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Continuous characterizations of weighted Besov spaces of variable smoothness and integrability

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Abstract. In this paper, we give continuous version of the weighted Besov spaces of variable smoothness and integrability and obtain their equivalent norms by Peetre's maximal functions.

1. Introduction

Spaces of variable integrability, also known as variable exponent space, are traced back to Orlicz [26, 27], and studied by Musielak [22] and Nakano [23, 24], but the modern development started with the work [18] of Kováčik and Rákosník and continued by the boundedeness of Hardy-Littlewood maximal operator on variable Lebesgue spaces [9, 10, 25]. Function spaces of variable smoothness were studied by Besov [4–6]. Then many variable spaces have appeared, such as variable Besov and Triebel-Lizorkin spaces [1, 2, 11, 13, 14, 17, 30, 35–38], weak Triebel-Lizorkin spaces with variable integrability, summability and smoothness [19], variable exponent Herz type Besov and Triebel-Lizorkin spaces [31], Morrey-Triebel-Lizorkin spaces with variable smoothness and integrability [34], variable Triebel-Lizorkin spaces associated with non-negative self-adjoint operators [32], Variable integral and smooth exponent Besov spaces associated to non-negative self-adjoint operators [33], weighted Besov spaces of variable smoothness and integrability [28]. As a continuation of [28], we will give equivalent norms of the weighted Besov spaces of variable smoothness and integrability by Peetre's maxiaml functions and continuous Fourier analytical tools.

The plan of the paper is as follows. In Section 2, we give continuous version of weighted Besov spaces of variable smoothness and integrability and their equivalent norms. Their proof will be given in Section 3. In Section 4, we show that the continuous version of weighted Besov spaces of variable smoothness and integrability is equivalent to that in [28]. In Section 5, we extend the results in Sections 2 and 4 for $w \in A_{p(\cdot)}$ to $w \in A_\infty$.

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2. Notations and main results

In this section, we first recall some definitions and notations. Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{Z}$ be the collection of all integers. Let $\mathbb{R}^n$ be n-dimensional Euclidean space, where $n \in \mathbb{N}$. Put $\mathbb{R} = \mathbb{R}^1$, whereas $\mathbb{C}$ is the complex plane. Let $p(\cdot)$ be a measurable function on $\mathbb{R}^n$ taking values in $[1, \infty)$, the Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) := \left\{ f \text{ is measurable on } \mathbb{R}^n : \rho_{p(\cdot)}(f/\lambda) < \infty \text{ for some } \lambda > 0 \right\},$$

where and what follows $\rho_{p(\cdot)} := \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx$. Then $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach function space equipped with the norm

$$\|f\|_{L^{p(\cdot)}} := \inf\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \le 1\}.$$

These spaces are generalization of the classical constant exponent Lebesgue spaces. The space $L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n)$ is defined by $L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n) := \{f : f\chi_K \in L^{p(\cdot)}(\mathbb{R}^n)$ for all compact subsets $K \subset \mathbb{R}^n\}$, where and what follows, χ_S denotes the characteristic function of a set $S \subset \mathbb{R}^n$. Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$, we denote $p^- := \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x)$, $p^+ := \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x)$. The set $\mathcal{P}(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p^- > 1$ and $p^+ < \infty$; $\mathcal{P}_0(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p^- > 0$ and $p^+ < \infty$. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then $p'(\cdot)$ be the conjugate exponent of $p(\cdot)$, that means $1/p(\cdot) + 1/p'(\cdot) = 1$.

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and w be a weight which is a nonnegative measurable function on $\mathbb{R}^n$. Then the weighted variable exponent Lebesgue space $L^{p(\cdot)}_w(\mathbb{R}^n)$ is the set of all complex-valued measurable functions f such that $fw \in L^{p(\cdot)}(\mathbb{R}^n)$. The space $L^{p(\cdot)}_w(\mathbb{R}^n)$ is a Banach space equipped with the norm

$$\|f\|_{L^{p(\cdot)}_w} := \|fw\|_{L^{p(\cdot)}}.$$

Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then the standard Hardy-Littlewood maximal function of f is defined by

$$\mathcal{M}f(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy, \ \forall x \in \mathbb{R}^n,$$

where the supremum is taken over all balls containing x in $\mathbb{R}^n$. In general, the Hardy-Littlewood maximal operator is not bounded on weighted variable Lebesgue spaces. But if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and satisfies the following global log-Hölder continuity condition and $w \in A_{p(\cdot)}$, then $\mathcal{M}$ is bounded on $L^{p(\cdot)}_w(\mathbb{R}^n)$ in [8].

Definition 1. *Let $\alpha(\cdot)$ be a real-valued measurable function on $\mathbb{R}^n$.*

(i) *The function $\alpha(\cdot)$ is locally* log*-Hölder continuous if there exists a constant $C_{\log(\alpha)}$ such that*

$$|\alpha(x) - \alpha(y)| \le \frac{C_{\log(\alpha)}}{\log(e + 1/|x - y|)}, \ x, y \in \mathbb{R}^n, \ |x - y| < \frac{1}{2}.$$

Denote by $C^{\log}_{\text{loc}}(\mathbb{R}^n)$ the set of all locally log*-Hölder continuous functions on $\mathbb{R}^n$.*

(ii) *The function $\alpha(\cdot)$ is* log*-Hölder continuous at the origin if there exists a constant C_2 such that*

$$|\alpha(x) - \alpha(0)| \le \frac{C_2}{\log(e + 1/|x|)}, \ \forall x \in \mathbb{R}^n.$$

Denote by $C^{\log}_0(\mathbb{R}^n)$ the set of all log*-Hölder continuous functions at the origin.*

(iii) *The function $\alpha(\cdot)$ is* log*-Hölder continuous at infinity if there exists $\alpha_\infty \in \mathbb{R}$ and a constant C_3 such that*

$$|\alpha(x) - \alpha_\infty| \le \frac{C_3}{\log(e + |x|)}, \ \forall x \in \mathbb{R}^n.$$

Denote by $C^{\log}_\infty(\mathbb{R}^n)$ the set of all log*-Hölder continuous functions at infinity.*

(iv) *The function $\alpha(\cdot)$ is global* log*-Hölder continuous if $\alpha(\cdot)$ are both locally* log*-Hölder continuous and* log*-Hölder continuous at infinity. Denote by $C^{\log}(\mathbb{R}^n)$ the set of all global* log*-Hölder continuous functions.*

Definition 2. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, a positive measurable function w is said to be in $A_{p(\cdot)}$, if there exists a positive constant C for all balls B in $\mathbb{R}^n$ such that*

$$|B|^{-1}\|w\chi_B\|_{L^{p(\cdot)}}\|w^{-1}\chi_B\|_{L^{p'(\cdot)}} \leqslant C.$$

In [8], Cruz-Uribe, Fiorenza and Neugebauer obtained the following result, which generalizes the result for constant exponent Lebesgue spaces in [21].

Lemma 1 (see [8, Theorem 1.5]). *If $p(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ and $w \in A_{p(\cdot)}$, then there is a positive constant C such that for each $f \in L_w^{p(\cdot)}(\mathbb{R}^n)$,*

$$\|(\mathcal{M}f)w\|_{L^{p(\cdot)}} \leq C\|fw\|_{L^{p(\cdot)}}.$$

Let $\{f_t\}_{0<t\leq 1}$ be a family of measurable functions when t is a continuous variable. We set

$$\rho_{\widetilde{\ell^{q(\cdot)}(L_w^{p(\cdot)})}}(\{f_t\}_{0<t\leq 1}) := \int_0^1 \inf\Big\{\lambda_t : \rho_{p(\cdot)}\Big(f_t w/\lambda_t^{\frac{1}{q(\cdot)}}\Big) \leq 1\Big\}\frac{dt}{t},$$

The norm is

$$\|\{f_t\}_{0<t\leq 1}\|_{\widetilde{\ell^{q(\cdot)}(L_w^{p(\cdot)})}} := \inf\Big\{\mu > 0 : \ \rho_{\widetilde{\ell^{q(\cdot)}(L_w^{p(\cdot)})}}(\{f_t\}_{0<t\leq 1}/\mu) \leq 1\Big\}.$$

In the following we denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$. For $f \in \mathcal{S}(\mathbb{R}^n)$, let $\mathcal{F}f$ or $\hat{f}$ denote the Fourier transform of f defined by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) := (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\xi}f(x)dx, \ \xi \in \mathbb{R}^n,$$

while $f^\vee(\xi) = \widehat{f}(-\xi)$ denote the inverse Fourier transform of f.

