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Sequential warped product submanifolds in nearly Kaehler manifolds

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Abstract. A new class of warped product manifolds which is known as sequential warped product manifolds have been defined in [15] and studied in detail in [9]. This article is dedicated to study sequential warped product submanifolds having factors holomorphic, totally real and pointwise slant submanifolds of nearly Kaehler manifolds. We obtained Chen's inequality for sequential warped product submanifolds involving second fundamental form and warping functions.

1. Introduction

The study on warped product manifolds is continuously growing day by day. Many authors are exploring this field in various settings. A wide range of applications of warped product manifolds emerged in Physics and Cosmology. To this fact, Mathematicians have been exploring it in the spaces with different stand point. Latest in the sequence is "Sequential warped product manifolds". These warped product manifolds were introduced by S. Shenawy [15] and a detailed study of curvatures was done in [9].
Warped product manifolds were first defined by Bishop and O'Neill [4] to examine the manifolds of negative curvature. It is well known that warped product manifolds are generalization of product manifolds. A warped product manifold $N_1 \times_f N_2$ is simply a product of two Riemmanian manifolds N_1 and N_2 with metric $g = g_{N_1} + f^2 g_{N_2}$ where (N_1, g_{N_1}) is base and (N_2, g_{N_2}) is fiber and f is positive valued smooth function on N_1.

In the early years of 21st century, warped product manifolds emerge more significantly when B. Y. Chen [5] characterize CR-submanifolds as warped product submanifolds in Kaehler manifold. He obtained a sharp inequality for the squared norm of the second fundamental form which is known as Chen's inequality. Later, many authors generalize Chen's inequality in different settings to characterize warped product manifolds and obtained its applications [7],[8],[12].

Apart from (single) warped product manifold, biwarped product and multiply warped product manifolds were also defined and studied thoroughly for their extrinsic properties (see [6],[11],[17]). We see that in all these types of warped products, the warping function (a positive valued smooth function) is taken on the base (which is a single manifold) of warped product. Now the question arises that what if

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the base manifold is itself a warped product. There are many space-time manifolds where base, fiber or both can be expressed as warped products. Some of them are Taub-Nut and stationary metrics (see [16]) and generalized Riemannian anti de Sitter T^2 black hole metrics (see [2]). The answer to this is sequential warped product.

Sequential warped products was defined in [15]. Later De et. al [9] explored its geometry by taking into account its curvature formulas. They also provide characterization for Killing and concircular vector fields on sequential warped product manifolds. In [13], Sahin studied these warped products in Kaehler manifolds and obtained an estimate in terms of second fundamental form. As nearly Kaehler manifolds are more general than Kaehler manifolds, it is natural to see whether sequential warped products exist in nearly Kaehler manifold and if it exist then what would be its geometry. In this paper, we establish that the sequential warped product manifolds with factors holomorphic, totally real and pointwise slant subamnifolds i.e. of type $(N_T \times_f N_\perp) \times_h N_\theta$ exist in nearly Kaehler manifold and we obtain a sharp inequality in terms of second fundamental form involving the warping functions and slant angle. Our result generalizes many existing results in different settings like CR-warped product, pointwise semi-slant warped product and biwarped product submanifolds in nearly Kaehler manifolds.

The paper is organized as follows: Section 2 is devoted to basic definitions, formulae and preliminary results which are required for the study of sequential warped products. In Section 3, we explore the existence of sequential warped product submanifolds in nearly Kaehler and prove our main results. Bibliography is given at the end of the paper.

2. Preliminaries

All manifolds, vector bundles, functions etc. are assumed to be of class C^∞. The set of locally defined sections of a vector bundle E is denoted by $\Gamma(E)$.

We know that nearly Kaehler manifolds are the most important class of almost Hermitian manifolds which are not integrable. An almost Hermitian manifold $\bar{M}$ is a *nearly Kaehler manifold* if its almost complex structure J satisfies

$$(\bar{\nabla}_U J)U = 0, \tag{1}$$

for all vector fields U on $\bar{M}$, where $\bar{\nabla}$ denotes the Levi-Civita connection on $\bar{M}$ and satisfies

$$(\bar{\nabla}_U J)V = \bar{\nabla}_U JV - J\bar{\nabla}_U V, \tag{2}$$

for any $U, V \in \Gamma(T\bar{M})$.

If the almost complex structure J is parallel with respect to the Levi-Civita connection $\bar{\nabla}$ on $\bar{M}$ i.e., $\bar{\nabla}J = 0$, the almost Hermitian manifold $\bar{M}$ is called a *Kaehler manifold*. If the Nijenhuis tensor of J vanishes, the nearly Kaehler manifold is a Kaehler manifold. The nearly Kaehler manifolds with dimension 4 are Kaehler manifolds.

Consider a Riemannian manifold M isometrically immersed in an almost Hermitian manifold $\bar{M}$. The Gauss and Weingarten formulas are respectively given by

$$\bar{\nabla}_U V = \nabla_U V + \sigma(U, V), \tag{3}$$

$$\bar{\nabla}_U \xi = -A_\xi U + \nabla^\perp_U \xi \tag{4}$$

for $U, V \in \Gamma(TM)$ and $\xi \in \Gamma(T^\perp M)$; where ∇ denotes the covariant differentiation with respect to the induced metric, σ the second fundamental form, $\nabla^\perp$ the normal connection, A_ξ the shape operator (corresponding to the normal vector field ξ) and TM (resp. $T^\perp M$) is the tangent (resp. normal) bundle of M. The relation between A_ξ and σ is given as

$$g(A_\xi U, V) = g(\sigma(U, V), \xi) \tag{5}$$

where g denotes the Riemannian metric on $\bar{M}$ as well as the induced metric on M.

Consider a submanifold M of an almost Hermitian manifold $\bar{M}$. The complex structure J when applied to the tangent bundle TM generates various distributions on M.

(i) A distribution D^T on a submanifold M of an almost Hermitian manifold $\bar{M}$ is called a *holomorphic* distribution if $JD^T \subseteq D^T$.

