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Starlikness associated with limacon

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Abstract. Let $\mathcal{S}^*_{lim}$ represent a subclass of analytic functions f defined in the unit disk such that $\frac{zf'(z)}{f(z)}$ lies in the interior of the region bounded by the limacon which is given by the equation $\left[(u-1)^2+v^2-\frac{1}{4}\right]^2-2\left[\left(u-1+\frac{1}{2}\right)^2+v^2\right]=0$. For this class, we obtain the structural formula, inclusion results and some radii problems for subclasses of starlike functions. Furthermore, we obtain sufficient conditions and coefficient bounds for this class of functions.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \qquad z\in\mathbb{D}, \tag{1}$$

which are analytic in the open unit disk $\mathbb{D}=\{z:|z|<1,\ z\in\mathbb{C}\}$. Let $\mathcal{S}$ denote the subclass of analytic functions $\mathcal{A}$ which are univalent in $\mathbb{D}$. A function f is in class $\mathcal{S}^*$ of starlike functions if it satisfies $Re\left(zf'(z)/f(z)\right)>0$ in $\mathbb{D}$. Similarly, a function f is in class $\mathcal{C}$ of convex functions if it satisfies $Re\left(1+zf''(z)/f'(z)\right)>0$ in $\mathbb{D}$. A function f is said to be subordinate to a function g written as $f\prec g$, if there exists a Schwarz function ω with $\omega(0)=0$ and $|\omega(z)|<1$ such that $f(z)=g(\omega(z))$. In particular, if g is univalent in $\mathbb{D}$ and $f(0)=g(0)$, then $f(\mathbb{D})\subset g(\mathbb{D})$. Ma and Minda [11] gave a unified presentation of various subclasses of starlike functions by using subordination, where they introduced

$$\mathcal{S}^*(\psi):=\left\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec\psi(z)\right\}.$$

Here ψ is an analytic and univalent function in $\mathbb{D}$ such that $\psi(\mathbb{D})$ is convex with $\psi(0)=1$ and $Re\{\psi'(z)\}>0$, $z\in\mathbb{D}$. For particular choices of function ψ, we obtain several classes of analytic and univalent functions. Some are given as follows:

i. $\mathcal{S}^*[A,B]:=\mathcal{S}^*\left(\frac{1+Az}{1+Bz}\right)$, $-1\le B<A\le 1$, see [5].

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ii. $\mathcal{S}_s^* := \mathcal{S}^*(1+\sin(z))$, see [3].
iii. $\mathcal{SL}^* := \mathcal{S}^*(\sqrt{1+z})$, see [23].
iv. $\mathcal{S}_{RL}^* := \mathcal{S}^*\left(\sqrt{2}-(\sqrt{2}-1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}}\right)$, see [14].
v. $\mathcal{S}_C^* := \mathcal{S}^*\left(1+\frac{4z}{3}+\frac{2z^2}{3}\right)$, see [21].
vi. $\mathcal{S}_e^* := \mathcal{S}^*(e^z)$, see [15].
vii. $\mathcal{S}_{\cos}^* := \mathcal{S}^*(\cos(z))$, see [2].
viii. $\mathcal{S}_l^* := \mathcal{S}^*(\sqrt{1+z^2}+z)$, see [18].
ix. $\mathcal{BS}(\alpha) := \mathcal{S}^*(1+\frac{z}{1-\alpha z^2}),\ 0\le\alpha\le 1$, see [6].
x. $\mathcal{S}_{q_c}^* := \mathcal{S}^*(\sqrt{1+cz},\ \ 0<c\le 1,$ see [22].
xi. $\mathcal{SL}^*(\alpha) = \mathcal{S}^*(\alpha+(1-\alpha)\sqrt{1+z}),\ 0\le\alpha\le 1,$ see [10].
Also see [8, 20, 26, 27, 29, 31].

Let the class $\mathcal{M}(\beta)$ consist of functions $f\in\mathcal{A}$ satisfying $Re(\frac{zf'(z)}{f(z)})<\beta,\ \beta>1$. Let

$$\mathcal{P}[A,B] := \left\{p(z)=1+\sum_{n=1}^{\infty}c_nz^n : p(z)\prec\frac{1+Az}{1+Bz}, -1\le B<A\le 1\right\}.$$

In particular, $\mathcal{P}[1-2\alpha,-1] := \mathcal{P}(\alpha)$, $(0\le\alpha<1)$ and $\mathcal{P}(0) := \mathcal{P}$, the well-known class of analytic functions with positive real part in $\mathbb{D}$. MacGregor [12] studied the class $\mathcal{W}$ of functions $f\in\mathcal{A}$ such that $f(z)/z\in\mathcal{P}$. Recently, Masih and Kanas [13] have studied the class $\mathcal{ST}_L(s)$ defined as

$$\mathcal{ST}_L(s) := \left\{f\in\mathcal{A} : \frac{zf'(z)}{f(z)}\prec\mathbb{L}_s(z)=(1+sz)^2,\quad 0<s\le\frac{1}{\sqrt{2}}\right\}.$$

The function $\mathbb{L}_s$ maps $\mathbb{D}$ onto a domain bounded by a limacon given by

$$\mathcal{D}_{\lim}(s) = \left\{u+iv\in\mathbb{C} : \left[(u-1)^2+v^2-s^4\right]^2 = 4s^2\left[\left(u-1+s^2\right)^2+v^2\right]\right\}.$$