Select a pair of Schwartz functions Φ and φ satisfying

$$\operatorname{supp}\mathcal{F}\Phi \subset \{x \in \mathbb{R}^n : |x| \leq 2\}, \quad \operatorname{supp}\mathcal{F}\varphi \subset \{x \in \mathbb{R}^n : 1/2 \leq |x| \leq 2\} \tag{1}$$

and

$$\mathcal{F}\Phi(\xi) + \int_0^1 \mathcal{F}\varphi(t\xi)\frac{dt}{t} = 1, \quad \xi \in \mathbb{R}^n. \tag{2}$$

Such a resolution (1) and (2) of unity can be constructed as follows. Let $\mu \in \mathcal{S}(\mathbb{R}^n)$ be such that $|\mathcal{F}\mu(\xi)| > 0$ for $1/2 < |\xi| < 2$. There exists $\eta \in \mathcal{S}(\mathbb{R}^n)$ with

$$\operatorname{supp}\mathcal{F}\eta \subset \{x \in \mathbb{R}^n : 1/2 < |x| < 2\}$$

such that

$$\int_0^\infty \mathcal{F}\mu(t\xi)\mathcal{F}\eta(t\xi)\frac{dt}{t} = 1, \quad \xi \neq 0,$$

see [7], [15] and [16]. We set $\mathcal{F}\varphi = \mathcal{F}\mu\mathcal{F}\eta$ and

$$\mathcal{F}\Phi(\xi) = \begin{cases} \int_1^\infty \mathcal{F}\varphi(t\xi)\frac{dt}{t} & \text{if} \quad \xi \neq 0, \\ 1 & \text{if} \quad \xi = 0. \end{cases}$$

Then $\mathcal{F}\Phi \in \mathcal{S}(\mathbb{R}^n)$, and as $\mathcal{F}\eta$ is supported in $\{x \in \mathbb{R}^n : 1/2 \leq |x| \leq 2\}$, we see that $\operatorname{supp}\mathcal{F}\Phi \subset \{x \in \mathbb{R}^n : |x| \leq 2\}$.

Now we define continuous version of weighted Besov spaces with variable exponents as follows.

Definition 3. *Let* $s(\cdot):\mathbb{R}^n\to\mathbb{R}$, $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, $q(\cdot)\in\mathcal{P}_0(\mathbb{R}^n)$ *and* w *be a weight.. Let* $\{\mathcal{F}\Phi,\mathcal{F}\varphi\}$ *be a resolution of unity as above and we put* $\varphi_t(\cdot)=t^{-n}\varphi(\cdot/t)$, $0<t\le 1$. *The Besov space* $\mathcal{B}^{s(\cdot),w}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *is the collection of all* $f\in\mathcal{S}'(\mathbb{R}^n)$ *such that*

$$\|f\|^{\Phi,\varphi}_{\mathcal{B}^{s(\cdot),w}_{p(\cdot),q(\cdot)}}:=\|\Phi*f\|_{L^{p(\cdot)}_w}+\|\{t^{-s(\cdot)}\varphi_t*f\}_{0<t\le1}\|_{\ell^{q(\cdot)}(\widetilde{L^{p(\cdot)}_w})}<\infty.$$

When $q=\infty$, the Besov space $\mathcal{B}^{s(\cdot),w}_{p(\cdot),\infty}(\mathbb{R}^n)$ consists of all distributions $f\in\mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|^{\Phi,\varphi}_{\mathcal{B}^{s(\cdot),w}_{p(\cdot),\infty}}:=\|\Phi*f\|_{L^{p(\cdot)}_w}+\sup_{t\in(0,1]}\|t^{-s(\cdot)}\varphi_t*f\|_{\ell^{q(\cdot)}(\widetilde{L^{p(\cdot)}_w})}<\infty.$$

Before going on, we dente by $A\lesssim B$ that there is a constant C such that $A\le CB$. If $A\lesssim B$ and $B\lesssim A$, we denote $A\approx B$. First, these spaces are independent of the choice of Φ and φ.

Theorem 1. *Let* $\{\mathcal{F}\Phi,\mathcal{F}\varphi\}$ *and* $\{\mathcal{F}\Psi,\mathcal{F}\psi\}$ *be two resolutions of unity satisfying (1) and (2). Let* $s(\cdot)\in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, $p(\cdot),q(\cdot)\in\mathcal{P}(\mathbb{R}^n)\cap C^{\log}(\mathbb{R}^n)$, *and* $w\in A_{p(\cdot)}$. *Then*

$$\|f\|^{\Phi,\varphi}_{\mathcal{B}^{s(\cdot),w}_{p(\cdot),q(\cdot)}}\approx\|f\|^{\Psi,\psi}_{\mathcal{B}^{s(\cdot),w}_{p(\cdot),q(\cdot)}}.$$

Let $a>0$, $s(\cdot)\in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ and $t>0$. For each $f\in\mathcal{S}'(\mathbb{R}^n)$, the Peetre maximal functions for f are defined by

$$\varphi^{*,a}_t t^{-s(\cdot)}f(x):=\sup_{y\in\mathbb{R}^n}\frac{t^{-s(y)}|\varphi_t*f(y)|}{(1+t^{-1}|x-y|)^a}$$

and

$$\Phi^{*,a}f(x):=\sup_{y\in\mathbb{R}^n}\frac{|\Phi*f(y)|}{(1+|x-y|)^a}.$$

Then these spaces can be characterized via the Peetre maximal functions.

Theorem 2. *Let* $s(\cdot)\in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, $p(\cdot),q(\cdot)\in\mathcal{P}(\mathbb{R}^n)\cap C^{\log}(\mathbb{R}^n)$, $w\in A_{p(\cdot)}$ *and* $a>n/p^-$. *Then*

$$\|f\|^{*}_{\mathcal{B}^{s(\cdot),w}_{p(\cdot),q(\cdot)}}:=\|\Phi^{*,a}\|_{L^{p(\cdot)}_w}+\|(\varphi^{*,a}t^{-s(\cdot)}f)_{0<t\le1}\|_{\ell^{q(\cdot)}(\widetilde{L^{p(\cdot)}_w})}$$

is an equivalent quasi-norm in $\mathcal{B}^{s(\cdot),w}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

The proofs of Theorems 1 and 2 will be given in the next section.

3. Proofs of Theorems 1 and 2

To prove Theorems 1 and 2, we need further preparation.

For a positive real number m, let η_m be the function as

$$\eta_m:=(1+|x|)^{-m}.$$

For $\nu\in\mathbb{N}_0$ and a positive real number m, we denote

$$\eta_{\nu,m}(x):=\frac{2^{n\nu}}{(1+2^\nu|x|)^m}.$$

Lemma 2 (see [36, Lemma 3.14]). *Let $p(\cdot)$, $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ and f be a measurable function on $\mathbb{R}^n$. If*

$$\left\| |f|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \geq 1,$$

then

$$\|f\|_{L^{p(\cdot)}}^{q^-} \leq \left\| |f|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}.$$

Lemma 3 (see [11, Lemma 6.1]). *If $\vartheta(\cdot) \in C_{\text{loc}}^{\log}(\mathbb{R}^n)$ and $R \geq C_{\log(\vartheta)}$, where $C_{\log(\vartheta)}$ is the constant in* (i) *of Definition 1 for $\vartheta = \alpha$, then*

$$2^{\vartheta(x)} \eta_{\nu,m+R}(x-y) \leq C 2^{\vartheta(y)} \eta_{\nu,m}(x-y)$$

with $C > 0$ independent of $x, y \in \mathbb{R}^n$ and $\nu \in \mathbb{N}_0$.

If $\vartheta \in C_{\text{loc}}^{\log}(\mathbb{R}^n)$ and $R \geq C_{\log(\vartheta)}$, then by Lemma 3 for $x, y \in \mathbb{R}^n$ and $\nu \in \mathbb{N}_0$,

$$2^{\vartheta(x)} \eta_{\nu,m+R} * f(x) \lesssim \eta_{\nu,m} * (2^{\vartheta(\cdot)} f)(x).$$

Lemma 4 (see [28, Lemma 4]). *If $p(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ and $w \in A_{p(\cdot)}$, then for every $m > n$ there exists $C > 0$ such that,*

$$\|\eta_{\nu,m} * g\|_{L_w^{p(\cdot)}} \leq C \|g\|_{L_w^{p(\cdot)}}$$

for all $\nu \geq 0$, functions $g \in L_w^{p(\cdot)}(\mathbb{R}^n)$.

Lemma 5 (see [11, Lemma A.6]). *Let $r > 0, \nu \in \mathbb{N}_0$ and $m \geq n+1$. Then there exists $C = C(r, m, n) > 0$ such that*

$$|g(x)| \leq C (\eta_{\nu,m} * |g|^r(x))^{\frac{1}{r}}$$

for all $x \in \mathbb{R}^n$, and every $g \in \mathcal{S}'(\mathbb{R}^n)$ with $\operatorname{supp} \mathcal{F} g \subset \{\xi : |\xi| \leq 2^{\nu+1}\}$.