(ii) A distribution $D^\perp$ on M is called *totally real* distribution if $JD^\perp \subseteq T^\perp M$.

A submanifold is said to be a CR-submanifold if it is endowed with a pair of orthogonal complementary distributions D^T and $D^\perp$ such that D^T is holomorphic and $D^\perp$ is totally real [3] .

(iii) Let D^θ be a distribution on a submanifold M of an almost Hermitian manifold $\bar{M}$. For any $x \in M$ and any non-zero vector $X \in D^\theta_x$, if the angle $\theta(X) \in [0, \pi/2]$ between JX and the vector space D^θ_x does not depend on the choice of $x \in M$ and $X \in D^\theta_x$, we say that D^θ is a *slant distribution* on M. The constant angle θ is called the *Wirtinger angle* of D^θ in M. Moreover, if the angle $\theta(X)$ is independent of the choice of $X \in D^\theta_x$ only, D^θ is called *pointwise slant distribution* on M. In this case θ is called slant function.

A submanifold M is called a *slant submanifold* if the tangent bundle $\Gamma(TM)$ is slant. Holomorphic and totally real submanifolds are special cases of slant submanifolds with Wirtinger angle 0 and $\pi/2$ respectively. Also, a submanifold is *pointwise slant submanifold* if the tangent bundle $\Gamma(TM)$ is pointwise slant.

Semi-slant and pointwise semi-slant are two another classes of submanifolds. If a submanifold is endowed with two orthogonal complementary distributions D^T and D^θ where D^T is holomorphic submanifold. The submanifold is called *semi-slant* if D^θ is slant and it is called *pointwise semi-slant* if D^θ is pointwise slant.

For any $x \in M$ and any $U \in T_xM$, JU can be decomposed as

$$JU = PU + FU, \quad PU \in T_xM \quad \text{and} \quad FU \in T^\perp_x M. \tag{6}$$

P and F are respectively the endomorphism $P : T_xM \longrightarrow T_xM$ and a normal valued linear map $F : T_xM \longrightarrow T^\perp_x M$ defined by (6). We also denote the $(1, 1)$ tensor field and the normal valued 1-form on M determined by P and F by the same letters. Similarly, for $\xi \in \Gamma(T^\perp M)$, we put

$$t\xi = tan(J\xi) \quad \text{and} \quad f\xi = nor(J\xi). \tag{7}$$

The covariant derivatives of these tensor fields are defined as:

$$(\bar{\nabla}_U P)V = \nabla_U PV - P\nabla_U V, \tag{8}$$

$$(\bar{\nabla}_U F)V = \nabla^\perp_U FV - F\nabla_U V, \tag{9}$$

$$(\bar{\nabla}_U t)\xi = \nabla_U t\xi - t\nabla^\perp_U \xi, \tag{10}$$

$$(\bar{\nabla}_U f)\xi = \nabla^\perp_U f\xi - f\nabla^\perp_U \xi. \tag{11}$$

If we denote by $(\tilde{\mathbb{M}}, J, g)$ a nearly Kaehler manifold and M a submanifold of $\tilde{\mathbb{M}}$. If $\mathcal{P}_U V$ (resp. $\mathcal{Q}_U V$) denote the tangential (resp. normal) part of $(\bar{\nabla}_U J)V$ for any $U, V \in \Gamma(T\tilde{\mathbb{M}})$, then it is straightforward to see that

$$\mathcal{P}_U V = (\bar{\nabla}_U P)V - A_{FV}U - t\sigma(U, V), \tag{12}$$

$$\mathcal{Q}_U V = (\bar{\nabla}_U F)V + \sigma(U, PV) - f\sigma(U, V). \tag{13}$$

It is easy to see that tensor fields $\mathcal{P}$ and $\mathcal{Q}$ satisfy the following:

$$g(\mathcal{P}_U V, W) = -g(V, \mathcal{P}_U W) \quad \text{and} \quad g(\mathcal{Q}_U V, \xi) = -g(V, \mathcal{P}_U \xi)$$

where $W \in \Gamma(TM)$ and $\xi \in \Gamma(T^\perp M)$.

The (Riemannian) product manifolds have been generalized by using warping functions to define warped product of manifolds viz. warped product, biwarped product, multiply warped product manifolds (see [6], [7], [17]).

Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds with Riemannian metrics g_1 and g_2 respectively and ψ be a positive differentiable function on N_1. If $\pi : N_1 \times N_2 \to N_1$ and $\eta : N_1 \times N_2 \to N_2$ are the projection maps given by $\pi(p, q) = p$ and $\eta(p, q) = q$ for every $(p, q) \in N_1 \times N_2$, then the *warped product manifold* $M = N_1 \times_\psi N_2$ is the product manifold $N_1 \times N_2$ equipped with the Riemannian metric g defined as

$$g(X, Y) = g_1(\pi_* X, \pi_* Y) + (\psi \circ \pi)^2 g_2(\eta_* X, \eta_* Y),$$

for all $X, Y \in \Gamma(TM)$, where $*$ denotes the tangent map. The function ψ is called the *warping function* of the warped product manifold. For a constant warping function, the warped product is trivial [4]. One can generalize this definition to multiply warped product manifolds as follows.

Let $\{N_i\}_{i=1,2,\cdots,k}$ be Riemannian manifolds with respective Riemannian metrics $\{g_i\}_{i=1,2,\cdots,k}$ and let $\{\psi_i\}_{i=2,3,\cdots,k}$ are positive real valued functions on N_1. Then the product manifold $M = N_1 \times N_2 \times \cdots \times N_k$ endowed with Riemannian metric g given by

$$g = \pi_1^*(g_1) + \sum_{i=2}^{k} (\psi_i \circ \pi_1)^2 \pi_i^*(g_i)$$

is called *multiply warped product manifold* denoted by $M = N_1 \times_{\psi_2} N_2 \times \cdots \times_{\psi_k} N_k$ where $\pi_i (i = 1, 2, \cdots, k)$ are the projection maps of M onto N_i respectively. The functions ψ_i are known as the *warping functions* [6]. If each of the warping function is constant, the warped product is simply a Riemannian product of manifolds, known as a *trivial multiply warped product manifold*.

