartheta}} \overline{\varphi_2} \Big| Y^{\frac{-1}{\vartheta}}(\tau) - \widetilde{Y}^{\frac{-1}{\vartheta}}(\tau) \Big| \\
=& -\vartheta \rho^{\frac{\vartheta_2}{\vartheta}} \underline{a_{22}} \Big| Y(\tau) - \widetilde{Y}(\tau) \Big| + \vartheta \rho^{\frac{1}{\vartheta}} \overline{a_{21}} \int_{-\overline{\xi}}^{0} k(s) \Big| X^{\frac{1}{\vartheta}}(\tau+s) - \widetilde{X}^{\frac{1}{\vartheta}}(\tau+s) \Big| \Delta s \\
&+ \vartheta \rho^{\frac{-1}{\vartheta}} \overline{\varphi_2} \Big| Y^{\frac{1}{\vartheta}}(\tau) - \widetilde{Y}^{\frac{1}{\vartheta}}(\tau) \Big| Y^{\frac{-1}{\vartheta}}(\tau) \widetilde{Y}^{\frac{-1}{\vartheta}}(\tau) \\
\leq& -\vartheta \rho^{\frac{\vartheta_2}{\vartheta}} \underline{a_{22}} \Big| Y(\tau) - \widetilde{Y}(\tau) \Big| + \vartheta \rho^{\frac{1}{\vartheta}} \overline{a_{21}} \int_{-\overline{\xi}}^{0} k(s) \Big| X^{\frac{1}{\vartheta}}(\tau+s) - \widetilde{X}^{\frac{1}{\vartheta}}(\tau+s) \Big| \Delta s \\
&+ \vartheta \rho^{\frac{-1}{\vartheta}} \overline{\varphi_2} \Big| Y^{\frac{1}{\vartheta}}(\tau) - \widetilde{Y}^{\frac{1}{\vartheta}}(\tau) \Big|,
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
&D^+ \left(\int_{-\overline{\xi}}^{0} k(s) \Big[\int_{\tau+s}^{\tau} \Big| X^{\frac{1}{\vartheta}}(\zeta) - \widetilde{X}^{\frac{1}{\vartheta}}(\zeta) \Big| \Delta\zeta \Big] \Delta s \right)^{\Delta} = \int_{-\overline{\xi}}^{0} k(s) \Delta s \cdot \Big| X^{\frac{1}{\vartheta}}(\tau) - \widetilde{X}^{\frac{1}{\vartheta}}(\tau) \Big| \\
&- \int_{-\overline{\xi}}^{0} k(s) \Big| X^{\frac{1}{\vartheta}}(\tau+s) - \widetilde{X}^{\frac{1}{\vartheta}}(\tau+s) \Big| \Delta s.
\end{aligned} \tag{26}$$

Noticing that, for constants $a, b > 0$ and $s \geq 1$, $q(s) = |a^s - b^s|$ is monotonically increasing, and $0 < \frac{\vartheta_1}{\vartheta}, \frac{\vartheta_2}{\vartheta}, \frac{1}{\vartheta} \leq 1$, it follows from (21), (24)-(26) and (H_3) that