Similar to Lemma 3 in [33], we have the following lemma.

Lemma 6. *Let $p(\cdot), q(\cdot)$ are positive functions on $\mathbb{R}^n$ such that $0 < p^-, q^- \ p^+, q^+ < \infty$. Let $\delta \in (0, \infty)$. Let w be a weight. For any sequence $\{g_j\}_{j=0}^{\infty}$ of nonnegative measurable functions on $\mathbb{R}^n$ denote*

$$G_j(x) := \sum_{k=0}^{\infty} 2^{-|k-j|\delta} g_k(x), \qquad x \in \mathbb{R}^n.$$

Then there is a positive constant $C = C(p(\cdot), q(\cdot), \delta)$ such that

$$\|\{G_j\}_{j=0}^{\infty}\|_{\ell^{q(\cdot)}(L_w^{p(\cdot)})} \leqslant C \|\{g_k\}_{k=0}^{\infty}\|_{\ell^{q(\cdot)}(L_w^{p(\cdot)})}.$$

The next lemma is Hardy type inequalities; see [20].

Lemma 7. *Let $s > 0$ and $\{\varepsilon_t\}_{0<t\leq 1}$ be a sequence of positive measurable functions when t is a continuous variable. Let*

$$\eta_t = t^s \int_t^1 \tau^{-s} \varepsilon_\tau \frac{d\tau}{\tau} \quad \text{and} \quad \delta_t = t^{-s} \int_0^t \tau^s \varepsilon_\tau \frac{d\tau}{\tau}.$$

Then there exists a constant $C > 0$ depending only on s such that

$$\int_0^1 \eta_t \frac{dt}{t} + \int_0^1 \delta_t \frac{dt}{t} \leq C \int_0^1 \varepsilon_t \frac{dt}{t}.$$

Lemma 8 (see [28, Lemma 6]). *Let $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, $w \in A_{p(\cdot)}$, and $m > n + C_{\log(1/q)}$, where $C_{\log(1/q)}$ is the constant in* (i) *of Definition 1 for $1/q = \alpha$. Then there exists a positive constant C such that*

$$\|\{\eta_{\nu,m} * f_\nu\}_{\nu=0}^\infty\|_{\ell^{q(\cdot)}(L_w^{p(\cdot)})} \le C\|\{f_\nu\}_{\nu=0}^\infty\|_{\ell^{q(\cdot)}(L_w^{p(\cdot)})}$$

holds for every sequence $\{f_\nu\}_{\nu=0}^\infty$ of locally Lebesgue integrable functions.

Lemma 9. *Let $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, $w \in A_{p(\cdot)}$ and $m > n + C_{\log(1/q)}$, where $C_{\log(1/q)}$ is the constant in* (i) *of Definition 1 for $1/q = \alpha$. Then there exists a positive constant C such that*

$$\|\{\eta_{t,m} * f_t\}_{0<t\le 1}\|_{\widetilde{\ell^{q(\cdot)}(L_w^{p(\cdot)})}} \le C\Big\|\{f_t\}_{0<t\le 1}\Big\|_{\widetilde{\ell^{q(\cdot)}(L_w^{p(\cdot)})}}$$

holds for every sequence functions $\{f_t\}_{0<t\le 1}$.

Proof. By the scaling argument, without loss of generality we may assume that $\|\{f_t\}_{0<t\le 1}\|_{\widetilde{\ell^{q(\cdot)}(L_w^{p(\cdot)})}} = 1$. We only prove that there exists a constant c such that

$$\int_0^1 \Big\||c\eta_{t,m} * f_t|^{q(\cdot)} w^{q(\cdot)}\Big\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \frac{dt}{t} \le 2.$$

This clearly follows from the inequality

$$\Big\||c\eta_{t,m} * f_t|^{q(\cdot)} w^{q(\cdot)}\Big\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \le \Big\||f_t|^{q(\cdot)} w^{q(\cdot)}\Big\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} + t := \delta_t$$

for any $t \in (0,1]$. This claim can be reformulated as showing that

$$\Big\|\delta_t^{-1} |c\eta_{t,m} * f_t|^{q(\cdot)} w^{q(\cdot)}\Big\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \le 1$$

which is equivalent to

$$c\Big\|\delta_t^{-\frac{1}{q(\cdot)}} \eta_{t,m} * f_t\Big\|_{L_w^{p(\cdot)}} \le 1, \quad t \in (0,1].$$

Since $1/q(\cdot)$ is log-Hölder continuous and $\delta_t \in (t, 1+t]$, then by Lemma 3, we have

$$\delta_t^{-1/q(\cdot)} |\eta_{t,m} * f_t| \le C|\eta_{t,m-C_{\log(1/q)}} * (\delta_t^{-1/q(\cdot)} f_t)|. \tag{3}$$

Then by (3) and Lemma 4, we obtain

$$\Big\|\delta_t^{-\frac{1}{q(\cdot)}} \eta_{t,m} * f_\nu\|_{L_w^{p(\cdot)}} \le \Big\|c\eta_{t,m-C_{\log(1/q)}} * (\delta_t^{-\frac{1}{q(\cdot)}} f_t)\Big\|_{L_w^{p(\cdot)}} \le \Big\|\delta_t^{-\frac{1}{q(\cdot)}} f_t\Big\|_{L_w^{p(\cdot)}}$$

with an appropriate choice of $c > 0$. Now the right-hand side is bounded if and only if

$$\Big\||f_t w|^{q(\cdot)}\Big\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \le \delta_t$$

which follows from the definition of δ_t. □

Lemma 10. *Let $0 < \alpha < \beta < \infty$, $w \in A_{p(\cdot)}$, $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ with $1/q(\cdot) \in C_{\mathrm{loc}}^{\log}(\mathbb{R}^n)$. Let*

$$g_t := \int_{\alpha t}^{\beta t} \eta_{\tau,m} * f_\tau(x) \frac{d\tau}{\tau}, \quad t \in (0,1],\ x \in \mathbb{R}^n.$$

(i) *Assume that* $0 < \beta t \le 1$. *The inequality*

$$\left\| |cg_t w|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \le \int_{\alpha t}^{\beta t} \left\| |f_\tau w|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \frac{d\tau}{\tau} + t, \quad t \in (0,1]$$

holds for every sequence of functions $(f_t)_{0<t\le 1}$ *and constant* $m > n + C_{\log(1/q)}$ *such that the first term on right-hand side is at most one, where the constant c independent of t, where* $C_{\log(1/q)}$ *is the constant in* (i) *of Definition 1 for* $1/q = \alpha$.
(ii) *The inequality*

$$\|\{g_t\}_{0<t\le 1}\|_{\ell^{q(\cdot)}(\widetilde{L_w^{p(\cdot)}})} \le C \|\{f_t\}_{0<t\le 1}\|_{\ell^{q(\cdot)}(\widetilde{L_w^{p(\cdot)}})}$$

holds for every sequence of functions $(f_t)_{0<t\le 1}$ *and constant* $m > n + C_{\log(1/q)}$ *such that the right-hand side is finite, where* $C_{\log(1/q)}$ *is the constant in* (i) *of Definition 1 for* $1/q = \alpha$.

Proof. (i). We put

$$\delta_t := \int_{\alpha t}^{\beta t} \left\| |c f_\tau w|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \frac{d\tau}{\tau} + t.$$

Since $1/q(\cdot)$ is log-Hölder continuous and $\delta_t \in (t, 1+t]$, then by Lemma 3, we have

$$\delta_t^{-1/q(\cdot)} |\eta_{\tau,m} * f_t| \le C |\eta_{\tau, m - C_{\log(1/q)}} * (\delta_t^{-1/q(\cdot)} f_t)|. \tag{4}$$

Then by (4) and Lemma 4, we obtain

$$\begin{aligned} \left\| c\delta_t^{-\frac{1}{q(\cdot)}} g_t \right\|_{L_w^{p(\cdot)}} &\le \int_{\alpha t}^{\beta t} \left\| c\delta_t^{-\frac{1}{q(\cdot)}} (\eta_{\tau,m} * f_\tau) \right\|_{L_w^{p(\cdot)}} \frac{d\tau}{\tau} \\ &\le \int_{\alpha t}^{\beta t} \left\| \eta_{\tau, m - C_{\log(1/q)}} * (c\delta_t^{-\frac{1}{q(\cdot)}} f_\tau) \right\|_{L_w^{p(\cdot)}} \frac{d\tau}{\tau} \\ &\le \int_{\alpha t}^{\beta t} \left\| c\delta_t^{-\frac{1}{q(\cdot)}} f_\tau \right\|_{L_w^{p(\cdot)}} \frac{d\tau}{\tau} \\ &= \int_{(\beta t, \alpha t] \cap E} \left\| c\delta_t^{-\frac{1}{q(\cdot)}} f_\tau \right\|_{L_w^{p(\cdot)}} \frac{d\tau}{\tau} + \int_{(\beta t, \alpha t] \cap E^c} \left\| c\delta_t^{-\frac{1}{q(\cdot)}} f_\tau \right\|_{L_w^{p(\cdot)}} \frac{d\tau}{\tau} \\ &:= F_1 + F_2, \end{aligned}$$

where

$$E := \left\{ \tau > 0 : \left\| |\delta_t^{-\frac{1}{q(\cdot)}} f_\tau w|^{q(\cdot)} \right\|_{L^{p(\cdot)/q(\cdot)}} \ge 1 \right\}.$$