As a particular case of multiply warped product manifolds, one can define biwarped product manifolds for $i = 3$. Multiply warped product manifolds reduces to (singly) warped product manifolds for $i = 2$.

We note that in multiply warped product manifolds, the warping functions are defined on the first factor N_1. Particularly, we consider the case of biwarped product manifolds in which the warping functions (say ψ_1 and ψ_2) are defined on N_1. Now if the function ψ_1 is defined on N_1 and the function ψ_2 is defined on $N_1 \times N_2$, in this case, we define the following.

Definition 2.1. *[15] Let N_i (for $i = 1, 2, 3$) be pseudo-Riemannian manifolds with pseudo-Riemannian metrics g_i respectively. Let $f : N_1 \to (0, \infty)$ and $h : N_1 \times N_2 \to (0, \infty)$ be two smooth functions on N_1 and $N_1 \times N_2$ respectively, then the sequential warped product is the product manifold $(N_1 \times N_2) \times N_3$ denoted by $(N_1 \times_f N_2) \times_h N_3$ endowed with the metric tensor $g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$.*

The positive valued functions f and h are called warping functions.

It is obvious that if (N_i, g_i) are Riemannian manifolds for $i = 1, 2, 3$ then the sequential warped product manifold $(N_1 \times_f N_2) \times_h N_3$ is a Riemannian manifold with Riemannian metric g.

Related to the geometry of the sequential warped product manifold, we have the following:

Proposition 2.2. *[15] Let $\bar{M} = (N_1 \times_f N_2) \times_h N_3$ be a sequential warped product manifold with metric g and if $X_i \in \Gamma(TN_i)$ (for $i = 1, 2, 3$). Then we have*

1. $\bar{\nabla}_{X_1} X_2 = \bar{\nabla}_{X_2} X_1 = X_1(\ln f) X_2$
2. $\bar{\nabla}_{X_3} X_1 = \bar{\nabla}_{X_1} X_3 = X_1(\ln h) X_3$
3. $\bar{\nabla}_{X_2} X_3 = \bar{\nabla}_{X_3} X_2 = X_2(\ln h) X_3$

A sequential warped product manifold is *proper* if the warping functions f and h are not constants i.e. they satisfy $X_1 \ln f \neq 0$, $X_1 \ln h \neq 0$ and $X_2 \ln h \neq 0$ for $X_1 \in \Gamma(TN_1)$ and $X_2 \in \Gamma(TN_2)$.
We also have the following consequences of Heipko's [10] characterization of warped product manifold.

Corollary 2.3. *Consider $M = (N_T \times_f N_\perp) \times_h N_\theta$ be a sequential warped product submanifold of a nearly Kaehler manifold $\tilde{\mathbb{M}}$ such that N_T is holomorphic, $N_\perp$ is totally real and N_θ is a pointwise proper slant submanifold of $\tilde{\mathbb{M}}$. We have the following:*
(a) N_T is a totally geodesic submanifold in $N_T \times_f N_\perp$.

(b) $N_\perp$ *is a spherical submanifold in* $N_T \times_f N_\perp$.
(c) $N_T \times_f N_\perp$ *is totally geodesic in* $(N_T \times_f N_\perp) \times_h N_\theta$.
(d) N_θ *is spherical submanifold in* $(N_T \times_f N_\perp) \times_h N_\theta$.

If M is an $n-$dimensional Riemannian manifold with the local orthonormal frame of the vector fields $\{e_1, e_2, \ldots, e_n\}$, the gradient of a function ψ is defined as

$$g(\nabla\psi, X) = X\psi, \tag{14}$$

for all $X \in \Gamma(TM)$. We also have

$$\|\nabla\psi\|^2 = \sum_{i=1}^{n} (e_i(\psi))^2. \tag{15}$$

3. Sequential warped product submanifolds

In this section, first we seek the existence of the sequential warped product submanifolds $M = (N_1 \times_f N_2) \times_h N_3$ for Riemannian submanifolds N_1, N_2 and N_3 in a nearly Kaehler manifold $\tilde{\mathbb{M}}$ with warping functions f on N_1 and h on $N_1 \times N_2$. For three submanifolds, there are 3! possible sequential warped product submanifolds. If we consider N_T, $N_\perp$ and N_θ as holomorphic submanifold, totally real submnaifold and proper pointwise slant submanifold respectively of $\tilde{\mathbb{M}}$, then following are the sequential warped product submanifold of $\tilde{\mathbb{M}}$ with N_T, $N_\perp$ and N_θ as factors of the warped product subamnifold of $\tilde{\mathbb{M}}$.

$$(i)\ (N_T \times_f N_\perp) \times_h N_\theta, \quad (ii)\ (N_\perp \times_f N_T) \times_h N_\theta,$$
$$(iii)\ (N_T \times_f N_\theta) \times_h N_\perp, \quad (iv)\ (N_\theta \times_f N_T) \times_h N_\perp,$$
$$(v)\ (N_\perp \times_f N_\theta) \times_h N_T, \quad (vi)\ (N_\theta \times_f N_\perp) \times_h N_T$$

In [13], B. Sahin investigated all possible sequential warped product submanifolds of the Kaehler manifold. He proved the non-existence of the sequential warped product submanifolds of the type (*ii*)-(*vi*) in Kaehler manifold. He established the existence of the sequential warped product submanifolds of the Kaehler manifold of the type (*i*) i.e. $(N_T \times_f N_\perp) \times_h N_\theta$. He provided an example and obtained some inequalities involving second fundamental form. Motivated by the results in [13], we study these sequential warped products in nearly Kaehler manifolds.