The class $\mathcal{S}_{lim}^* = \mathcal{ST}_L(\frac{1}{\sqrt{2}})$ is studied in [33] and defined as

$$\mathcal{S}_{lim}^* := \left\{f\in\mathcal{A} : \frac{zf'(z)}{f(z)}\prec\mathbb{L}_{\frac{1}{\sqrt{2}}}(z)=\left(1+\frac{1}{\sqrt{2}}z\right)^2\right\}.$$

A function f is said to be in the class $\mathcal{S}_{lim}^*$ if there exists an analytic function h, satisfying $h(z)\prec h_0(z) = \left(1+\frac{1}{\sqrt{2}}z\right)^2$ such that

$$f(z) = z\,exp\left(\int_0^z\frac{h(t)-1}{t}dt\right). \tag{2}$$

We now give few examples of the functions in the class $\mathcal{S}_{lim}^*$. Let $q_1(z)=1+\frac{z}{2},\ \ q_2(z)=\frac{21+19z}{21+z}$ and $q_3(z)=1+\sin(z)$. The function $\mathbb{L}_{\frac{1}{\sqrt{2}}}(z)$ is univalent in $\mathbb{D}$, $h_i(0)=\mathbb{L}_{\frac{1}{\sqrt{2}}}(0)=1, (i=1,2,3)$ and $h_i(\mathbb{D})\subset\mathbb{L}_{\frac{1}{\sqrt{2}}}(\mathbb{D})$. This implies that $h_i(z)\prec\mathbb{L}_{\frac{1}{\sqrt{2}}}(z)$. Hence by using (2), we obtain functions in the class $\mathcal{S}_{lim}^*$ corresponding to every function $h_i(z)$, $(i=1,2,3)$ respectively as follows:

$$f_1(z)=ze^{\frac{z}{2}}\quad f_2(z)=z\left(\frac{z+21}{21}\right)^{18}\quad f_3(z)=z+z^2+\frac{z^3}{2}+\frac{z^4}{9}-\frac{z^5}{72}-\cdots.$$

If we take $h(z)=\mathbb{L}_{\frac{1}{\sqrt{2}}}(z)$, then the function

$$f_*(z)=ze^{\sqrt{2}z+\frac{1}{4}z^2} \tag{3}$$

plays the role of extremal function for many problems in the class $\mathcal{S}^*_{lim}$.

We use the following lemma to establish our results.

Lemma 1.1. *[19] If $p \in \mathcal{P}[A,B]$, then, for $|z| = r$,*

$$\left| p(z) - \frac{1 - ABr^2}{1 - B^2r^2} \right| \leq \frac{(A-B)r}{1 - B^2r^2}.$$

In particular, if $p \in \mathcal{P}(\alpha)$, then, for $|z| = r$,

$$\left| p(z) - \frac{(1 + (1-2\alpha))r^2}{1 - r^2} \right| \leq \frac{2(1-\alpha)r}{1 - r^2},$$

and

$$\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2(1-\alpha)r}{(1-r)(1+(1-2\alpha)r)}.$$

2. Inclusion Results

This section deals with inclusion relations between the class $\mathcal{S}^*_{lim}$ and certain subclasses of starlike functions.

Theorem 2.1. *For $\mathcal{S}^*_{lim}$, the following inclusion relations hold:*
(i) $\mathcal{SL}^*(\alpha) \subset \mathcal{S}^*_{lim}$, for $\alpha \geq \frac{3-2\sqrt{2}}{2}$.
(ii) $\mathcal{S}^*_{q_c} \subset \mathcal{S}^*_{lim}$, for $0 < c \leq \frac{12\sqrt{2}-13}{4}$.
(ii) $\mathcal{S}^*[1-\alpha, 0] \subset \mathcal{S}^*_{lim}$, for $\frac{3-2\sqrt{2}}{2} \leq \alpha \leq 1$.

Proof. (i) To show the function $f \in \mathcal{SL}^*(\alpha)$ lies in the class $\mathcal{S}^*_{lim}$, we use the result due to Khattar et al. [10, Lemma 2.1], which gives

$$\alpha < Re\left(\frac{zf'(z)}{f(z)}\right) < \alpha + (1-\alpha)\sqrt{2}.$$

The function $f \in \mathcal{S}^*_{lim}$ if either $\alpha \geq \frac{3-2\sqrt{2}}{2}$ or $\alpha + (1-\alpha)\sqrt{2} \leq \frac{3+2\sqrt{2}}{2}$. Thus, $f \in \mathcal{S}^*_{lim}$ for $\alpha \geq \frac{3-2\sqrt{2}}{2}$.
(ii) Let $f \in \mathcal{S}^*_{q_c}$ $(0 < c \leq 1)$. Then $\frac{zf'(z)}{f(z)} \prec \sqrt{1+cz}$ and