By Lemma 2, we have

$$F_1 \le \int_{(\beta t, \alpha t] \cap E} \left\| |\delta_t^{-\frac{1}{q(\cdot)}} f_\tau w|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}^{\frac{1}{q^-}} \frac{d\tau}{\tau} \le \delta_t^{-1} \int_{\alpha t}^{\beta t} \left\| |f_\tau w|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \frac{d\tau}{\tau} \le 1$$

and

$$F_2 \le \int_{\alpha t}^{\beta t} \left\| \delta_t^{-\frac{1}{q(\cdot)}} f_\tau \right\|_{L_w^{p(\cdot)}} \frac{d\tau}{\tau} \le \int_{\alpha t}^{\beta t} \frac{d\tau}{\tau} = \log \frac{\beta}{\alpha}.$$

(ii). By the scaling argument, without loss of generality we may assume that $\|(f_t)_{0<t\le 1}\|_{\ell^{q(\cdot)}(\widetilde{L_w^{p(\cdot)}})} = 1$. we will show that

$$\int_0^1 \left\| |cg_t|^{q(\cdot)} w^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \frac{dt}{t} \le 2 \quad \text{whenever} \quad \left\| |f_t|^{q(\cdot)} w^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} = 1.$$

By Lemma 7 and (i), we obtain the desired result. □

Lemma 11 (see [3, Lemma 2.8]). *Let $0 < r < 1$ and $m > \max\{n, n/r\}$, and $\{\mathcal{F}\Phi, \mathcal{F}\varphi\}$ be a resolution of unuty,*

$$\mathcal{F}\Phi(\xi) + \int_0^1 \mathcal{F}\varphi(t\xi)\frac{dt}{t} = 1, \quad \xi \in \mathbb{R}^n.$$

(i) *Let $\theta \in \mathcal{S}(\mathbb{R}^n)$ be such that $\operatorname{supp} \mathcal{F}\theta \subset \{\xi \in \mathbb{R}^n : |\xi| \le 2\}$. Then there exists a constant $C > 0$ such that*

$$|\theta * f|^r \le C\eta_{1,mr} * |\Phi * f|^r + C\int_{1/4}^1 \eta_{1,mr} * |\varphi_\tau * f|^r \frac{d\tau}{\tau}$$

for any $f \in \mathcal{S}'(\mathbb{R}^n)$, where $\varphi_\tau = \tau^{-n}\varphi(\cdot/\tau)$.

(ii) *Let $\omega \in \mathcal{S}(\mathbb{R}^n)$ be such that $\operatorname{supp} \mathcal{F}\omega \subset \{\xi \in \mathbb{R}^n : 1/2 \le |\xi| \le 2\}$. There exists a constant $C > 0$ such that*

$$|\omega_t * f|^r \le C\eta_{1,mr} * |\Phi * f|^r + C\int_{1/4}^{\min\{1,4t\}} \eta_{\tau,mr} * |\varphi_\tau * f|^r \frac{d\tau}{\tau}$$

for any $f \in \mathcal{S}'(\mathbb{R}^n)$ and $0 < t \le 1$, where $\omega_\tau = t^{-n}\omega(\cdot/t)$.

Proof of Theorem 1. It is sufficient to show that there exists a constant $C > 0$ such that for all $\|f\|^{\Psi,\psi}_{\mathcal{B}^{s(\cdot),w}_{p(\cdot),q(\cdot)}} < \infty$ we have

$$\|f\|^{\Phi,\varphi}_{\mathcal{B}^{s(\cdot),w}_{p(\cdot),q(\cdot)}} \le C\|f\|^{\Psi,\psi}_{\mathcal{B}^{s(\cdot),w}_{p(\cdot),q(\cdot)}}$$

By the scaling argument, without loss of generality we may assume that $\|f\|^{\Psi,\psi}_{\mathcal{B}^{s(\cdot),w}_{p(\cdot),q(\cdot)}} = 1$. We will show that

$$\|\Phi * f\|_{L^{p(\cdot)}_w} \lesssim 1 \text{ and } \int_0^1 \left\| |ct^{-s(\cdot)}(\varphi_t * f)w|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \frac{dt}{t} \le 1$$

for some positive constant c. We have

$$\mathcal{F}\Phi(\xi) = \mathcal{F}\Phi(\xi)\mathcal{F}\Psi(\xi) + \int_{1/4}^1 \mathcal{F}\Phi(\xi)\mathcal{F}\psi(\tau\xi)\frac{d\tau}{\tau}$$

and

$$\mathcal{F}\varphi(t\xi) = \int_{t/4}^{\min\{1,4t\}} \mathcal{F}\varphi(t\xi)\mathcal{F}\psi(\tau\xi)\frac{d\tau}{\tau} + \begin{cases} 0, & \text{if } \ 0 < t < \frac{1}{4}, \\ \mathcal{F}\varphi(t\xi)\mathcal{F}\Psi(\xi), & \text{if } \ \frac{1}{4} \le t \le 1 \end{cases}$$

for any $\xi \in \mathbb{R}^n$. Then

$$\Phi * f = \Phi * \Psi * f + \int_{1/4}^1 \Phi * \psi_\tau * f \frac{d\tau}{\tau}$$

and

$$\varphi_t * f = \int_{t/4}^{\min\{1,4t\}} \varphi_t * \psi_\tau * f \frac{d\tau}{\tau} + \begin{cases} 0, & \text{if } \ 0 < t < \frac{1}{4} \\ \varphi_t * \Psi * f, & \text{if } \ \frac{1}{4} \le t \le 1. \end{cases}$$

For $m > n$ and $\frac{1}{4} < \tau < 1$, by Lemma 11, we have

$$|\Phi * \psi_\tau * f| \lesssim |\eta_{0,m} * \psi_\tau * f|$$

$$\lesssim \eta_{0,m} * \tau^{-s(\cdot)}|\psi_\tau * f|$$

and

$$|\Phi * \Psi * f| \lesssim \eta_{0,m} * |\Psi * f|.$$

Thus,

$$|\Phi * f| \lesssim \eta_{0,m} * |\Psi * f| + \int_{1/4}^{1} \eta_{0,m} * \tau^{-s(\cdot)}|\psi_\tau * f| =: \eta_{0,m} * |\Psi * f| + g.$$

By Lemma 4, we obtain

$$\left\|\eta_{0,m} * |\Psi * f|\right\|_{L_w^{p(\cdot)}} \leq C\|\Psi * f\|_{L_w^{p(\cdot)}} \lesssim 1.$$

For some suitable $c_1 > 0$, then by Lemma 10,

$$\left\||c_1 g w|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \leq \int_{1/4}^{1} \left\||c_1 t^{-s(\cdot)}(\varphi_t * f) w|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \frac{dt}{t} \leq 1$$

which is equivalent to

$$\|c_1 g\|_{L_w^{p(\cdot)}} \leq 1.$$

Therefore,

$$\|\Phi * f\|_{L_w^{p(\cdot)}} \lesssim 1.$$

With an appropriate choice of $c > 0$ and any $t \in (0, 1]$, then by Lemma 4

$$\left\||c\varphi_t * \Psi * f|^{q(\cdot)} w^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \leq 1.$$

For $m > n$, $t \in (0, 1/4]$, we obtain

$$|\varphi_t * f| \lesssim \int_{t/4}^{4t} \eta_{\tau,m} * |\psi_\tau * f| \frac{d\tau}{\tau}.$$

For some suitable $c > 0$, then by Lemma 10, we obtain

$$\int_0^{\frac{1}{4}} \left\||ct^{-s(\cdot)}(\varphi_t * f) w|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \frac{dt}{t} \leq 1.$$

Interchanging the roles of (Φ, φ) and (Ψ, ψ) we obtain the desired result. □

Proof of Theorem 2. It is easy to see that for any $f \in \mathcal{S}'(\mathbb{R}^n)$ with $\|f\|^*_{\mathcal{B}^{s(\cdot),w}_{p(\cdot),q(\cdot)}} < \infty$ and any $x \in \mathbb{R}^n$ we have

$$t^{-s(x)}|\varphi_t * f(x)| \leq \varphi_t^{*,a} t^{-s(\cdot)} f(x).$$

Therefore, we obtain

$$\|f\|_{\mathcal{B}^{s(\cdot),w}_{p(\cdot),q(\cdot)}} \leq \|f\|^*_{\mathcal{B}^{s(\cdot),w}_{p(\cdot),q(\cdot)}}.$$