Let us consider $M = (N_T \times_f N_\perp) \times_h N_\theta$ a sequential warped product submanifold of a nearly Kaehler manifold $(\tilde{\mathbb{M}}, J, g)$ with warping functions f on N_T and h on $N_T \times_f N_\perp$ such that N_T is holomorphic, $N_\perp$ a totally real and N_θ a pointwise proper slant submanifold of $\tilde{\mathbb{M}}$. Thus, the tangent bundle TM of M has the following direct sum decomposition

$$TM = D^T \oplus D^\perp \oplus D^\theta,$$

where D^T is holomorphic distribution, $D^\perp$ is totally real and D^θ is pointwise proper slant distribution with the slant function θ. The normal bundle $T^\perp M$ of M is decompounded as

$$T^\perp M = JD^\perp \oplus FD^\theta \oplus \nu,$$

where ν is the orthogonal complementary distribution of $JD^\perp \oplus FD^\theta$ in $T^\perp M$. It is easy to see that ν is an invariant subbundle of $T^\perp M$ with respect to J.

Throughout, we denote by X, Y the vector fields tangential to the submanifold N_T, by Z etc, the vector fields tangential to $N_\perp$ and by W etc, the vector fields tangential to N_θ.

Theorem 3.1. *There do not exist proper sequential warped product submanifolds of nearly Kaehler manifold of the form* $(N_\theta \times_f N_\perp) \times_h N_T$.

Proof. On using formula (3), we can write

$$g(\sigma(X,JX),JZ) = g(\bar{\nabla}_{JX}X,JZ) = -g(J\bar{\nabla}_{JX}X,Z). \tag{16}$$

As $JX \in D$ for $X \in D$, using (2) and (3), we get

$$g(J\bar{\nabla}_{JX}X,Z) = -g((\bar{\nabla}_{JX}J)X,Z) - g(\nabla_{JX}Z,JX).$$

Now, in view of Proposition 2.2, the last term of the above equation leads to

$$g(J\bar{\nabla}_{JX}X,Z) = -g((\bar{\nabla}_{JX}J)X,Z) - Z\ln hg(X,X) \tag{17}$$

Using (16) and (17), we obtain

$$g(\sigma(X,JX),JZ) = Z\ln hg(X,X) + g((\bar{\nabla}_{JX}J)X,Z) \tag{18}$$

If X is replaced by JX in the above equation, we get

$$-g(\sigma(X,JX),JZ) = Z\ln hg(X,X) - g((\bar{\nabla}_X J)JX,Z) \tag{19}$$

Adding (18) and (19) and using the nearly Kaehler condition (2), we obtain

$$Z\ln h\, g(X,X) = 0.$$

As X is arbitrary vector field on N_T, it follows from the above equation that h is constant on $N_\perp$, that is the warped product $(N_\theta \times_f N_\perp) \times_h N_T$ is not proper. □

In [12], V. A. Khan et. al established the following:

Theorem 3.2. *[12] In a nearly Kaehler manifold $\tilde{\mathbb{M}}$, the proper warped product submanifolds of the form $M = N \times_\psi N_T$, where N and N_T are respectively Riemannian and holomorphic submanifolds of $\tilde{M}$ and ψ is the warping function on N, do not exist.*

In view of the above result, we can conclude that

Corollary 3.3. *If N_T, $N_\perp$ and N_θ are the holomorphic, totally real and pointwise proper slant submanifolds then (proper) sequential warped product submanifolds of type (ii) and (iv) in a nearly Kaehler manifold are non-existent.*

Since with base as totally real submanifold $N_\perp$ and fiber as pointwise slant submanifold N_θ, $N_\perp \times_f N_\theta$ is a warped product submanifold which is itself a Riemannian submanifold, we can use Theorem 3.2 for the sequential warped product of type (v) to deduce that

Corollary 3.4. *The proper sequential warped product subamnifolds $(N_\perp \times_f N_\theta) \times_h N_T$ in a nearly Kaehler manifold do not exist.*

The sequential warped product submanifolds of type (i) i.e. $(N_T \times_f N_\perp) \times_h N_\theta$ do exist in Kaehler manifold [13]. Therefore we study these warped products in nearly Kaehler manifold and obtain a sharpe inequality involving the second fundamental form and warping functions. In a Kaehler manifold, the proper sequential warped product submanifolds of type (iii) do not exist [13]. In a nearly Kaehler manifold, they are subject to investigate and will be studied separately.

We start with the following lemmas which will be helpful in proving our main result.

Lemma 3.5. *Let $(N_T \times_f N_\perp) \times_h N_\theta$ be a sequential warped product submanifold of a nearly Kaehler manifold $\tilde{\mathbb{M}}$ where N_T, $N_\perp$ and N_θ are respectively the holomorphic, totally real and pointwise slant submanifolds of $\tilde{\mathbb{M}}$, then we have the following identities:*

$$g(\sigma(X,Y),JZ) = 0 \tag{20}$$

and

$$g(\sigma(X,Y),FW) = 0 \tag{21}$$

for $X, Y \in \Gamma(TN_T)$, $Z \in \Gamma(TN_\perp)$ and $W \in \Gamma(TN_\theta)$.

Proof. On using (2) and (3), we have

$$\begin{aligned} g(\sigma(X,Y),JZ) &= g(\bar{\nabla}_X Y, JZ) \\ &= -g(\bar{\nabla}_X JY, Z) + g((\bar{\nabla}_X J)Y, Z) \\ &= g(\nabla_X Z, JY) + g((\bar{\nabla}_X J)Y, Z) \end{aligned}$$