$$\sqrt{1-c} < Re\left(\frac{zf'(z)}{f(z)}\right) < \sqrt{1+c}.$$

We see that $\sqrt{1+c} < \sqrt{2} < 2 < \frac{3+2\sqrt{2}}{2}$. Thus the function $f \in \mathcal{S}^*_{lim}$ if $\sqrt{1-c} \geq \frac{3-2\sqrt{2}}{2}$. This gives $c \leq \frac{12\sqrt{2}-13}{4}$.
(iii) Proceeding as in part (ii), we see that the function $f \in \mathcal{S}^*[1-\alpha, 0]$ lies in the class $\mathcal{S}^*_{lim}$ if

$$\frac{3-2\sqrt{2}}{2} \leq \alpha < Re\left(\frac{zf'(z)}{f(z)}\right) < 2 - \alpha \leq \frac{3+2\sqrt{2}}{2},$$

which holds for $\alpha \geq \frac{3-2\sqrt{2}}{2}$. □

3. Radius problems for the class $\mathcal{S}^*_{lim}$

Lemma 3.1. *Let* $\frac{3-2\sqrt{2}}{2} < a < \frac{3+2\sqrt{2}}{2}$*. Then the following inclusions hold:*

$$\{\omega \in \mathbb{C} : |\omega - a| < r_a\} \subseteq \mathcal{D}_{\lim}\left(\frac{1}{\sqrt{2}}\right) \subseteq \{\omega \in : |\omega - a| < R_a\},$$

where

$$r_a = \begin{cases} a - \frac{3-2\sqrt{2}}{2}, & \frac{3-2\sqrt{2}}{2} < a \le \frac{3}{2}, \\ \frac{3+2\sqrt{2}}{2} - a, & \frac{3}{2} \le a < \frac{3+2\sqrt{2}}{2}, \end{cases}$$

and R_a *be given by*

$$R_a = \begin{cases} \frac{3+2\sqrt{2}}{2} - a, & \frac{3-2\sqrt{2}}{2} < a \le \frac{4+3\sqrt{2}}{4+2\sqrt{2}}, \\ \sqrt{\frac{a(2a-1)^2}{4(a-1)}}, & \frac{4+3\sqrt{2}}{4+2\sqrt{2}} \le a < \frac{3+2\sqrt{2}}{2}. \end{cases}$$

Proof. Let us first consider the square of distance from $(a, 0)$ to a point on the boundary $\mathcal{D}_{\lim}\left(\frac{1}{\sqrt{2}}\right)$, which is given by

$$h(t) = \left(a - \left(1 + \sqrt{2}\cos(t) + \frac{1}{2}\cos(2t)\right)\right)^2 + \left(\sqrt{2}\sin(t) + \frac{1}{2}\sin(2t)\right)^2, \quad -\pi \le t \le \pi.$$

In order to show that $|\omega - a| < r_a$ is the largest disk contained in $\mathcal{D}_{\lim}\left(\frac{1}{\sqrt{2}}\right)$, we need only to show that $\min_{0\le t\le\pi} \sqrt{h(t)} = r_a$. Since $h(t) = h(-t)$, it is sufficient to consider the range $0 \le t \le \pi$. We suppose that $\left(3 - 2\sqrt{2}\right)/2 < a \le \left(4 + 3\sqrt{2}\right)/\left(4 + 2\sqrt{2}\right)$. It is easy to see that $h'(t) = 0$ has two roots 0 and π. Also $h'(t) < 0$ for $0 < t < \pi$. This implies

$$\min_{0\le t\le\pi} \sqrt{h(t)} = \sqrt{h(\pi)} = a - \frac{3}{2} + \sqrt{2} \text{ and } \max_{0\le t\le\pi} \sqrt{h(t)} = \sqrt{h(0)} = \frac{3}{2} + \sqrt{2} - a.$$

We also suppose that $\left(4 + 3\sqrt{2}\right)/\left(4 + 2\sqrt{2}\right) < a \le 3/2$. Then $h'(t) = 0$ has three roots namely 0, $t_0 \in (0, \pi)$ and π. The root t_0 depends upon a. The graph of $h'(t)$ shows that $h'(t) > 0$ for $(0, t_0)$ and $h'(t) < 0$ for (t_0, π). Hence we conclude that

$$\min_{0\le t\le\pi} \sqrt{h(t)} = \sqrt{h(\pi)} = a - \frac{3}{2} + \sqrt{2}.$$

After simple calculations, we obtain $t_0 = \cos^{-1}\left(\frac{-\sqrt{2}(3-2a)}{4(1-a)}\right)$. Therefore

$$\max_{0\le t\le\pi} \sqrt{h(t)} = \sqrt{h(t_0)} = \sqrt{\frac{a(2a-1)^2}{4(a-1)}}.$$

Now for $3/2 < a < \frac{3+2\sqrt{2}}{2}$, the equation $h'(t) = 0$ has three roots namely 0, $t_1 \in (0, \pi)$ and π. The root t_1 depends upon a. The graph of $h(t)$ reveals that it is increasing in the interval $(0, t_1)$ and decreasing in (t_1, π) but $h(0) < h(\pi)$. Hence

$$\min_{0\le t\le\pi} \sqrt{h(t)} = \sqrt{h(0)} = \frac{3}{2} + \sqrt{2} - a \text{ and } \max_{0\le t\le\pi} \sqrt{h(t)} = \sqrt{h(t_0)} = \sqrt{\frac{a(2a-1)^2}{4(a-1)}}.$$