Next, we will prove that there is a constant $C > 0$ such that for each $f \in \mathcal{B}^{s(\cdot),w}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$

$$\|f\|^*_{\mathcal{B}^{s(\cdot),w}_{p(\cdot),q(\cdot)}} \le C\|f\|_{\mathcal{B}^{s(\cdot),w}_{p(\cdot),q(\cdot)}}. \tag{5}$$

By the argument in the proof [3, Theorem 3.5], we obtain

$$\varphi_t^{*,a} t^{-s(\cdot)} f(x) \le C'\Big(\eta_{t,ap^-} * (t^{-s(\cdot)}|\varphi_t * f|^{p^-})(x)\Big)^{\frac{1}{p^-}}$$

for all $f \in \mathcal{B}^{s(\cdot),w}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and any $t > 0$ and any $s > \max\{n/p^-, a + C_{\log(s)}\}$, where $C' > 0$ is independent of x, t and f and $C_{\log(1/q)}$ is the constant in (i) of Definition 1 for $1/q = \alpha$. By Lemma 9, we get the desired estimate (5). □

4. Consistency of two versions

It is noted that if $s(\cdot) \in L^\infty(\mathbb{R}^n) \cap C^{\log}_{\text{loc}}(\mathbb{R}^n)$, $p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$ and $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C^{\log}_{\text{loc}}(\mathbb{R}^n)$, and $w \equiv 1$, then $\mathcal{B}^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, see [3].

In this section we discuss the relation between the spaces in Section 2 and the spaces in [28]. Let us recall the definition of the spaces in [28]

Definition 4. *Let Ψ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $\Psi(x) = 1$ for $|x| \le 1$ and $\Psi(x) = 0$ for $|x| \ge 2$. We put $\mathcal{F}\phi_0(x) = \Psi(x)$, $\mathcal{F}\phi_1(x) = \Psi(\frac{x}{2}) - \Psi(x)$ and $\mathcal{F}\phi_\nu(x) = \mathcal{F}\phi_1(2^{-\nu+1}x)$, $\nu = 2, 3, \cdots$. Then $\{\mathcal{F}\phi_\nu\}_{\nu\in\mathbb{N}_0}$ is a smooth dyadic resolution of unity,*

$$\sum_{\nu=0}^{\infty} \mathcal{F}\phi_\nu(x) = 1 \text{ for all } x \in \mathbb{R}^n.$$

Thus the Littlewood–Paley decomposition holds, that is for each $f \in \mathcal{S}'(\mathbb{R}^n)$

$$f = \sum_{\nu=0}^{\infty} \phi_\nu * f \text{ with convergence in } \mathcal{S}'(\mathbb{R}^n).$$

Definition 5. *Let $p(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ and $s(\cdot) \in L^\infty(\mathbb{R}^n)$. Furthermore, let $\{\phi_j\}_{j=0}^\infty$ be the system in Definition 4 and w be a weight. The weighted Besov space with variable exponents $B^{s(\cdot),w}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is the collection of $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|f\|_{B^{s(\cdot),w}_{p(\cdot),q(\cdot)}} := \|\{2^{js(\cdot)}\phi_j * f\}_{j=0}^\infty\|_{\ell^{q(\cdot)}(L^{p(\cdot)}_w)} < \infty,$$

where the norm is

$$\|\{f_j\}_{j=0}^\infty\|_{\ell^{q(\cdot)}(L^{p(\cdot)}_w)} := \inf\Big\{\lambda > 0 : \ \rho_{\ell^{q(\cdot)}(L^{p(\cdot)}_w)}((f_j/\lambda)_j) \le 1\Big\},$$

and

$$\rho_{\ell^{q(\cdot)}(L^{p(\cdot)}_w)}(\{f_j\}_{j=0}^\infty) := \sum_{j=0}^{\infty} \inf\Big\{\lambda_j : \rho_{p(\cdot)}\Big(f_j w/\lambda_j^{\frac{1}{q(\cdot)}}\Big) \le 1\Big\}.$$

If $q = \infty$, the norm is

$$\|\{f_j\}_{j=0}^\infty\|_{\ell^\infty(L^{p(\cdot)}_w)} := \Big\|\{\|f_j\|_{L^{p(\cdot)}_w}\}_{j=0}^\infty\Big\|_{\ell^\infty}.$$

Theorem 3. *Let* $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, $w \in A_{p(\cdot)}$ *and* $s(\cdot) \in C^{\log}_{\rm loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. *Then*

$$\mathcal{B}^{s(\cdot),w}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = B^{s(\cdot),w}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$$

in the sense of equivalent quasi-norms.

Proof. First, we show that

$$\mathcal{B}^{s(\cdot),w}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow B^{s(\cdot),w}_{p(\cdot),q(\cdot)}(\mathbb{R}^n).$$

To do so, let $\{\Phi, \varphi\}$ obey (1) and (2) and $\{\mathcal{F}\psi_j\}_{j\in\mathbb{N}_0}$ be a resolution of unity. Let $f \in \mathcal{B}^{s(\cdot),w}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ with

$$\|f\|_{\mathcal{B}^{s(\cdot),w}_{p(\cdot),q(\cdot)}} \leq 1.$$

Then, we have

$$\varphi_t * f = \int_{2^{-\nu-2}}^{\min\{1,2^{2-\nu}\}} \psi_\nu * \varphi_t * f \frac{dt}{t} + \begin{cases} 0, & \text{if} \quad \nu \geq 2, \\ \psi_\nu * \Phi * f, & \text{if} \quad \nu = 0,1,2. \end{cases}$$

By Lemma 4 for some suitable positive constant c, we obtain

$$\left\| |c(\psi_\nu * \Phi * f) w|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \leq 1, \quad \nu = 0,1,2.$$

By Lemma 10, we have

$$\left\| |c_1(2^{\nu s(\cdot)}\psi_\nu * f) w|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \leq \int_{2^{-\nu-2}}^{\min\{1,2^{2-\nu}\}} \left\| |t^{s(\cdot)}(\psi_t * f) w|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \frac{dt}{t} + 2^\nu, \quad \nu \geq 2,$$

with an appropriate choice of c_1. Taking the sum over $\nu \geq 2$, we obtain

$$\|f\|_{B^{s(\cdot),w}_{p(\cdot),q(\cdot)}} \lesssim 1.$$

Thus we have

$$\mathcal{B}^{s(\cdot),w}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow B^{s(\cdot),w}_{p(\cdot),q(\cdot)}(\mathbb{R}^n).$$

Next, we prove that

$$B^{s(\cdot),w}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow \mathcal{B}^{s(\cdot),w}_{p(\cdot),q(\cdot)}(\mathbb{R}^n).$$

Again, let $\{\Phi, \varphi\}$ obey (1) and (2) and $\{\mathcal{F}\psi_j\}_{j\in\mathbb{N}_0}$ be a resolution of unity. Let $f \in B^{s(\cdot),w}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ with

$$\|f\|_{B^{s(\cdot),w}_{p(\cdot),q(\cdot)}} \leq 1.$$

We have

$$\begin{aligned} \varphi_t * f &= \sum_{\nu=0}^{\infty} \varphi_t * \psi_\nu * f \\ &= \sum_{\nu=[\log_2(\frac{1}{2t})]}^{\left[\log_2(\frac{4}{t})\right]+1} \varphi_t * \psi_\nu * f + \begin{cases} 0, & \text{if} \quad 0 < t \leq \frac{1}{4}, \\ \psi_0 * \Phi * f, & \text{if} \quad t > \frac{1}{4}. \end{cases} \end{aligned}$$

and

$$\Phi * f = \sum_{\nu=0}^{2} \Phi * \psi_\nu * f.$$

If $\nu < 0$, we put $\psi_\nu * f = 0$. By Lemma 4, we obtain

$$\left\| |c(\Phi * \psi_\nu * f)w|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \leq 1, \quad \nu = 0, 1, 2,$$

which yields,

$$\left\| |(\Phi * f)w|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \leq 1$$

for some suitable positive constant c. Let $t \in [2^{-i}, 2^{-i+1}]$, $i \in \mathbb{N}$. We have