$$\begin{aligned}
D^+V^\Delta(\tau) \leq & -\vartheta\rho^{\frac{\vartheta_1}{\vartheta}}\underline{a_{11}}\Big|X(\tau)-\widetilde{X}(\tau)\Big| + \vartheta\rho^{\frac{-1}{\vartheta}}\overline{\varphi_1}\Big|X^{\frac{1}{\vartheta}}(\tau)-\widetilde{X}^{\frac{1}{\vartheta}}(\tau)\Big| - \vartheta\rho^{\frac{\vartheta_2}{\vartheta}}\underline{a_{22}}\Big|Y(\tau)-\widetilde{Y}(\tau)\Big| \\
& + \vartheta\rho^{\frac{1}{\vartheta}}\overline{a_{21}}\int_{-\overline{\xi}}^{0}k(s)\Big|X^{\frac{1}{\vartheta}}(\tau+s)-\widetilde{X}^{\frac{1}{\vartheta}}(\tau+s)\Big|\Delta s + \vartheta\rho^{\frac{-1}{\vartheta}}\overline{\varphi_2}\Big|Y^{\frac{1}{\vartheta}}(\tau)-\widetilde{Y}^{\frac{1}{\vartheta}}(\tau)\Big| \\
& + \vartheta\rho^{\frac{1}{\vartheta}}\overline{a_{21}}\int_{-\overline{\xi}}^{0}k(s)\Delta s\cdot\Big|X^{\frac{1}{\vartheta}}(\tau)-\widetilde{X}^{\frac{1}{\vartheta}}(\tau)\Big| - \vartheta\rho^{\frac{1}{\vartheta}}\overline{a_{21}}\int_{-\overline{\xi}}^{0}k(s)\Big|X^{\frac{1}{\vartheta}}(\tau+s)-\widetilde{X}^{\frac{1}{\vartheta}}(\tau+s)\Big|\Delta s \\
= & -\vartheta\rho^{\frac{\vartheta_1}{\vartheta}}\underline{a_{11}}\Big|X(\tau)-\widetilde{X}(\tau)\Big| + \vartheta\rho^{\frac{-1}{\vartheta}}\overline{\varphi_1}\Big|X^{\frac{1}{\vartheta}}(\tau)-\widetilde{X}^{\frac{1}{\vartheta}}(\tau)\Big| - \vartheta\rho^{\frac{\vartheta_2}{\vartheta}}\underline{a_{22}}\Big|Y(\tau)-\widetilde{Y}(\tau)\Big| \\
& + \vartheta\rho^{\frac{-1}{\vartheta}}\overline{\varphi_2}\Big|Y^{\frac{1}{\vartheta}}(\tau)-\widetilde{Y}^{\frac{1}{\vartheta}}(\tau)\Big| + \vartheta\rho^{\frac{1}{\vartheta}}\overline{a_{21}}\int_{-\overline{\xi}}^{0}k(s)\Delta s\cdot\Big|X^{\frac{1}{\vartheta}}(\tau)-\widetilde{X}^{\frac{1}{\vartheta}}(\tau)\Big| \\
\leq & -\vartheta\rho^{\frac{\vartheta_1}{\vartheta}}\underline{a_{11}}\Big|X(\tau)-\widetilde{X}(\tau)\Big| + \vartheta\rho^{\frac{-1}{\vartheta}}\overline{\varphi_1}\Big|X(\tau)-\widetilde{X}(\tau)\Big| - \vartheta\rho^{\frac{\vartheta_2}{\vartheta}}\underline{a_{22}}\Big|Y(\tau)-\widetilde{Y}(\tau)\Big| \\
& + \vartheta\rho^{\frac{-1}{\vartheta}}\overline{\varphi_2}\Big|Y(\tau)-\widetilde{Y}(\tau)\Big| + \vartheta\rho^{\frac{1}{\vartheta}}\overline{a_{21}}\int_{-\overline{\xi}}^{0}k(s)\Delta s\cdot\Big|X(\tau)-\widetilde{X}(\tau)\Big| \\
= & \vartheta\Big[-\rho^{\frac{\vartheta_1}{\vartheta}}\underline{a_{11}} + \rho^{-\frac{1}{\vartheta}}\overline{\varphi_1} + \rho^{\frac{1}{\vartheta}}\overline{a_{21}}\int_{-\overline{\xi}}^{0}k(s)\Delta s\Big]\Big|X(\tau)-\widetilde{X}(\tau)\Big| \\
& + \vartheta\Big[-\rho^{\frac{\vartheta_2}{\vartheta}}\underline{a_{22}} + \rho^{-\frac{1}{\vartheta}}\overline{\varphi_2}\Big]\Big|Y(\tau)-\widetilde{Y}(\tau)\Big| < 0.
\end{aligned} \tag{27}$$

Thus, from (22), (23) and (27), we know that $V(\tau)$ is positive definite and $D^+V^\Delta(\tau) < 0$, for all $\tau \geq 0$. Therefore, according to Lyapunov stability theory, we conclude that the ω-periodic solution $(\widetilde{X}(\tau), \widetilde{Y}(\tau))^T$ of (20) is globally asymptotically stable. The proof is completed. □

5. An example and its simulation

In this section, we consider a nonlinear commensalism Ayala-Gilpin ecosystem with distributed lags on time scale $\mathbb{T} = \mathbb{R}$

$$\begin{cases} \frac{d\mathcal{X}(\tau)}{d\tau} = \mathcal{X}(\tau)\Big[r_1(\tau) - a_{11}(\tau)\mathcal{X}^{\vartheta_1}(\tau)\Big] - \varphi_1(\tau), \\ \frac{d\mathcal{Y}(\tau)}{d\tau} = \mathcal{Y}(\tau)\Big[r_2(\tau) - a_{22}(\tau)\mathcal{Y}^{\vartheta_2}(\tau) + a_{21}(\tau)\int_{-\xi(\tau)}^{0}k(s)\mathcal{X}(\tau+s)ds\Big] - \varphi_2(\tau), \end{cases} \tag{28}$$