This completes the result. □

Theorem 3.2. *The radius of starlikeness of order α for the class $\mathcal{S}^*_{lim}$ is given by*

$$R_{\mathcal{S}^*(\alpha)}(\mathcal{S}^*_{lim}) = \begin{cases} \sqrt{1-2\alpha}, & 0 < \alpha \leq \frac{1}{4}, \\ \sqrt{2}(1-\sqrt{\alpha}), & \frac{1}{4} \leq \alpha < 1. \end{cases}$$

Proof. Let $f \in \mathcal{S}^*_{lim}$. Then $f \in \mathcal{S}^*(\psi)$, where $\psi(z) = 1 + \sqrt{2}z + z^2/2$ and we notice that

$$Re\frac{zf'(z)}{f(z)} \geq \min_{|z|=r} Re\psi(z) = \begin{cases} 1-\sqrt{2}r + \frac{r^2}{2}, & r \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{2}(1-r^2), & r \geq \frac{1}{\sqrt{2}}. \end{cases} \tag{4}$$

Case(i) Let $0 < \alpha \leq \frac{1}{4}$. Then $\frac{1}{\sqrt{2}} \leq \rho < 1$, where $\rho := R_{\mathcal{S}^*(\alpha)}(\mathcal{S}^*_{lim})$. Let $|z| = r < \rho$. If $r \leq \frac{1}{\sqrt{2}}$, then we have

$$Re\frac{zf'(z)}{f(z)} \geq 1-\sqrt{2}r + \frac{r^2}{2} \geq \frac{1}{4} \geq \alpha.$$

If $\rho > r > \frac{1}{\sqrt{2}}$, then from (4), we write

$$Re\frac{zf'(z)}{f(z)} \geq \frac{1}{2}(1-r^2) \geq \frac{1}{2}(1-\rho^2) = \alpha.$$

Case(ii) Let $\frac{1}{4} \leq \alpha < 1$. Then $0 < \rho \leq \frac{1}{\sqrt{2}}$. Let $|z| = r < \rho$. Then by using (4), it follows that

$$Re\frac{zf'(z)}{f(z)} \geq 1-\sqrt{2}r + \frac{r^2}{2} \geq 1-\sqrt{2}\rho + \frac{\rho^2}{2} = \alpha.$$

The result is sharp for the function f_* given by (3). □

Theorem 3.3. *The $\mathcal{SL}^*$-radius for the class $\mathcal{S}^*_{lim}$ is given by*

$$R_{\mathcal{SL}^*}(\mathcal{S}^*_{lim}) = -\sqrt{2} + 2^{\frac{3}{4}} \simeq 0.2676.$$

Proof. Let $f \in \mathcal{S}^*_{lim}$. Then for $z = re^{i\theta}$, $-\pi < \theta \leq \pi$, we can write

$$\left|\psi(z) - 1\right|^2 = \frac{1}{4}r^2(r^2 + 4\sqrt{2}\cos(t) + 8) < (\sqrt{2}-1)^2.$$

This implies that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \sqrt{2}-1, \quad |z| = r < -\sqrt{2} + 2^{\frac{3}{4}} \simeq 0.2676.$$

Now using a result due to Ali et al. [1, Lemma 2.2], we have the required result. Result is sharp for the function f_* given by (3).
□

Theorem 3.4. *The $\mathcal{M}(\beta)(\beta > 1)$ radius for the class $\mathcal{S}^*_{lim}$ is given by*

$$R_{\mathcal{M}(\beta)}(\mathcal{S}^*_{lim}) = \begin{cases} \sqrt{2}(\sqrt{\beta}-1), & 1 < \beta \leq \frac{3+2\sqrt{2}}{2}, \\ 1, & \beta \geq \frac{3+2\sqrt{2}}{2}. \end{cases}$$

Proof. Let $f \in \mathcal{S}^*_{lim}$. Then it is easy to deduce that

$$\max_{|z|=r} Re\psi(z) = \psi(r) = 1 + \sqrt{2}r + \frac{1}{2}r^2.$$

To prove our result, we have the following two cases:
Case(i). If $1 < \beta \leq \frac{3+2\sqrt{2}}{2}$, then for $|z| = r$, we have

$$Re\frac{zf'(z)}{f(z)} \leq \max_{|z|=r} Re\psi(z) < \beta.$$

Solving above relation, we obtain $r < \sqrt{2}(\sqrt{\beta} - 1)$.
Case (ii). If $\beta \geq \frac{3+2\sqrt{2}}{2}$, then for $|z| = r$, we have

$$Re\frac{zf'(z)}{f(z)} \leq 1 + \sqrt{2}r + \frac{1}{2}r^2 < \frac{3+2\sqrt{2}}{2} \leq \beta.$$