Now, applying Proposition 2.2, it takes the form

$$g(\sigma(X,Y),JZ) = (X \ln f) g(Z, JY) + g((\bar{\nabla}_X J)Y, Z).$$

Hence,

$$g(\sigma(X,Y),JZ) = g((\bar{\nabla}_X J)Y, Z).$$

The left hand side in the above equation is symmetric in X and Y while the right hand side is skew-symmetric, therefore

$$g(\sigma(X,Y),JZ) = 0.$$

This proves (20). Now to prove (21), by the use of (3) and (6), we have

$$g(\sigma(X,Y),FW) = g(\bar{\nabla}_X Y, JW) - g(\nabla_X Y, PW).$$

By applying Heipko's characterization for the sequential warped product $(N_T \times_f N_\perp) \times_h N_\theta$, N_T is totally geodesic submanifold in $N_T \times_f N_\perp$ resulting in $\nabla_X Y \in \Gamma(TN_T)$. With this fact the above equation becomes

$$g(\sigma(X,Y),FW) = g(\bar{\nabla}_X Y, JW)$$

Now, in a similar way as in the proof of equation (20), we get (21). □

On a nearly Kaehler manifold $\tilde{\mathbb{M}}$, for any $U, V \in \Gamma(T\tilde{\mathbb{M}})$

$$\mathcal{P}_U V + \mathcal{P}_V U = 0. \tag{22}$$

Now, for $X \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\perp)$, using (12), we have

$$\mathcal{P}_X Z + \mathcal{P}_Z X = (\bar{\nabla}_X P)Z + (\bar{\nabla}_Z P)X - A_{FZ}X - 2th(X,Z).$$

Further, using (22), (8) and the fact that $PZ = 0$ for $Z \in \Gamma(TN_\perp)$, we have

$$\nabla_Z PX - P\nabla_Z X - P\nabla_X Z - A_{FZ}X - 2th(X,Z) = 0.$$

By the use of Proposition 2.2, the above equation reduces to

$$(PX \ln f)Z = A_{FZ}X + 2th(X,Z) \tag{23}$$

where f is the warping function on N_T. Proceeding in the same way as above, we can get

$$(PX \ln h)W - (X \ln h)PW = A_{FW}X + 2th(X,W) \tag{24}$$

for any $X \in \Gamma(TN_T)$ and $W \in \Gamma(TN_\theta)$ and h being the warping function on $N_T \times N_\perp$.

Lemma 3.6. *For a sequential warped product submanifold $(N_T \times_f N_\perp) \times_h N_\theta$ of a nearly Kaehler manifold $\tilde{\mathbb{M}}$ where N_T, $N_\perp$ and N_θ are respectively the holomorphic, totally real and pointwise slant submanifolds of $\tilde{\mathbb{M}}$, we have*

$$g(\sigma(X,Z),FW) = g(\sigma(X,W),FZ) = 0 \tag{25}$$

for $X \in \Gamma(TN_T)$, $Z \in \Gamma(TN_\perp)$ and $W \in \Gamma(TN_\theta)$.

Proof. By taking the inner product of $W \in \Gamma(TN_\theta)$ in (23), we get

$$g(\sigma(X,W),FZ) = 2g(\sigma(X,Z),FW) \tag{26}$$

Again taking the inner product of $Z \in \Gamma(TN_\perp)$ in (24), we get

$$g(\sigma(X,Z),FW) = 2g(\sigma(X,W),FZ). \tag{27}$$

(25) follows from (26) and (27). □

Lemma 3.7. *Let $(N_T \times_f N_\perp) \times_h N_\theta$ be a sequential warped product submanifold of a nearly Kaehler manifold $\tilde{\mathbb{M}}$ where N_T, $N_\perp$ and N_θ are respectively the holomorphic, totally real and pointwise slant submanifolds of $\tilde{\mathbb{M}}$. Then*
(i) for $Z_1, Z_2 \in \Gamma(TN_\perp)$, we have

$$g(\sigma(X,Z_1),FZ_2) = -(JX \ln f)g(Z_1,Z_2), \tag{28}$$

(ii) for $W_1, W_2 \in \Gamma(TN_\theta)$, we have

$$g(\sigma(X,W_1),FW_2) = \frac{1}{3}(X \ln h)g(PW_1,W_2) - (PX \ln h)g(W_1,W_2) \tag{29}$$

Proof. If $Z_1, Z_2 \in \Gamma(TN_\perp)$, by using (23), we can write

$$(PX \ln f)g(Z_1,Z_2) = g(\sigma(X,Z_2),FZ_1) - 2g(\sigma(X,Z_1),FZ_2). \tag{30}$$

Interchanging Z_1 and Z_2 in the above equation and subtracting the two equations, we get

$$g(\sigma(X,Z_1),FZ_2) = g(\sigma(X,Z_2),FZ_1). \tag{31}$$

From (30) and (31), we get (28). This proves part (*i*).
Now to prove part (*ii*), we take any $W_1, W_2 \in \Gamma(TN_\theta)$. On using (24), we obtain

$$(PX \ln h)g(W_1,W_2) - (X \ln h)g(PW_1,W_2) = g(\sigma(X,W_2),FW_1) - 2g(\sigma(X,W_1),FW_2) \tag{32}$$

Interchanging W_1 and W_2 and then adding and subtracting with the above equation, we arrive at the following

$$g(\sigma(X,W_1),FW_2) + g(\sigma(X,W_2),FW_1) = -2(PX \ln h)g(W_1,W_2) \tag{33}$$

$$g(\sigma(X,W_1),FW_2) - g(\sigma(X,W_2),FW_1) = \frac{2}{3}(X \ln h)g(PW_1,W_2) \tag{34}$$

The above two equations give (29). □

Let $M = (N_T \times_f N_\perp) \times_h N_\theta$ be a sequential warped product submanifold of a nearly Kaehler manifold $\tilde{\mathbb{M}}$ with dim $(M) = n$, dim $(\tilde{\mathbb{M}}) = \tilde{n}$, $\dim(N_T) = p = 2m$, $\dim(N_\perp) = q$ and $\dim(N_\theta) = r = 2s$. For any local orthonormal frame $\{\tilde{e}_i\}, i = 1,2,\cdots,n$ of the tangent bundle and $\{E_k\}, k = 1,2,\cdots,(\tilde{n}-n)$ of the normal bundle of the manifold M, the norm of the second fundamental form σ is defined as

$$||\sigma||^2 = \sum_{i,j=1}^{n} g(\sigma(\tilde{e}_i,\tilde{e}_j),\sigma(\tilde{e}_i,\tilde{e}_j)) = \sum_{i,j=1}^{n}\sum_{k=1}^{(\tilde{n}-n)} g(\sigma(\tilde{e}_i,\tilde{e}_j),E_k)^2. \tag{35}$$

