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On generalized implicit equilibrium problems

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Abstract. This paper is devoted to the investigation of the generalized implicit equilibrium problems with weak conditions in general space. Sufficient conditions for the set of solutions to be compact and convex are given. Our results improve some recent results in this field.

1. Introduction

Given a nonempty set K and a scalar bifunction $f : K \times K \to \mathbb{R} = (-\infty, \infty)$ such that $f(x, x) \geq 0$ for all $x \in K$, the scalar equilibrium problem (EP, for short) for f is to find $z \in K$ such that $f(z, y) \geq 0$ for all $y \in K$. It is well known that (EP) is closely related to games theory, mechanics and physics, economics and finance, operations research, variational inequality and complementarity problems, as well as optimization and control problems(see, for more details , Takahashi[19] , Blum and Oettli[4], and Noor and Oettli[15] and the references therein. At the same time, this problem has been generalized to the vector case (see [2, 4, 10, 11]) as follows. Let X and Y be Hausdorff topological vector spaces (tvs, for short), K a nonempty, closed, and convex subset of X, and C a pointed, closed, and convex cone in Y with $intC \neq \emptyset$. Given a vector valued bifunction $f : K \times K \to Y$, the vector equilibrium problem (VEP) for f consists in finding $x \in K$ such that

$$f(x, y) \notin -intC, \quad \forall y \in K.$$

It is well known that vector equilibrium problems provide a unified model for several classes of problems, for example, vector variational inequality problems, vector complementarity problems, vector optimization problems, and vector saddle point problems; see, for more details, [4]-[3],[8]-[21] and the references therein.

The implicit vector equilibrium problem (IVEP), which is a generalization of (EP), (VEP), and implicit variational inequality and complementarity problems, was introduced by Huang et al. [13] as follows:

Given a vector valued bifunction $f : K \times K \to Y$ and $g : K \to K$, find $x \in K$ such that

$$f(g(x), y) \notin -intC, \quad \forall y \in K.$$

If $T : K \to L(X, Y), \theta : K \times K \to X$, and $g : K \to K$, then (IVEP) reduces to the generalized vector variational inequality (GWI) of finding $x \in K$ such that

$$\langle T(g(x)), \theta(y, g(x))\rangle \notin -intC(x), \quad \forall y \in K,$$

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where $L(X,Y)$ denotes the space of all continuous linear operators from X to Y, $\langle T(z), y\rangle$ denotes the evaluation of the linear operator $T(z)$ at y.

The generalized vector equilibrium problem was first introduced in 1997 [1] as follows.

Let K a nonempty, closed, and convex subset of (tvs) X, and C a closed, and convex cone in Y with $intC \neq \emptyset$. Let $F : K \times K \to 2^Y$ be a set-valued mapping. The generalized vector equilibrium problem (GVEP) for F consists in finding $x \in K$ such that

$$F(x,y) \nsubseteq -intC, \quad \forall y \in K.$$

The authors of [16] considered the generalized implicit operator equilibrium problem (GIOEP) which consists of finding $f^* \in K$ such that

$$F(h(f^*), g) \nsubseteq -intC(f^*), \forall g \in K,$$

where $F : K \times K \longrightarrow 2^Y$ is a set-valued mapping, $h : K \longrightarrow K$ is a mapping, X and Y are two Hausdorff topological vector spaces, $L(X,Y)$ is the space of all continuous linear operators from X to Y, $K \subseteq L(X,Y)$ is a non-empty convex set, $C : K \longrightarrow 2^Y$ is a set-valued mapping such that for each $f \in K$, $C(f)$ is a closed and convex cone in Y with $intC(f) \neq \emptyset$($intC(f)$ is the interior of $C(f)$), 2^Y denotes the set of all non-empty subsets of Y. This paper is motivated and inspired by the recent paper [16] and its aim is to extend the results given in [16] to the setting of Hausdorff topological vector spaces with mild assumptions and relaxing some conditions.

In the rest of this section we recall some definitions and results that we need in the next section.