This result is sharp for the function f_* given by (3). □

Theorem 3.5. *The $\mathcal{S}^*_{lim}$ radii for the classes $\mathcal{SL}^*$, $\mathcal{S}^*_{RL}$, $\mathcal{S}^*_C$, $\mathcal{S}^*_{q_c}$, $\mathcal{SL}^*(\alpha)$, $\mathcal{BS}(\alpha)$ and $\mathcal{W}$ are given by*
(i) $R_{\mathcal{S}^*_{lim}}(\mathcal{SL}^*) = \frac{12\sqrt{2}-13}{4} \approx 0.99264$,
(ii) $R_{\mathcal{S}^*_{lim}}(\mathcal{S}^*_{RL}) = \frac{63\sqrt{2}+65}{158} \approx 0.975287$,
(iii) $R_{\mathcal{S}^*_{lim}}(\mathcal{S}^*_C) = \frac{-2+\sqrt{7+6\sqrt{2}}}{2} \approx 0.9675671$,
(iv) $R_{\mathcal{S}^*_{lim}}\left(\mathcal{S}^*_{q_c}\right) = \frac{4\sqrt{2}-5}{4c}$,
(v) $R_{\mathcal{S}^*_{lim}}(\mathcal{SL}^*(\alpha)) = \frac{4\alpha-13+4\sqrt{2}(3-2\alpha)}{4(\alpha-1)^2}$,
(vi) $R_{\mathcal{S}^*_{lim}}(\mathcal{BS}(\alpha)) = \frac{-1-2\sqrt{2}+\sqrt{9+4\sqrt{2}+49\alpha}}{7\alpha}$,
(vii) $R_{\mathcal{S}^*_{lim}}(\mathcal{W}) = \frac{\sqrt{13-4\sqrt{2}}-2}{2\sqrt{2}-1}$.

Proof. (i) For the functions $f \in \mathcal{SL}^*$, we have $\frac{zf'(z)}{f(z)} \prec \sqrt{1+z}$. Thus for $|z| = r$, we have by Lemma 3.1 that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1 - \sqrt{1-r} \leq 1 - \frac{3-2\sqrt{2}}{2},$$

whenever the inequality $r \leq \frac{12\sqrt{2}-13}{4}$ holds. The result is sharp for the function

$$f_1(z) = \frac{4z\exp(2\sqrt{1+z}-2}{(1+\sqrt{1+z})^2}.$$

Since $\frac{zf_1'(z)}{f_1(z)} = \sqrt{1+z} = \frac{3-2\sqrt{2}}{2}$ at point $z = R_{\mathcal{S}^*_{lim}}(\mathcal{SL}^*)$.
(ii) For functions $f \in \mathcal{S}^*_{RL}$, we have

$$\frac{zf'(z)}{f(z)} \prec \sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}}.$$

This implies that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1 - \sqrt{2} + (\sqrt{2}-1)\sqrt{\frac{1+r}{1-2(\sqrt{2}-1)r}} \leq \frac{2\sqrt{2}-1}{2}$$

provided

$$r \leq \frac{63\sqrt{2}+65}{158} = R_{\mathcal{S}^*_{lim}}(\mathcal{S}^*_{RL}).$$

To show the sharpness of the result, we consider the following function defined as

$$f_2(z) = z\ \exp\left(\int_0^z \frac{q_0(t)-1}{1}dt\right),$$

where

$$q_0(z) = \sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}}.$$

At point $z = R_{\mathcal{S}^*_{\cos}}(\mathcal{S}^*_{RL})$, we have

$$\frac{zf_2'(z)}{f_2(z)} = \sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}} = \frac{3-2\sqrt{2}}{2}.$$

This shows that the result is sharp.
(iii) For the functions $f \in \mathcal{S}^*_C$, we have $\frac{zf'(z)}{f(z)} \prec 1 + \frac{4z}{3} + \frac{2z^2}{3}$. Thus for $|z| = r$, we have by Lemma 3.1

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{4r}{3} + \frac{2r^2}{3} \leq \frac{3+2\sqrt{2}}{2} - 1$$

whenever the inequality $r \leq \frac{-2+\sqrt{7+6\sqrt{2}}}{2}$ holds. Consider the function

$$f_3(z) = z\exp\frac{4z+z^z}{3}.$$

Since $\frac{zf_3'(z)}{f_3(z)} = 1 + \frac{4z}{3} + \frac{2z^2}{3}$, so $f_3 \in \mathcal{S}^*_C$ and at point $z = R_{\mathcal{S}^*_{lim}}(\mathcal{S}^*_C)$, we have $\frac{zf_3'(z)}{f_3(z)} - 1 = \frac{2\sqrt{2}-1}{2}$. Hence the result is sharp.

(iv) For $0 < c \leq 1 - \frac{3-2\sqrt{2}}{2}$, the $\mathcal{S}^*_{lim}$-radius for the class $\mathcal{S}^*_{q_c}$ is 1 by Theorem 2.1 (ii). Let us now assume that $1-(3-2\sqrt{2})/2 < c \leq 1$. Since $f \in \mathcal{S}^*_{q_c}$, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1 - \sqrt{1-cr}.$$

By using Lemma 3.1, we get $1 - \sqrt{1-cr} \leq (3-2\sqrt{2})/2$ and this simplifies to $r \leq \frac{4\sqrt{2}-5}{4c}$.

(v) Let $f \in \mathcal{SL}^*(\alpha)$. Then $\frac{zf'(z)}{f(z)} \prec \alpha + (1-\alpha)\sqrt{(1+z)}$. Now we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \left|\alpha + (1-\alpha)\sqrt{1+z} - 1\right| \leq (1-\alpha)(1-\sqrt{1-r}) \leq 1 - \frac{3-2\sqrt{2}}{2}.$$

This holds for $r \leq \frac{4\alpha-13+4\sqrt{2}(3-2\alpha)}{4(\alpha-1)^2}$. The result is sharp for the function

$$f_4(z) = z\ \exp\left(\int_0^z \frac{q_1(t)-1}{1}dt\right),$$

where $q_1(z) = \alpha + (1-\alpha)\sqrt{(1+z)}$ and $zf'(z)/f(z) = \frac{3-2\sqrt{2}}{2}$ for $z = \frac{4\alpha-13+4\sqrt{2}(3-2\alpha)}{4(\alpha-1)^2}$.