$$\begin{aligned} t^{-s(\cdot)}|\psi_t * f| &\lesssim \sum_{\nu=[\log_2(\frac{1}{2t})]}^{[\log_2(\frac{4}{t})]+1} t^{-s(\cdot)}\eta_{t,m} * |\psi_\nu * f| \\ &\lesssim \sum_{\nu=i-3}^{i-1} 2^{(i-\nu)}s^{-}\eta_{\nu,m-C_{\log}(s)} * 2^{\nu s(\cdot)}|\psi_\nu * f| \\ &= c\sum_{j=-3}^{-1} 2^{-js^{-}}\eta_{j+i,m-C_{\log}(s)} * 2^{(j+i)s(\cdot)}|\psi_{j+i} * f|, \end{aligned}$$

where $m > n + C_{\log}(s) + C_{\log}(1/q)$, $C_{\log}(s)$ and $C_{\log(1/q)}$ are the constant in (i) of Definition 1 for $s = \alpha$ and $1/q = \alpha$. Now by Lemma 8

$$\begin{aligned} &\int_0^1 \left\| |ct^{-s(\cdot)}(\varphi_t * f)w|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \frac{dt}{t} \\ &= \sum_{i=0}^{\infty} \int_{2^{-i}}^{2^{1-i}} \left\| |t^{-s(\cdot)}(\varphi_t * f)w|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \frac{dt}{t} \\ &\leq \sum_{j=-3}^{-1} 2^{-js^{-}} \sum_{i=0}^{\infty} \left\| (c\eta_{j+i,m-C_{\log}(s)} * 2^{(j+i)s(\cdot)}|\psi_{j+i} * f|w)^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \\ &\leq \sum_{\nu=0}^{\infty} \left\| (c\eta_{\nu,m-C_{\log}(s)} * 2^{\nu s(\cdot)}|\psi_\nu * f|w)^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \\ &\leq c\|f\|_{B^{s(\cdot),w}_{p(\cdot),q(\cdot)}} \leq 1 \end{aligned}$$

for some suitable positive constant c. This finishes the proof. □

Finally, we give a characterization of the space $B^{s(\cdot),w}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ via the so-called local means. Let $k_0, k \in \mathcal{S}(\mathbb{R}^n)$ and $S \geq -1$ an integer such that for $\varepsilon > 0$

$$\mathcal{F}k_0(\xi) > 0 \quad \text{for } |\xi| < 2\varepsilon, \tag{6}$$

$$\mathcal{F}k(\xi) > 0 \quad \text{for } \frac{\varepsilon}{2} < |\xi| < 2\varepsilon \tag{7}$$

and

$$\int_{\mathbb{R}^n} x^\alpha k(x)dx = 0 \quad \text{for any } |\alpha| \leq S, \tag{8}$$

where (6) and (7) are Tauberian conditions, (8) are moment conditions on k. We recall the notation

$$k_t(x) = t^{-n}k(t^{-1}x), \quad k_j(x) = k_{2^{-j}}(x), \quad \text{for } t > 0 \text{ and } j \in \mathbb{N}.$$

For any $a > 0$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we define

$$k_j^{*,a}t^{-s(\cdot)}f(x) := \sup_{y\in\mathbb{R}^n} \frac{t^{-s(y)}|k_t * f(y)|}{(1+t^{-1}|x-y|)^a}, \quad j \in \mathbb{N}_0.$$

Theorem 4. *Let k_0 and k obey (6)-(8). Let $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, $w \in A_{p(\cdot)}$, $s(\cdot) \in C^{\log}_{\text{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $a > n/p^-$ and $s^+ < S+1$. Then*

$$\|f\|'_{B^{s(\cdot),w}_{p(\cdot),q(\cdot)}} := \|k_0^{*,a}f\|_{L^{p(\cdot)}_w} + \|(k_t^{*,a}t^{-s(\cdot)}f)_{0<t\le 1}\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)}_w)}}$$

is an equivalent quasi-norms on $B^{s(\cdot),w}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Proof. Let $\varepsilon > 0$. Take any pair of functions φ_0 and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\mathcal{F}\varphi_0(\xi) > 0 \quad \text{for } |\xi| < 2\varepsilon,$$

$$\mathcal{F}\varphi(\xi) > 0 \quad \text{for } \frac{\varepsilon}{2} < |\xi| < 2\varepsilon.$$

We prove that there exists a constant c such taht for any $f \in B^{s(\cdot),w}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$

$$\|f\|'_{B^{s(\cdot),w}_{p(\cdot),q(\cdot)}} \le c\|\varphi_0^{*,a}\|_{L^{p(\cdot)}_w} + \left\|(\varphi_j^{*,a}2^{js(\cdot)}f)_{j\ge 1}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}_w)}. \tag{9}$$

For any $f \in B^{s(\cdot),w}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, any $x \in \mathbb{R}^n$ and any $2^{-i} \le t \le 2^{1-i}, i \in \mathbb{N}_0$, By the argument in the proof of [3, Theorem 4.6], we obtain

$$\begin{aligned} k_t^{*,a}t^{-s(\cdot)}f(x) &\lesssim 2^{-i(S+1-s^+)}\varphi_0^{*,a}f(x) + C\sum_{j=1}^{\infty}\min\{2^{(j-i)(S+1-s^+)}, 2^{i-j}\}\varphi_j^{*,a}2^{js(\cdot)}f(x) \\ &= C\sum_{j=0}^{\infty}\min\{2^{(j-i)(S+1-s^+)}, 2^{i-j}\}\varphi_j^{*,a}2^{js(\cdot)}f(x) \\ &= C\Psi_i(x). \end{aligned}$$

Assume that the right hand side of (9) is less than or equal one. We have

$$\begin{aligned} \int_0^1 \left\||k_t^{*,a}t^{-s(\cdot)}fw|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \frac{dt}{t} &= \sum_{i=0}^{\infty}\int_{2^{-i}}^{2^{1-i}} \left\||k_t^{*,a}t^{-s(\cdot)}fw|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \\ &\le \sum_{i=0}^{\infty}\left\||c\Psi_i w|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \end{aligned}$$

for some positive constant c. The last term on the right hand side is less than or equal one if and only if

$$\|(c_1\Psi_i)_{i\ge 0}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}_w)} \le 1$$

for some suitable positive constant c_1, which follows by Lemma 6 and the fact that $s^+ < S+1$. Similarly, we get for any $x \in \mathbb{R}^n$

$$k_0^{*,a}f(x) \le C\varphi_0^{*,a}f(x) + C\sum_{j=1}^{\infty}2^{-j}\varphi_j^{*,a}2^{js(\cdot)}f(x).$$

Therefore, taking the $L_w^{p(\cdot)}$-norm and using the embedding $\ell^{q(\cdot)}(L_w^{p(\cdot)}) \hookrightarrow \ell^\infty(L_w^{p(\cdot)})$, we obtain (9). Now, let $\{\mathcal{F}\varphi_j\}_{j\in\mathbb{N}_0}$ be a resolution of unity. By (9) and Theorem 1 in [29], we have

$$\|f\|'_{B^{s(\cdot),w}_{p(\cdot),q(\cdot)}} \lesssim \|\varphi_0^{*,a} f\|_{L_w^{p(\cdot)}} + \left\|(\varphi_j^{*,a} 2^{js(\cdot)} f)_{j\geq 1}\right\|_{\ell^{q(\cdot)}(L_w^{p(\cdot)})} \lesssim \|f\|_{B^{s(\cdot),w}_{p(\cdot),q(\cdot)}}.$$

Next, we prove the oppositive inequality. To do so, let $\{\mathcal{F}\varphi_j\}_{j\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n)$ be such that

$$\operatorname{supp} \mathcal{F}\varphi \subset \{\xi \in \mathbb{R}^n : \varepsilon/2 \leq |\xi| \leq 2\varepsilon\}$$

and

$$\operatorname{supp} \mathcal{F}\varphi_0 \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2\varepsilon\}, \quad \varepsilon > 0,$$

with $\varphi_j = 2^{jn}\varphi(2^j\cdot)$, $j \in \mathbb{N}$. We will prove that

$$\|\varphi_0 * f\|_{L_w^{p(\cdot)}} + \left\|(2^{js(\cdot)}(\varphi_j * f))_{j\geq 1}\right\|_{\ell^{q(\cdot)}(L_w^{p(\cdot)})} \lesssim \|f\|'_{B^{s(\cdot),w}_{p(\cdot),q(\cdot)}}. \tag{10}$$

By the argument in the proof of [3, (4.10) of Theorem 4.6], we obtain

$$2^{js(x)r}|\varphi_j * f(x)|^r \lesssim \int_{2^{-j-2}}^{2^{-j+2}} (k_\tau^{*,a}\tau^{-s(\cdot)} f(x))^r \frac{d\tau}{\tau}, \quad x \in \mathbb{R}^n. \tag{11}$$

Similarly, we obtain

$$|\varphi_0 * f(x)|^r \lesssim (k_0^{*,a} f(x))^r + \int_{\frac{1}{4}}^{1} (k_\tau^{*,a}\tau^{-s(\cdot)} f(x))^r \frac{d\tau}{\tau}.$$