where $r_1(\tau) = 8 + 2\cos(3\tau)$, $r_2(\tau) = 6 + \sin(2\tau)$, $a_{11}(\tau) = 5 + 2\sin(\tau)$, $a_{22}(\tau) = 3 + \cos(2\tau)$, $a_{21}(\tau) = \frac{3+\cos(3\tau)}{10}$, $\xi(t) = k(\tau) = \frac{2+\cos(\tau)}{4}$, $\varphi_1(\tau) = \frac{3+\cos(2\tau)}{7}$, $\varphi_2(\tau) = \frac{4+\sin(\tau)}{7}$, $\vartheta_1 = \frac{1}{2}$, $\vartheta_2 = \frac{1}{\sqrt{2}}$. Take the initial functions $\mathcal{X}(\tau) = 7 + \sin(t)$, $\mathcal{Y}(\tau) = \frac{2+\cos(\tau)}{7}$, $\tau \in [-\overline{\xi}, 0]$.

Obviously, $r_1(\tau)$, $r_2(\tau)$, $a_{11}(\tau)$, $a_{22}(\tau)$, $a_{21}(\tau)$, $\xi(\tau)$, $k(\tau)$, $\varphi_1(\tau)$ and $\varphi_2(\tau)$ are all positive periodic functions with period $\omega = 2\pi$. That is, the conditions (H_1) holds. A direct calculation leads $\overline{r_1} = 10$, $\underline{r_1} = 6$, $\overline{r_2} = 7$, $\underline{r_2} = 5$, $\overline{a_{11}} = 7$, $\underline{a_{11}} = 3$, $\overline{a_{22}} = 4$, $\underline{a_{22}} = 2$, $\overline{a_{21}} = \frac{2}{5}$, $\underline{a_{21}} = \frac{1}{5}$, $\overline{\varphi_1} = \frac{4}{7}$, $\underline{\varphi_1} = \frac{2}{7}$, $\overline{\varphi_2} = \frac{5}{7}$, $\underline{\varphi_2} = \frac{3}{7}$, $\overline{\xi} = \frac{3}{4}$, $\int_{-\overline{\xi}}^{0}k(s)ds \approx 0.5454$. Solving the following equation

$$f(\mathcal{Z}) = (\underline{a_{11}}e^{-\omega\vartheta_1\overline{r_1}})e^{(1+\vartheta_1)\mathcal{Z}} - \overline{r_1}e^{\mathcal{Z}} + \underline{\varphi_1} = 0,$$

we obtain the only two roots $L^- \approx -3.5553$, $L^+ \approx 65.2398$. From the equation below

$$g(\mathcal{Z}) = (\underline{a_{22}}e^{-\omega\vartheta_2\overline{r_2}})e^{(1+\vartheta_2)\mathcal{Z}} - \Big[\overline{r_2} + \overline{a_{21}}e^{L^+}\int_{-\overline{\xi}}^{0}k(s)ds\Big]e^{\mathcal{Z}} + \underline{\varphi_2} = 0,$$

we similarly get its only two roots $M^- \approx -64.5646$, $M^+ \approx 133.1119$. Consequently, we obtain an open bounded subsets $\Omega \subset \mathbb{R}^2$ defined by

$$\Omega = \left\{(\mathcal{X}(\tau), \mathcal{Y}(\tau))^T : 1.4737 \times 10^{-29} < \mathcal{X}(\tau) < 2.1542 \times 10^{28}, 7.2224 \times 10^{-48} < \mathcal{Y}(\tau) < 6.4530 \times 10^{57}\right\}.$$

So much for that, we verify that the condition (H_2) holds as follows:

$$\vartheta_1(\underline{a_{11}}e^{-\omega\vartheta_1\overline{r_1}})^{-\frac{1}{\vartheta_1}}\left(\frac{\overline{r_1}}{1+\vartheta_1}\right)^{\frac{1+\vartheta_1}{\vartheta_1}} \approx 3.1914 \times 10^{28} > \underline{\varphi_1} = \frac{2}{7},$$

$$\vartheta_2(\underline{a_{22}}e^{-\omega\vartheta_2\overline{r_2}})^{-\frac{1}{\vartheta_2}}\left(\frac{\overline{r_2} + \overline{a_{21}}e^{L^+}\int_{-\overline{\xi}}^{0} k(s)ds}{1+\vartheta_2}\right)^{\frac{1+\vartheta_2}{\vartheta_2}} \approx 2.5538 \times 10^{84} > \underline{\varphi_2} = \frac{3}{7}.$$

So far as, we have verified the conditions (H_1) and (H_2) are all true. It follows from Theorem 3.5 that (28) has at least an 2π-periodic positive solutions $(\widetilde{\mathcal{X}}(\tau), \widetilde{\mathcal{Y}}(\tau))^T \in \Omega$.