For the local frame of othonormal vector fields $\{\tilde{e}_i\}$ for tangent bundle of M, we adopt the following convention of indices:
For $1 \le i \le p$, $\tilde{e}_i = e_i$, for $p+1 \le i \le p+q$, $\tilde{e}_i = \bar{e}_i$ and for $p+q \le i \le p+q+r$, $\tilde{e}_i = \hat{e}_i$. Moreover
$D^T = \text{span}\{e_1, e_2, \cdots, e_p\}$ where $p = 2m$ and $e_{m+i} = Je_i, i = 1,2,\cdots,m$
$D^\perp = \text{span}\{\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_q\}$

$D^\theta = \text{span}\{\hat{e_1}, \hat{e_2}, \cdots, \hat{e_r}\}$ where $r = 2s$ and $\hat{e}_{s+k} = \sec\theta P\hat{e_k}$, $k = 1, 2, \cdots, s$.
The local orthonormal frame $\{E_k\}$ for normal bundle $T^\perp M$ will be given as
$JD^\perp = \text{span}\{J\bar{e_1}, J\bar{e_2}, \cdots, J\bar{e_q}\}$
$FD^\theta = \text{span}\{\csc\theta F\hat{e_1}, \csc\theta F\hat{e_2}, \cdots, \csc\theta F\hat{e_r}\}$
$\nu = \text{span}\{e_1^*, e_2^*, \cdots, e_t^*\}$ where $t = \tilde{n} - q - r$.
Therefore, the tangent bundle of M is spanned by $\{e_1, e_2, \cdots, e_p, \bar{e_1}, \bar{e_2}, \cdots, \bar{e_q}, \hat{e_1}, \hat{e_2}, \cdots, \hat{e_r}\}$ with $\dim(M) = p + q + r$.

Now we prove Chen's inequality for sequential warped product submanifold of a nearly Kaehler manifold. From now we consider the ranges of indices as follows:

$$i, j = 1, 2, \cdots p;\ \alpha, \beta, \gamma = 1, 2, \cdots q;\ k, l, u = 1, 2, \cdots r$$

Theorem 3.8. *Let $(N_T \times_f N_\perp) \times_h N_\theta$ be a $(p + q + r)$-dimensional sequential warped product submanifold of a nearly Kaehler manifold $\tilde{\mathbb{M}}$ such that N_T, $N_\perp$ and N_θ are respectively the holomorphic, totally real and pointwise slant submanifolds of $\tilde{\mathbb{M}}$, then*

$$||\sigma||^2 \geq 2q||\nabla \ln f||^2 + 2r(\csc^2\theta + \frac{1}{9}\cot^2\theta)||\nabla^T \ln h||^2 \tag{36}$$

where $\nabla \ln f$ is the gradient of $\ln f$ on M and $\nabla^T \ln h$ is the gradient of $\ln h$ on N_T. If the equality in (36) holds identically, we obtain
(i) $N_T \times_f N_\perp$ is totally geodesic in $\tilde{\mathbb{M}}$ if and only if $g(\sigma(D^T, D^\perp), JD^\perp) = 0$.
(ii) N_θ is totally umbilical in $\tilde{\mathbb{M}}$ with mean curvature vector $-(\nabla \ln h)$.
(iii) M is minimal in $\tilde{\mathbb{M}}$.

Proof. For the adapted frame of orthonormal vector fields, (35) can be written as

$$\begin{aligned}
||\sigma||^2 &= \sum_{i,j}\sum_{\alpha} g(\sigma(e_i, e_j), J\bar{e_\alpha})^2 + \sum_{i,j}\sum_{k} g(\sigma(e_i, e_j), \csc\theta F\hat{e_k})^2 \\
&+ \sum_{i,j}\sum_{v} g(\sigma(e_i, e_j), e_v^*)^2 + \sum_{\alpha,\beta,\gamma} g(\sigma(\bar{e_\alpha}, \bar{e_\beta}), J\bar{e_\gamma})^2 \\
&+ \sum_{\alpha,\beta}\sum_{k} g(\sigma(\bar{e_\alpha}, \bar{e_\beta}), \csc\theta F\hat{e_k})^2 + \sum_{\alpha,\beta}\sum_{v} g(\sigma(\bar{e_\alpha}, \bar{e_\beta}), e_v^*)^2 \\
&+ \sum_{k,l}\sum_{\alpha} g(\sigma(\hat{e_k}, \hat{e_l}), J\bar{e_\alpha})^2 + \sum_{k,l,u} g(\sigma(\hat{e_k}, \hat{e_l}), \csc\theta F\hat{e_u})^2 \\
&+ \sum_{k,l}\sum_{v} g(\sigma(\hat{e_k}, \hat{e_l}), e_v^*)^2 + 2\{\sum_{i}\sum_{\alpha,\beta} g(\sigma(e_i, \bar{e_\alpha}), J\bar{e_\beta})^2 \\
&+ \sum_{i}\sum_{\alpha}\sum_{k} g(\sigma(e_i, \bar{e_\alpha}), \csc\theta F\hat{e_k})^2 + \sum_{i}\sum_{\alpha}\sum_{v} g(\sigma(e_i, \bar{e_\alpha}), e_v^*)^2 \\
&+ \sum_{i}\sum_{k}\sum_{\alpha} g(\sigma(e_i, \hat{e_k}), J\bar{e_\alpha})^2 + \sum_{i}\sum_{k,l} g(\sigma(e_i, \hat{e_k}), \csc\theta F\hat{e_l})^2 \\
&+ \sum_{i}\sum_{k}\sum_{v} g(\sigma(e_i, \hat{e_k}), e_v^*)^2 + \sum_{\alpha,\beta}\sum_{k} g(\sigma(\bar{e_\alpha}, \hat{e_k}), J\bar{e_\beta})^2 \\
&+ \sum_{i}\sum_{k,l} g(\sigma(\bar{e_\alpha}, \hat{e_k}), \csc\theta F\hat{e_l})^2 + \sum_{\alpha}\sum_{k}\sum_{v} g(\sigma(\bar{e_\alpha}, \hat{e_k}), e_v^*)^2\}
\end{aligned}$$