(vi) For $f \in (\mathcal{BS}(\alpha)$, we have $zf'(z)/f(z) \prec 1 + z/(1-\alpha z^2$, which gives

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{r}{1-\alpha r^2}$$

for$|z| < r$. By using Lemma 3.1, we get $r/(1-\alpha r^2) \leq 1-(3-2\sqrt{2})/2$ and it simplifies to $r \leq \frac{-1-2\sqrt{2}+\sqrt{9+4\sqrt{2}+49\alpha}}{7\alpha} = R_{\mathcal{S}^*_{lim}}(\mathcal{BS}(\alpha))$, for $0 < \alpha < 1$. For sharpness, consider the function f_5 given by

$$f_5(z) = z\left(\frac{1+\sqrt{\alpha}z}{1-\sqrt{\alpha}z}\right)^{1/(2\sqrt{\alpha})}.$$

At $z = -R_{\mathcal{S}^*_{lim}}(\mathcal{BS}(\alpha))$, the quantity $zf_5'(z)/f_5(z) = \frac{3-2\sqrt{2}}{2}$.

(vii) Let $f \in \mathcal{W}$. Then $\frac{f(z)}{z} \in \mathcal{P}$ for all $z \in \mathbb{D}$. Let us define a function $p \in \mathcal{P}$ such that $p(z) = f(z)/z$. Then

$$\frac{zf'(z)}{f(z)} = 1 + \frac{zp'(z)}{p(z)}.$$

Thus from Lemma 1.1, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{2r}{1-r^2}.$$

By using Lemma 3.1, the function $f \in \mathcal{S}^*_{lim}$ for $|z| < r$ if $2r/(1-r^2) < 1-\frac{3-2\sqrt{2}}{2}$. This simplifies to $r \leq \frac{\sqrt{13-4\sqrt{2}}-2}{2\sqrt{2}-1}$. The result is sharp for the function $f_6(z) = z(1+z)/(1-z)$. For this function, we have

$$\frac{zf_6'(z)}{f_6(z)} = \frac{3-2\sqrt{2}}{2} \quad at \quad z = \frac{\sqrt{13-4\sqrt{2}}-2}{2\sqrt{2}-1}.$$

□

Theorem 3.6. *Let $-1 \leq B < A \leq 1$, with $B < 0$. Let*

$$R_1 = min\left(1, \frac{1}{\sqrt{3B^2-2AB}}\right),\ R_2 = min\left(1, \frac{-1}{B}\right),\ R_3 = min\left(1, \frac{1+2\sqrt{2}}{2A-3B-2B\sqrt{2}}\right).$$

Then $\mathcal{S}^*_{lim}$ radius for the class $\mathcal{S}^*[A,B]$ is given by

$$R_{\mathcal{S}^*_{lim}}(\mathcal{S}^*[A,B]) = \begin{cases} R_2, & if R_2 \leq R_1, \\ R_3, & if R_2 > R_1. \end{cases}$$

Proof. Let $f \in \mathcal{S}^*[A,B]$, then by Lemma 1.1, we have

$$\left|\frac{zf'(z)}{f(z)} - \frac{1-ABr^2}{1-B^2r^2}\right| \leq \frac{(A-B)r}{1-B^2r^2}.$$

We determine numbers R_1, R_2 and R_3. Now $r \leq R_1$, if and only if $\frac{1-ABr^2}{1-B^2r^2} \leq \frac{3}{2}$. This yields us $r \leq \frac{1}{\sqrt{3B^2-2AB}}$. We determine R_2 such that $r \leq R_2$ if and only if

$$\frac{(A-B)r}{1-B^2r^2} \leq \frac{1-ABr^2}{1-B^2r^2} - \frac{3-2\sqrt{2}}{2}.$$

The above relation gives us $r \leq \frac{-1}{B}$. We determine R_3 such that $r \leq R_3$ if and only if

$$\frac{(A-B)r}{1-B^2r^2} \leq \frac{3+2\sqrt{2}}{2} - \frac{1-ABr^2}{1-B^2r^2}.$$

A simple calculation yields $r \leq \frac{1+2\sqrt{2}}{2A-3B-2B\sqrt{2}}$. □

4. Coefficient Bounds

We need the following lemma to prove our results.

Lemma 4.1. *[17] Let $p \in \mathcal{P}$ and be of the form*

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \qquad z \in \mathbb{D}. \tag{5}$$

Then

1. $|c_n| \leq 2$,
2. $|c_n - v c_k c_{n-k}| \leq 2, \quad v \in [0,1], \quad n > k$,
3. $\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1^2|}{2}$.