Let $\theta > 0$ be such that $\max\{1, (1/p)^+/(1/q)^-\} < \theta < q^-/r$. By Hölder's and Minkowski's inequalities, we obtain

$$\begin{aligned}\left\||c2^{js(\cdot)}(\varphi_j * f)w|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} &\leq \left(\int_{2^{-j-2}}^{2^{-j+2}} \left\||k_\tau^{*,a}\tau^{-s(\cdot)} fw|^{q(\cdot)}\right\|^{\frac{1}{\theta}}_{L^{\frac{p(\cdot)}{q(\cdot)}}} \frac{d\tau}{\tau}\right)^\theta \\ &\leq \int_{2^{-j-2}}^{2^{-j+2}} \left\||k_\tau^{*,a}\tau^{-s(\cdot)} fw|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \frac{d\tau}{\tau}.\end{aligned}$$

So, we have

$$\sum_{j=1}^{\infty} \left\||c2^{js(\cdot)}(\varphi_j * f)w|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \leq 1$$

with an appropriate choice of $c > 0$ such that the left hand side of (11) it at most one. Similarly we obtain

$$\left\||c(\varphi_0 * f)w|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \leq 1.$$

Hence, we obtain (10). Now let $\{\mathcal{F}\varphi_j\}_{j\in\mathbb{N}_0}$ be a resolution of unity. By (9), we obtain

$$\|f\|_{B^{s(\cdot),w}_{p(\cdot),q(\cdot)}} \lesssim \|\varphi_0 * f\|_{L_w^{p(\cdot)}} + \left\|(2^{js(\cdot)}\varphi_j * f)_{j\geq 1}\right\|_{\ell^{q(\cdot)}(L_w^{p(\cdot)})} \lesssim \|f\|'_{B^{s(\cdot),w}_{p(\cdot),q(\cdot)}}.$$

This finishes the proof. □

5. With weights in A_∞

For a wight w and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, we define $\tilde{L}_w^{p(\cdot)} = \{f : \|fw^{1/p(\cdot)}\|_{L^{p(\cdot)}} < \infty\}$ and $\|f\|_{\tilde{L}_w^{p(\cdot)}} = \|fw^{1/p(\cdot)}\|_{L^{p(\cdot)}}$.

Definition 6. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, a nonnegative measurable function w is said to be in $\tilde{A}_{p(\cdot)}$, if*

$$\|w\|_{\tilde{A}_{p(\cdot)}} = \sup_B |B|^{-p_B} \|w\chi_B\|_{L^1} \|w^{-1}\chi_B\|_{L^{p'(\cdot)/p(\cdot)}} < \infty,$$

where p_B is the harmonic average of $p(\cdot)$ over B, namely, $p_B := \left(\frac{1}{|B|}\int_B \frac{1}{p(x)}dx\right)^{-1}$. The set $\tilde{A}_{p(\cdot)}$ consists of all $\tilde{A}_{p(\cdot)}$ weights.

Diening and Hästö obtained the following lemma in [12].

Lemma 12. *If $p(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ and w is a weight, then the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on $\tilde{L}^{p(\cdot)}(w)$ if and only if $w \in \tilde{A}_{p(\cdot)}$.*

Due to Definition of $\tilde{A}_{p(\cdot)}$ and [12, Lemma 3.1], we have the next lemma.

Lemma 13. *Suppose $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ and $p(\cdot) \le q(\cdot)$. Then we have*

$$A_1 \subset A_{p^-} \subset \tilde{A}_{p(\cdot)} \subset \tilde{A}_{q(\cdot)} \subset A_{q^+} \subset A_\infty.$$

By Lemmas 12, 13 and [11, Lemma 3.2], we have the next lemma.

Lemma 14. *If $p(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ and $w \in A_\infty$, then for every $m > n$ there exists $C > 0$ such that,*

$$\|\eta_{\nu,m} * g\|_{\tilde{L}_w^{p(\cdot)}} \le C\|g\|_{\tilde{L}_w^{p(\cdot)}}$$

for all $\nu \ge 0$, functions $g \in \tilde{L}_w^{p(\cdot)}(\mathbb{R}^n)$.

Lemma 15. *Let $p(\cdot)$, $q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, $w \in A_\infty$ and $m > n + C_{\log(1/q)}$, where $C_{\log(1/q)}$ is the constant in* (i) *of Definition 1 for $1/q = \alpha$. Then there exists a positive constant C such that*

$$\|\{\eta_{t,m} * f_t\}_{0<t\le 1}\|_{\widetilde{\ell^{q(\cdot)}(\tilde{L}_w^{p(\cdot)})}} \le C\left\|\{f_t\}_{0<t\le 1}\right\|_{\widetilde{\ell^{q(\cdot)}(\tilde{L}_w^{p(\cdot)})}}$$

holds for every sequence functions $\{f_t\}_{0<t\le 1}$.

Proof. By the scaling argument, without loss of generality we may assume that $\|\{f_t\}_{0<t\le 1}\|_{\widetilde{\ell^{q(\cdot)}(\tilde{L}_w^{p(\cdot)})}} = 1$. We only prove that there exists a constant c such that

$$\int_0^1 \left\| |c\eta_{t,m} * f_t|^{q(\cdot)} w^{q(\cdot)/p(\cdot)} \right\|_{\tilde{L}^{\frac{p(\cdot)}{q(\cdot)}}} \frac{dt}{t} \le 2.$$

This clearly follows from the inequality

$$\left\| |c\eta_{t,m} * f_t|^{q(\cdot)} w^{q(\cdot)/p(\cdot)} \right\|_{\tilde{L}^{\frac{p(\cdot)}{q(\cdot)}}} \le \left\| |f_t|^{q(\cdot)} w^{q(\cdot)/p(\cdot)} \right\|_{\tilde{L}^{\frac{p(\cdot)}{q(\cdot)}}} + t := \delta_t$$

for any $t \in (0, 1]$. This claim can be reformulated as showing that

$$\left\| \delta_t^{-1} |c\eta_{t,m} * f_t|^{q(\cdot)} w^{q(\cdot)/p(\cdot)} \right\|_{\tilde{L}^{\frac{p(\cdot)}{q(\cdot)}}} \le 1$$

which is equivalent to

$$c\|\delta_t^{-\frac{1}{q(\cdot)}} \eta_{t,m} * f_t\|_{\tilde{L}_w^{p(\cdot)}} \le 1, \quad t \in (0, 1].$$

Since $1/q(\cdot)$ is log-Hölder continuous and $\delta_t \in (t, 1+t]$, then by Lemma 3, we have

$$\delta_t^{-1/q(\cdot)} |\eta_{t,m} * f_t| \le C|\eta_{t,m-C_{\log(1/q)}} * (\delta_t^{-1/q(\cdot)} f_t)|. \tag{12}$$

Then by (12) and Lemma 14, we obtain

$$\|\delta_t^{-\frac{1}{q(\cdot)}} \eta_{t,m} * f_\nu\|_{\tilde{L}_w^{p(\cdot)}} \le \|c\eta_{t,m-C_{\log(1/q)}} * (\delta_t^{-\frac{1}{q(\cdot)}} f_t)\|_{\tilde{L}_w^{p(\cdot)}} \le \|\delta_t^{-\frac{1}{q(\cdot)}} f_t\|_{\tilde{L}_w^{p(\cdot)}}$$

with an appropriate choice of $c > 0$. Now the right-hand side is bounded if and only if

$$\left\| |f_t|^{q(\cdot)} w^{q(\cdot)/p(\cdot)} \right\|_{\tilde{L}^{\frac{p(\cdot)}{q(\cdot)}}} \le \delta_t$$

which follows from the definition of δ_t. □

Lemma 16. *Let $0 < \alpha < \beta < \infty$, $w \in A_\infty$, $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ with $1/q(\cdot) \in C_{\rm loc}^{\log}(\mathbb{R}^n)$. Let*

$$g_t := \int_{\alpha t}^{\beta t} \eta_{\tau,m} * f_\tau(x) \frac{d\tau}{\tau}, \quad t \in (0,1], x \in \mathbb{R}^n.$$