Using the observations in Lemma 3.5, Lemma 3.6 and Lemma 3.7, the above expression reduces to

$$\begin{aligned}
\|\sigma\|^2 = &\sum_{i,j}\sum_{v} g(\sigma(e_i,e_j),e_v^*)^2 + \sum_{\alpha,\beta,\gamma} g(\sigma(\bar{e_\alpha},\bar{e_\beta}),J\bar{e_\gamma})^2 \\
&+ \sum_{\alpha,\beta}\sum_{k} g(\sigma(\bar{e_\alpha},\bar{e_\beta}),\csc\theta F\hat{e_k})^2 + \sum_{\alpha,\beta}\sum_{v} g(\sigma(\bar{e_\alpha},\bar{e_\beta}),e_v^*)^2 \\
&+ \sum_{k,l}\sum_{\alpha} g(\sigma(\hat{e_k},\hat{e_l}),J\bar{e_\alpha})^2 + \sum_{k,l,u} g(\sigma(\hat{e_k},\hat{e_l}),\csc\theta F\hat{e_u})^2 \\
&+ \sum_{k,l}\sum_{v} g(\sigma(\hat{e_k},\hat{e_l}),e_v^*)^2 + 2\{\sum_{i}\sum_{\alpha,\beta}(Je_i\ln f)^2 g(\bar{e_\alpha},\bar{e_\beta})^2 \\
&+ \sum_{i}\sum_{\alpha}\sum_{v} g(\sigma(e_i,\bar{e_\alpha}),e_v^*)^2 \\
&+ \csc^2\theta\sum_{i}\sum_{k,l}\{\frac{1}{3}(e_i\ln h)g(P\hat{e_k},\hat{e_l}) - (Je_i\ln h)g(\hat{e_k},\hat{e_l})\}^2 \\
&+ \sum_{i}\sum_{k}\sum_{v} g(\sigma(e_i,\hat{e_k}),e_v^*)^2 + \sum_{\alpha,\beta}\sum_{k} g(\sigma(\bar{e_\alpha},\hat{e_k}),J\bar{e_\beta})^2 \\
&+ \sum_{i}\sum_{k,l} g(\sigma(\bar{e_\alpha},\hat{e_k}),\csc\theta F\hat{e_l})^2 + \sum_{\alpha}\sum_{k}\sum_{v} g(\sigma(\bar{e_\alpha},\hat{e_k}),e_v^*)^2\}.
\end{aligned}$$

A sharp inequality for $\|\sigma\|^2$ is given as (noting that all the terms are positive in the above expression)

$$\begin{aligned}
\|\sigma\|^2 \geq &2\sum_{i}\sum_{\alpha,\beta}(Je_i\ln f)^2 g(\bar{e_\alpha},\bar{e_\beta})^2 \\
&+ 2\csc^2\theta\sum_{i}\sum_{k,l}\{(Je_i\ln h)g(\hat{e_k},\hat{e_l}) + \frac{1}{3}(e_i\ln h)g(\hat{e_k},P\hat{e_l})\}^2.
\end{aligned}$$

If we denote by $\nabla^T\ln h$, the gradient of $\ln h$ on N_T then by direct computations using adapted frame, we derive

$$\|\sigma\|^2 \geq 2q\|\nabla\ln f\|^2 + 2\csc^2\theta(r\|\nabla^T\ln h\|^2 + \frac{1}{9}r\cos^2\theta\|\nabla^T\ln h\|^2)$$

or

$$\|\sigma\|^2 \geq 2q\|\nabla\ln f\|^2 + 2r(\csc^2\theta + \frac{1}{9}\cot^2\theta)\|\nabla^T\ln h\|^2$$

which is (36).
If the equality case in (36) holds identically, we have

$$\sigma(D^T,D^T) = \{0\},\ \sigma(D^\perp,D^\perp) = \{0\},\ \sigma(D^\theta,D^\theta) = \{0\},\ \sigma(D^\perp,D^\theta) = \{0\}, \tag{37}$$

$$\sigma(D^T,D^\theta) \subset FD^\theta,\ \sigma(D^T,D^\perp) \subset JD^\perp. \tag{38}$$

From Corollary 2.3, we know that $N_T \times_f N_\perp$ is totally geodesic in M. By the use of Lemma 3.5, 3.6 and equation (37), (38), we conclude that $N_T \times_f N_\perp$ is totally geodesic in $\tilde{\mathbb{M}}$ if and only if $g(\sigma(D^T,D^\perp),JD^\perp) = 0$. Again using Corollary 2.3, we know that N_θ is totally umbilical in M i.e. we can write

$$\sigma'(W_1,W_2) = g(W_1,W_2)H' \tag{39}$$

for $W_1, W_2 \in TN_\theta$ where σ' and H' are the second fundamental form and mean curavture vector of N_θ in M. If the second fundamental form of N_θ in $\tilde{\mathbb{M}}$ is denoted by σ^0, then we have

$$\sigma^0(W_1, W_2) = \sigma'(W_1, W_2) + \sigma(W_1, W_2).$$

Using (37) and (39), we get

$$\sigma^0(W_1, W_2) = g(W_1, W_2)H' \tag{40}$$

which shows that N_θ is totally umbilical in $\tilde{\mathbb{M}}$. Using Proposition 2.2, for any $U \in TN_T$ or $TN_\perp$, it is easy to find

$$\begin{aligned} g(\sigma'(W_1, W_2), U) &= g(\nabla_{W_1} W_2, U) \\ &= -(U \ln h) g(W_1, W_2). \end{aligned}$$