Theorem 4.2. *Let $f \in \mathcal{S}^*_{lim}$ and be of the form (1). Then*

$$|a_2| \leq \sqrt{2}, \quad |a_3| \leq \frac{5}{4}, \quad |a_4| \leq \frac{7\sqrt{2}}{12}, \quad |a_5| \leq \frac{97}{32} - \frac{41\sqrt{2}}{24}.$$

Proof. Since $f \in \mathcal{S}^*_{lim}$, therefore

$$\frac{zf'(z)}{f(z)} = 1 + \sqrt{2}\omega(z) + \frac{\omega(z)^2}{2}, \tag{6}$$

where ω is analytic with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{D}$. Now for $p \in \mathcal{P}$, we can write

$$\omega(z) = \frac{p(z) - 1}{p(z) + 1}.$$

Let p be of the form (5). Then from (6), we have

$$a_2 = \frac{\sqrt{2}}{2} c_1, \quad a_3 = \left(\frac{5}{16} - \frac{\sqrt{2}}{8} \right) c_1{}^2 + \frac{\sqrt{2}}{4} c_2, \quad a_4 = \frac{\sqrt{2}}{6} c_3 - \left(\frac{1}{6} - \frac{11\sqrt{2}}{96} \right) c_1{}^3 + \left(\frac{1}{3} - \frac{\sqrt{2}}{6} \right) c_1 c_2 \tag{7}$$

and

$$a_5 = \left(\frac{167}{1536} - \frac{29\sqrt{2}}{384} \right) c_1{}^4 - \left(\frac{31}{96} - \frac{41\sqrt{2}}{192} \right) c_2 c_1{}^2 + \left(\frac{11}{48} - \frac{\sqrt{2}}{8} \right) c_1 c_3 + \left(\frac{3}{32} - \frac{\sqrt{2}}{16} \right) c_2{}^2 + \frac{\sqrt{2}}{8} c_4. \tag{8}$$

The first two bounds follow from Lemma 4.1 (1). For the third bound, we have from (7) that

$$\begin{aligned} |a_4| &\leq \frac{1}{3} \left[\frac{|c_3|}{\sqrt{2}} + \left(\frac{11\sqrt{2}}{16} - \frac{1}{\sqrt{2}} \right) |c_1| |c_2| + \left(1 - \frac{11\sqrt{2}}{16} \right) \left| c_2 - \frac{c_1^2}{2} \right| |c_1| \right] \\ &\leq \frac{1}{3} \left[\sqrt{2} + 2 \left(\frac{11\sqrt{2}}{16} - \frac{1}{\sqrt{2}} \right) |c_1| + \left(1 - \frac{11\sqrt{2}}{16} \right) \left(2 - \frac{|c_1^2|}{2} \right) |c_1| \right], \end{aligned}$$

where we have used Lemma 4.1 (1) and Lemma 4.1 (3). Let $|c_1| = x \in [0,2]$. Then

$$|a_4| \leq \frac{1}{3} \left[\sqrt{2} + 2 \left(\frac{11\sqrt{2}}{16} - \frac{1}{\sqrt{2}} \right) x + \left(1 - \frac{11\sqrt{2}}{16} \right) \left(2 - \frac{x^2}{2} \right) x \right] = \varphi(x).$$

It is easy to see that the maximum of φ exists at $x = 2$, therefore we have the required result. These results are sharp for the function f_* defined in (3) . Let $l_1 = \frac{167}{1536} - \frac{29\sqrt{2}}{384}$, $l_2 = \frac{31}{96} - \frac{41\sqrt{2}}{192}$, $l_3 = \frac{11}{48} - \frac{\sqrt{2}}{8}$, $l_4 = \frac{3}{32} - \frac{\sqrt{2}}{16}$, $l_5 = \frac{\sqrt{2}}{8}$ with $l_i > 0$, $i = 1, 2, \cdots 5$ and $\frac{l_2}{l_3} \in (0, 1)$. Then

$$\begin{aligned} |a_5| &= \left| l_1 \, c_1{}^4 - l_2 \, c_2 c_1{}^2 + l_3 c_1 c_3 + l_4 \, c_2{}^2 + l_5 c_4 \right| \\ &\leq l_1 \, |c_1|^4 + l_3 \, |c_1| \left| c_3 - \frac{l_2}{l_3} \, c_1 c_2 \right| + l_4 \, |c_2|^2 + l_5 \, |c_4| \, . \end{aligned}$$

Now using Lemma 4.1 (1) and Lemma 4.1 (2), we obtain required result. □

Theorem 4.3. *Let $f \in \mathcal{S}^*_{lim}$ and of the form* (1) *. Then*

$$\sum_{n=2}^{\infty} [4n^2 - (17 + 2\sqrt{2})] |a_n|^2 \leq 13 + 12\sqrt{2}.$$

Proof. Let $f \in \mathcal{S}^*_{lim}$. Then $zf'(z)/f(z) = 1 + \sqrt{2}\omega(z) + \frac{1}{2}\omega(z)^2$, where ω is an analytic function with $|\omega(z)| < 1$ for all $z \in \mathbb{D}$. Using the identity $[2 + 2\sqrt{2}\omega(z) + \omega^2(z)]f(z) = 2zf'(z)$, we have

$$\begin{aligned} 2\pi \sum_{n=1}^{\infty} |a_n|^2 r^{2n} &= \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^2 d\theta \geq \int_0^{2\pi} \left| \frac{2re^{i\theta} f'(re^{i\theta})}{2 + 2\sqrt{2}\omega(re^{i\theta}) + \omega^2(re^{i\theta})} \right|^2 d\theta \\ &\geq \frac{4}{17 + 12\sqrt{2}} \int_0^{2\pi} \left| re^{i\theta} f'(re^{i\theta}) \right|^2 d\theta = \frac{8\pi}{17 + 12\sqrt{2}} \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n}. \end{aligned}$$