A subset C of Y is called a pointed and convex cone if and only if $C + C \subseteq C, tC \subseteq C, \forall t \geq 0$, and $C \cap -C = \{0_Y\}$ (see, for instance, [1, 4, 6–8]) The domain of a set-valued mapping $W : X \longrightarrow 2^Y$is defined as

$$D(W) = \{x \in X : W(x) \neq \emptyset\}$$

and its graph is defined as

$$Gr(W) = \{(x,z) \in X \times Y : z \in W(x)\}.$$

Also W is said to be closed if its graph, that is, $Gr(W)$, is a closed subset of $X \times Y$.
A set-valued mapping $T : X \longrightarrow 2^Y$ is called upper semicontinuous (in short *u.s.c.*) at $x_0 \in X$ if for every open set $V \subseteq Y$ containing $T(x_0)$ there exists an open set $U \subseteq X$ containing x_0 such that $T(u) \subseteq V$, for all $u \in U$. The mapping T is said to be lower semicontinuous (in short *l.s.c.*) at $x_0 \in X$ if for every open set $V \subseteq Y$ with $T(x_0) \cap V \neq \emptyset$ there exists an open set $U \subseteq X$ containing x_0 such that $T(u) \cap V \neq \emptyset, \forall u \in U$. The mapping T is continuous at x_0 if it is both *u.s.c.* and *l.s.c.* at x_0. Moreover, T is *u.s.c.* (*l.s.c.*) on X if T is *u.s.c.*(*l.s.c.*) at each point of X. We need the following basic definitions and results in the sequel.

Lemma 1.1. *([20]) Let X and Y be two Hausdorff topological spaces and $T : X \longrightarrow 2^Y$ be a mapping. The following statements are true:*
(i) For any given $x_0 \in X$ if T has compact value at x_0 (i.e., $T(x_0)$ is a compact), then T is u.s.c. at $x_0 \in X$ if and only if for any net $\{x_\alpha\} \subseteq X$ with $x_\alpha \longrightarrow x_0$ and for every $y_\alpha \in T(x_\alpha)$, there exist $y_0 \in T(x_0)$ and a subnet $\{y_{\alpha_\beta}\} \subseteq \{y_\alpha\}$ such that $y_{\alpha_\beta} \longrightarrow y_0$;
(ii) T is l.s.c. at $x_0 \in X$ if and only if for any net $\{x_\alpha\} \subseteq X$ with $x_\alpha \longrightarrow x_0$ and for any $y_0 \in T(x_0)$, there exists $y_\alpha \in T(x_\alpha)$ such that $y_\alpha \longrightarrow y_0$.

Definition 1.2. *[21] Let K be a non-empty subset of topological vector space X. A set-valued mapping $T : K \to 2^X$ is called a KKM mapping if for every finite subset $\{x_1, x_2, ..., x_n\}$ of K, $Co\{x_1, x_2, ..., x_n\}$ is contained in $\bigcup_{i=1}^n T(x_i)$, where Co denotes the convex hull.*

The KKM-mappings were first considered by Knaster, Kuratowski and Mazurkiewicz (KKM) ([21]) in 1920, in order to guarantee the finite intersection property for values of the mapping .

Lemma 1.3. *([5]) Let K be a nonempty subset of a topological vector space X and $F : K \longrightarrow 2^X$ be a KKM-mapping with closed values in K. Assume that there exists a nonempty compact convex subset B of K such that $\cap_{x\in B}F(x)$ is compact. Then $\cap_{x\in K}F(x) \neq \emptyset$.*

Remark that if $F : K \longrightarrow 2^X$ is a KKM-mapping with closed values in K, then the family $\{G(x) : x \in X\}$ of sets has the finite intersection property.

The following results are important in the next section.

Lemma 1.4. *Let $f : X \longrightarrow 2^Y$ be a nonempty set-valued mapping. If f is u.s.c. with closed values then it is closed (that is, its graph is closed).*

Lemma 1.5. *Sum of two upper semicontinuous functions is upper semicontinuous.*

Proposition 1.6. *If F is u.s.c. at x_0, then $-F$ is u.s.c. at x_0.*

Proof. Let W_0 be an open set. Then there exists V_{x_0} such that

$$F(y) \subseteq W_0, \forall y \in V_{x_0}.$$

If W' is an open set and $-F(x_0) \subseteq W'$, then $-F(x_0) \subseteq W'$. Hence $F(x_0) \subseteq -W'$, thus there exists $V'_{x_0}, s.t.F(y) \subseteq -W', \forall y \in V'_{x_0}$, so $-F(y) \subseteq W', \forall y \in V'_{x_0}$. This completes the proof. □

Definition 1.7. *Let $f : K \times K \longrightarrow Y$ be a vector valued bifunction and $g : K \longrightarrow K$.*