(i) *Assume that $0 < \beta t \le 1$. The inequality*

$$\left\| |c g_t w^{1/p(\cdot)}|^{q(\cdot)} \right\|_{\tilde{L}^{\frac{p(\cdot)}{q(\cdot)}}} \le \int_{\alpha t}^{\beta t} \left\| |f_\tau w^{1/p(\cdot)}|^{q(\cdot)} \right\|_{\tilde{L}^{\frac{p(\cdot)}{q(\cdot)}}} \frac{d\tau}{\tau} + t, \quad t \in (0,1]$$

holds for every sequence of functions $(f_t)_{0<t\le 1}$ and constant $m > n + C_{\log(1/q)}$ such that the first term on right-hand side is at most one, where the constant c independent of t and $C_{\log(1/q)}$ is the constant in (i) *of Definition 1 for $1/q = \alpha$.*
(ii) *The inequality*

$$\|\{g_t\}_{0<t\le 1}\|_{\widetilde{\ell^{q(\cdot)}(\tilde{L}_w^{p(\cdot)})}} \le C \|\{f_t\}_{0<t\le 1}\|_{\widetilde{\ell^{q(\cdot)}(\tilde{L}_w^{p(\cdot)})}}$$

holds for every sequence of functions $(f_t)_{0<t\le 1}$ and constant $m > n + C_{\log(1/q)}$ such that the right-hand side is finite, where $C_{\log(1/q)}$ is the constant in (i) *of Definition 1 for $1/q = \alpha$.*

Proof. (i) We put

$$\delta_t := \int_{\alpha t}^{\beta t} \left\| |c f_\tau w^{1/p(\cdot)}|^{q(\cdot)} \right\|_{\tilde{L}^{\frac{p(\cdot)}{q(\cdot)}}} \frac{d\tau}{\tau} + t.$$

Since $1/q(\cdot)$ is log-Hölder continuous and $\delta_t \in (t, 1+t]$, then by Lemma 3, we have

$$\delta_t^{-1/q(\cdot)} |\eta_{\tau,m} * f_t| \le C|\eta_{\tau,m-C_{\log(1/q)}} * (\delta_t^{-1/q(\cdot)} f_t)|. \tag{13}$$

Then by (13) and Lemma 14, we obtain

$$\begin{aligned}
\left\| c\delta_t^{-\frac{1}{q(\cdot)}} g_t \right\|_{\tilde{L}_w^{p(\cdot)}} &\le \int_{\alpha t}^{\beta t} \left\| c\delta_t^{-\frac{1}{q(\cdot)}} (\eta_{\tau,m} * f_\tau) \right\|_{\tilde{L}_w^{p(\cdot)}} \frac{d\tau}{\tau} \\
&\le \int_{\alpha t}^{\beta t} \left\| \eta_{\tau,m-C_{\log(1/q)}} * (c\delta_t^{-\frac{1}{q(\cdot)}} f_\tau) \right\|_{\tilde{L}_w^{p(\cdot)}} \frac{d\tau}{\tau} \\
&\le \int_{\alpha t}^{\beta t} \left\| c\delta_t^{-\frac{1}{q(\cdot)}} f_\tau \right\|_{\tilde{L}_w^{p(\cdot)}} \frac{d\tau}{\tau} \\
&= \int_{(\beta t, \alpha t] \cap E} \left\| c\delta_t^{-\frac{1}{q(\cdot)}} f_\tau \right\|_{\tilde{L}_w^{p(\cdot)}} \frac{d\tau}{\tau} + \int_{(\beta t, \alpha t] \cap E^c} \left\| c\delta_t^{-\frac{1}{q(\cdot)}} f_\tau \right\|_{\tilde{L}_w^{p(\cdot)}} \frac{d\tau}{\tau}
\end{aligned}$$

$$:= F_1 + F_2,$$

where

$$E := \left\{\tau > 0 : \left\| |\delta_t^{-\frac{1}{q(\cdot)}} f_\tau w^{1/p(\cdot)}|^{q(\cdot)} \right\|_{\tilde{L}^{p(\cdot)/q(\cdot)}} \ge 1 \right\}.$$

By Lemma 2, we have

$$F_1 \le \int_{(\beta t, \alpha t] \cap E} \left\| |\delta_t^{-\frac{1}{q(\cdot)}} f_\tau w^{1/p(\cdot)}|^{q(\cdot)} \right\|_{\tilde{L}^{\frac{p(\cdot)}{q(\cdot)}}}^{\frac{1}{q^-}} \frac{d\tau}{\tau} \le \delta_t^{-1} \int_{\alpha t}^{\beta t} \left\| |f_\tau w^{1/p(\cdot)}|^{q(\cdot)} \right\|_{\tilde{L}^{\frac{p(\cdot)}{q(\cdot)}}} \frac{d\tau}{\tau} \le 1$$

and

$$F_2 \le \int_{\alpha t}^{\beta t} \left\| \delta_t^{-\frac{1}{q(\cdot)}} f_\tau \right\|_{\tilde{L}_w^{p(\cdot)}} \frac{d\tau}{\tau} \le \int_{\alpha t}^{\beta t} \frac{d\tau}{\tau} = \log \frac{\beta}{\alpha}.$$

(ii) By the scaling argument, without loss of generality we may assume that $\|(f_t)_{0<t\le 1}\|_{\ell^{q(\cdot)}(\widetilde{\tilde{L}_w^{p(\cdot)}})} = 1$. we show that

$$\int_0^1 \left\| |cg_t|^{q(\cdot)} w^{q(\cdot)/p(\cdot)} \right\|_{\tilde{L}^{\frac{p(\cdot)}{q(\cdot)}}} \frac{dt}{t} \le 2 \quad \text{whenever} \quad \left\| |f_t|^{q(\cdot)} w^{q(\cdot)/p(\cdot)} \right\|_{\tilde{L}^{\frac{p(\cdot)}{q(\cdot)}}} = 1.$$

By Lemma 7 and (i), we obtain the desired result. □

Similar to Definitions 3 and 5, we denote

$$\|f\|^{\Phi,\varphi}_{\tilde{\mathcal{B}}^{s(\cdot),w}_{p(\cdot),q(\cdot)}} := \|\Phi * f\|_{\tilde{L}_w^{p(\cdot)}} + \|\{t^{-s(\cdot)}\varphi_t * f\}_{0<t\le 1}\|_{\ell^{q(\cdot)}(\widetilde{\tilde{L}_w^{p(\cdot)}})}.$$

$$\|f\|_{\tilde{B}^{s(\cdot),w}_{p(\cdot),q(\cdot)}} := \|\{2^{js(\cdot)}\phi_j * f\}_{j=0}^\infty\|_{\ell^{q(\cdot)}(\tilde{L}_w^{p(\cdot)})}.$$

Theorem 5. *Let $\{\mathcal{F}\Phi, \mathcal{F}\varphi\}$ and $\{\mathcal{F}\Psi, \mathcal{F}\psi\}$ be two resolutions of unity satisfying (1) and (2). Let $s(\cdot) \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$, and $w \in A_\infty$. Then*

$$\|f\|^{\Phi,\varphi}_{\tilde{\mathcal{B}}^{s(\cdot),w}_{p(\cdot),q(\cdot)}} \approx \|f\|^{\Psi,\psi}_{\tilde{\mathcal{B}}^{s(\cdot),w}_{p(\cdot),q(\cdot)}}.$$

Proof. The proof of Theorem 5 is similar to that of Theorem 1. In fact, by repeating the argument that used in the proof of Theorem 1, where Lemmas 4 and 10 are replaced by Lemmas 14 and 16 respectively, we can prove it. The detail is omitted. □

Theorem 6. *Let $s(\cdot) \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$, $w \in A_\infty$ and $a > n/p^-$ Then*

$$\|f\|^*_{\tilde{\mathcal{B}}^{s(\cdot),w}_{p(\cdot),q(\cdot)}} := \|\Phi^{*,a}\|_{\tilde{L}_w^{p(\cdot)}} + \|(\varphi^{*,a} t^{-s(\cdot)} f)_{0<t\le 1}\|_{\ell^{q(\cdot)}(\widetilde{\tilde{L}_w^{p(\cdot)}})}$$

is an equivalent quasi-norm in $\mathcal{B}^{s(\cdot),w}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Proof. The proof of Theorem 6 is similar to that of Theorem 2. In fact, by repeating the argument that used in the proof of Theorem 2, where Lemma 9 is replaced by Lemma 15, we can prove it. The detail is omitted. □

Theorem 7. *Let k_0 and k obey* (6)-(8). *Let $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, $w \in A_\infty$, $s(\cdot) \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $a > n/p^-$ and $s^+ < S + 1$. Then*

$$\|f\|'_{\tilde{B}^{s(\cdot),w}_{p(\cdot),q(\cdot)}} := \|k_0^{*,a} f\|_{\tilde{L}_w^{p(\cdot)}} + \|(k_t^{*,a} t^{-s(\cdot)} f)_{0<t\le 1}\|_{\ell^{q(\cdot)}(\widetilde{\tilde{L}_w^{p(\cdot)}})}$$

is an equivalent quasi-norms on $\tilde{B}^{s(\cdot),w}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Proof. The proof of Theorem 7 is similar to that of Theorem 4. In fact, by repeating the argument that used in the proof of Theorem 2, where Lemmas 4 and 9 are replaced by Lemmas 14 and 15 respectively, we can prove it. The detail is omitted. □

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