Now $0 < r < 1$ and $a_1 = 1$. Therefore, we have

$$\sum_{n=2}^{\infty} [4n^2 - (17 + 2\sqrt{2})] |a_n|^2 r^{2n} \leq 0.$$

On taking $r \to 1^-$. We obtain the required result. □

5. Sufficient conditions for class $\mathcal{S}^*_{lim}$

In this section, we find some sufficient conditions for class $\mathcal{S}^*_{lim}$. We use Fukui and Sakaguchi [4] lemma to prove our result. For applications of this lemma see [16].

Lemma 5.1. *[4] Let $w(z) = a_p z^p + a_{p+1} z^{p+1} + \cdots$, $a_p \neq 0$, $1 \leq p$ be analytic in $\mathbb{D}$. If the maximum of $|w(z)|$ on the circle $|z| = r < 1$ is attained at $z = z_0$, then $z_0 w_0'(z_0)/w(z_0)$ is a real number and*

$$\frac{z_0 w_0'(z_0)}{w(z_0)} \geq p.$$

Theorem 5.2. *Let $p(z) = 1 + c_n z^n + \cdots$ be analytic in $\mathbb{D}$ with $c_n \neq 0$. Suppose that $p(z) \neq 0$ and $p(z) \neq -1$ for every z in $\mathbb{D}$. Also we suppose that $p(z) \neq 1$ for every z in $\mathbb{D} \backslash \{0\}$. Let*

$$\left| \frac{zp'(z)}{p(z)} \right| < \frac{2n}{\sqrt{2} + 1}.$$

Then $p(z) \prec \left(1 + \frac{1}{\sqrt{2}} z^n\right)^2$.

Proof. Consider the function ω such that $\omega^n(z) = -c_n z^n$. Then $\omega(z) = \sqrt[n]{-c_n}z$. It is clear that ω is analytic in $\mathbb{D}$. This implies that

$$p(z) = \left(1 + \frac{1}{\sqrt{2}}\omega^n(z)\right)^2.$$

Now by using Lemma 5.2, there exists a point $z_0 \in \mathbb{D}$ such that $|\omega(z)| < |z_0|$ for $|z| < |z_0|$ and

$$|\omega(z_0)| = |z_0|, \quad \omega(z_0) = e^{i\theta}, \quad 0 \le \theta < 2\pi. \tag{9}$$

Then

$$\frac{z_0\omega'(z_0)}{\omega(z_0)} = k \ge 1. \tag{10}$$

From the above relations, it is clear that $\omega(z_0) \neq \pm 1$. After simple calculations, we obtain

$$\frac{zp'(z)}{p(z)} = \frac{2nz\omega'(z)\omega^{n-1}(z)}{\sqrt{2} + \omega^n(z)}. \tag{11}$$

Using (9) and (10) in (11), we obtain

$$\left|\frac{z_0p'(z_0)}{p(z_0)}\right| = \left|\frac{2nz_0'\omega(z_0)\omega^{n-1}(z_0)}{\sqrt{2}+\omega^n(z_0)}\right| \ge \left|\frac{2nk\omega^n(z_0)}{\sqrt{2}+\omega^n(z_0)}\right| \ge \frac{2n}{\sqrt{2}+1}.$$

This is a contradiction. Hence result is proved. □

Corollary 5.3. *Let $zf'(z)/f(z) = 1 + c_1z + \cdots$ such that $zf'(z)/f(z) \neq -1$ and $zf'(z)/f(z) \neq 0$ be analytic in $\mathbb{D}$. Also suppose that*

$$\left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| < \frac{2}{\sqrt{2}+1}.$$

Then $f \in \mathcal{S}^*_{lim}$.

6. Conclusion

In this paper, we have investigated a subclass of starlike functions f such that $\frac{zf'(z)}{f(z)}$ lies in the interior of the region bounded by the limacon which is given by the equation $\left[(u-1)^2 + v^2 - \frac{1}{4}\right]^2 - 2\left[\left(u - 1 + \frac{1}{2}\right)^2 + v^2\right] = 0$. We have studied certain radii problems, inclusion results, coefficient bounds and sufficient conditions for this class of functions.

Basic (or q-) calculus plays an important role in geometric function theory. Recently, by making use of the concept of basic (or q-) calculus, various families of q-extensions of starlike functions were introduced (see, for example [7, 9, 28, 30, 32]).

In view of the concept of basic (or q-) calculus and limacon domain studied in this paper, q-analogue of the class $\mathcal{S}^*_{lim}$ can be defined and studied. Furthermore, various classes of analytic functions can be studied by using basic (or q-) calculus and limacon domain.

At the same time, it is worth to mention that the current trend of trivially and inconsequentially translating q-results to the corresponding (p, q) results leads to no more than a straightforward and shallow publication (see [24, pp. 340], see also [25, pp. 1511-1512]).

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