By using (14), we obtain

$$\sigma'(W_1, W_2) = -g(W_1, W_2)\nabla \ln h.$$

Therefore, N_θ is totally umbilical in $\tilde{\mathbb{M}}$ with mean curvature vector $-(\nabla \ln h)$.
Moreover from (37), it is clear that M is minimal in $\tilde{\mathbb{M}}$. □

If we consider $\dim(N_\theta) = 0$, i.e. M is CR-warped product submanifold in a nearly Kaehler manifold. We have the following:

Corollary 3.9. *Let $M = N_T \times_f N_\perp$ be a $(p + q)$-dimensional CR-warped product submanifold of a nearly Kaehler manifold $\tilde{\mathbb{M}}$ such that N_T and $N_\perp$ are respectively the holomorphic and totally real submanifolds of $\tilde{\mathbb{M}}$, then*

$$\|\sigma\|^2 \geq 2q\|\nabla \ln f\|^2 \tag{41}$$

where $\nabla \ln f$ is the gradient of $\ln f$ and q is the dimension of $N_\perp$. If the equality in (41) holds identically, we obtain
(i) N_T is totally geodesic submanifold in $\tilde{\mathbb{M}}$.
(ii) $N_\perp$ is totally umbilical in $\tilde{\mathbb{M}}$.
(iii) M is a minimal submanifold of $\tilde{\mathbb{M}}$.

The above characterization was obtained in [1], [14].
The following was proved in [12].

Theorem 3.10. *[12] Let $M = N_T \times_f N_\theta$ be a semi-slant warped product submanifold of a nearly Kaehler manifold $\tilde{\mathbb{M}}$ such that N_T and N_θ are respectively the holomorphic and slant submanifolds of $\tilde{\mathbb{M}}$, then σ satisfies*

$$\|\sigma\|^2 \geq 2r\csc^2\theta\{1 + \frac{\cos^4\theta}{9}\}\|\nabla \ln f\|^2 \tag{42}$$

where $\nabla \ln f$ is the gradient of $\ln f$ and r is the dimension of N_θ.

If we consider $\dim(N_\perp) = 0$ in the sequential warped product, i.e. M is a pointwise semi-slant warped product submanifold in a nearly Kaehler manifold. Under this condition, Theorem 3.8 implies the following:

Theorem 3.11. *Let $N_T \times_h N_\theta$ be a pointwise semi-slant warped product submanifold of a nearly Kaehler manifold $\tilde{\mathbb{M}}$ such that N_T and N_θ are respectively the holomorphic and pointwise slant submanifolds of $\tilde{\mathbb{M}}$, then*

$$\|\sigma\|^2 \geq 2r(\csc^2\theta + \frac{1}{9}\cot^2\theta)\|\nabla \ln h\|^2 \tag{43}$$

where $\nabla \ln h$ is the gradient of $\ln h$ and r is the dimension of N_θ. If the equality in (43) holds identically, N_T is totally geodesic in $\tilde{\mathbb{M}}$ and N_θ is totally umbilical in $\tilde{\mathbb{M}}$. Also, M is minimal in $\tilde{\mathbb{M}}$.

Proof. Inequality (43) follows directly from (36) assuming the submanifold as pointwise semi-slant submanifold. In view of (37), it is easy to verify that if the equality holds in (43), N_T is totally geodesic in $\tilde{\mathbb{M}}$ and N_θ is totally umbilical in $\tilde{\mathbb{M}}$ with mean curvature vector $-\nabla \ln h$. Again from (37), M is a minimal submanifold in $\tilde{\mathbb{M}}$. □

Thus Theorem 3.11 is an improved version of Theorem 3.10 proved in [12] as can be seen in the following remark.

Remark 3.12. *For $\theta \in (0, \pi/2)$,*

$$\csc^2\theta\Big\{1 + \frac{\cos^4\theta}{9}\Big\} < (\csc^2\theta + \frac{1}{9}\cot^2\theta)$$

This means the inequality in Theorem 3.11 is more sharp than in Theorem 3.10.

Now we consider both the warping functions f and h are on N_T i.e. in this case sequential warped product $(N_T \times_f N_\perp) \times_h N_\theta$ change into biwarped product submanifold $N_T \times_f N_\perp \times_h N_\theta$ with $f, h : N_T \to (0, \infty)$. In a similar manner as in Theorem 3.8, we find the following inequality for biwarped product submanifolds in nearly Kaehler manifold.

Corollary 3.13. *Let $N_T \times_f N_\perp \times_h N_\theta$ be a $(p + q + r)$-dimensional biwarped product submanifold of a nearly Kaehler manifold $\tilde{\mathbb{M}}$ such that N_T, $N_\perp$ and N_θ are respectively the holomorphic, totally real and pointwise slant submanifolds of $\tilde{\mathbb{M}}$, then*

$$||\sigma||^2 \geq 2q||\nabla \ln f||^2 + 2r(\csc^2\theta + \frac{1}{9}\cot^2\theta)||\nabla \ln h||^2 \tag{44}$$

where $\nabla \ln f$ and $\nabla \ln h$ are the gradients of $\ln f$ and $\ln h$ respectively and q and r are the dimensions of $N_\perp$ and N_θ respectively. If the equality in (44) *holds identically, we obtain*
(i) N_T is totally geodesic in $\tilde{\mathbb{M}}$.
(ii) $N_\perp$ and N_θ are totally umbilical in $\tilde{\mathbb{M}}$ with mean curvature vectors $-(\nabla \ln f)$ and $-(\nabla \ln h)$ respectively.

The same result was obtained in [18].

4. Conclusion

In this study on warped product manifolds, we investigated a new class of warped product manifolds namely sequential warped product submanifolds with holomorphic, totally real and pointwise slant factor and ambient manifold a nearly Kaehler manifold. We looked into all possible type of products and discussed in detail the sequential warped product of type $(N_T \times_f N_\perp) \times_h N_\theta$. We obtained an inequality having the squared norm of the second fundamental form and the warping functions and slant function. This inequality generalizes many existing results in other leading submanifolds embedded in nearly Kaehler manifold.

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