Next, we demonstrate that the periodic positive solution $(\widetilde{\mathcal{X}}(\tau), \widetilde{\mathcal{Y}}(\tau))^T$ is globally asymptotically stable. In fact, choose $\rho = 1.5 \times 10^{-29}$, $\vartheta = 120$, we yield

$$-\rho^{\frac{\vartheta_1}{\vartheta}}\underline{a_{11}} + \rho^{-\frac{1}{\vartheta}}\overline{\varphi_1} + \rho^{\frac{1}{\vartheta}}\overline{a_{21}}\int_{-\overline{\xi}}^{0} k(s)ds \approx -1.1551 < 0, \quad -\rho^{\frac{\vartheta_2}{\vartheta}}\underline{a_{22}} + \rho^{-\frac{1}{\vartheta}}\overline{\varphi_2} \approx -0.1108 < 0.$$

Hence the condition (H_3) holds. By Theorem 4.2, we conclude that the periodic solution $(\widetilde{\mathcal{X}}(\tau), \widetilde{\mathcal{Y}}(\tau))^T$ is globally asymptotically stable. In addition, By ddesd toolbox in MATLAB, the simulations of solution to example (28) is given as shown in Figure 1.

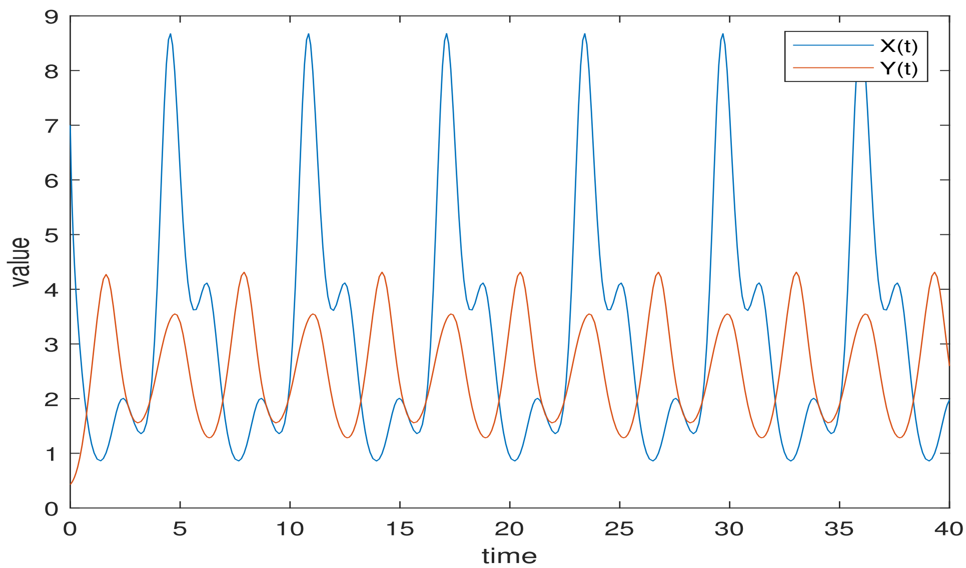

Figure 1: Existence and global asymptotic stability of solution $(\widetilde{\mathcal{X}}(\tau), \widetilde{\mathcal{Y}}(\tau))^T$ to (28).

6. Conclusions

The Ayala-Gilpin ecosystem model is an important and well-known differential equation. The study of the dynamic behavior and properties of this ecosystem can provide a theoretical basis for the governance and protection. This work deals with a classical nonlinear periodic commensalism Ayala-Gilpin ecosystem (1) with distributed lags and control terms on time scales. By employing some inequality techniques, we first build a priori estimates of the existence region of solutions. Based on the theory coincidence degree in nonlinear analysis, we obtain some sufficient criteria to ensure the existence of periodic solutions to (1). Secondly, we also establish the global asymptotical stability by constructing some Lyapunov functionals, Finally, a numerical example and its simulation is provided to verify the correctness and availability of our main results. In recent years, some scholars have begun to apply fractional order differential equation models and diffusion partial differential equation models to study the dynamic behavior of ecosystems. Aroused by the latest published articles [10, 29–38], we determined to use fractional calculus and reaction-diffusion differential equation theory to explore the dynamics of some ecosystems in the future.

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