(i) $f(x, y)$ is a $(\pm Q)-$ function with respect to y if, for any given $x \in K$

$$f(x, ty_1 + (1 - t)y_2) \in tf(x, y_1) + (1 - t)f(x, y_2) \pm Q;$$

for all $y_1, y_2 \in K$ and $t \in [0, 1]$, where Q is a closed and convex cone of Y such that $intQ \neq \emptyset$;

(ii) g is a affine mapping if, for any $y_1, y_2 \in K$ and $t \in R$,

$$g(ty_1 + (1 - t)y_2) = tg(y_1) + (1 - t)g(y_2).$$

.

2. Main results

The results of this section theorem can be viewed as an extension, improvement and repairmen of the Theorem 3.1 given in [16] by relaxing or weakening some assumptions and it is implicit version of Corollary 2 in [1] from locally convex spaces to topological vector spaces and relaxing conditions (iv)-(vi) of it. Moreover, it is set-valued version of Theorem 3.1 and Theorem 3.2 in [18] with mild assumptions.

Theorem 2.1. *Let K be a non-empty convex subset of X and $h : K \longrightarrow K$ be a mapping and $F : K \times K \longrightarrow 2^Y$ be a set-valued mapping. Suppose that the following assumptions hold:*

(a) The set-valued mapping $x \longrightarrow F(g(x), y)$ is u.s.c. with compact values, for all $y \in K$;

(b) The mapping $x \longrightarrow Y \setminus -intC(x)$ is u.s.c.;

(c) There exists a set- valued mapping $G : K \times K \longrightarrow 2^Y$ such that;

(i) $G(h(x), x) \nsubseteq -intC(x), \forall x \in K$;

(ii) $G(h(x), y) - F(h(x), y) \subseteq -intC(x), \forall x, y \in K$;

(iii) $\{y \in K : G(x, y) \subseteq -intC(x)\}$ *is convex*, $\forall x \in K$.

Then the solution set of GIVEP is nonempty, i.e. there exists $x^* \in K$ *such that* $F(h(x^*), y) \nsubseteq -intC(x^*), \forall y \in K$. *Moreover, the solution set is compact if the following condition is satisfied:*

(d) There exist a nonempty, compact and convex subset B of K, such that for each $x \in K \setminus B$, *there exists* $y \in B$ *such that*

$$F(h(x), y) \subseteq -intC(f).$$

Proof. *Let D be an arbitrary compact and convex subset of K.*
Define $T : D \longrightarrow 2^D$ *by*

$$T(y) = \{x \in D : F(h(x), y) \nsubseteq -intC(x)\}, \forall y \in D.$$

We show that T satisfies all the assumptions of Lemma1.3. We first prove that $T(y)$ *is closed, for all* $y \in K$. *For this, let* $\{x_\alpha\}$ *be a net in* $T(y)$ *such that* $x_\alpha \longrightarrow x^*$. *Define the mapping* $H_y : D \longrightarrow 2^Y$ *by*

$$H_y(x) = F(h(x), y).$$

It follows from $x_\alpha \in T(y)$ *that* $H_y(x_\alpha) \nsubseteq -intC(x_\alpha)$. *Hence, for each* α,

$$\exists z_\alpha \in H_y(x_\alpha) \quad s.t. \quad z_\alpha \in Y \setminus -intC(x_\alpha) = W(x_\alpha),$$

by (a) there exist $z \in H_y(x^*)$ *and a subnet* $\{z_{\alpha_\beta}\}$ *such that* $z_{\alpha_\beta} \longrightarrow z$. *Also* $(x_{\alpha_\beta}, z_{\alpha_\beta}) \longrightarrow (x^*, z)$ *and* $(x_{\alpha_\beta}, z_{\alpha_\beta}) \in G_r H_y$. *By Lemma1.4,*

$$(x^*, z) \in G_r H_y \text{ and } z \in F(h(x^*, y)) \qquad (1).$$

On the other hand $z_{\alpha_\beta} \in w(x_{\alpha_\beta})$ *and* $(x_{\alpha_\beta}, z_{\alpha_\beta}) \in G_r W$. *Since* $W(x_{\alpha_\beta})$ *is closed, by (b) and Lemma1.4, we conclude that* $(x^*, z) \in G_r W$, *thus*

$$z \in W(x^*) = Y \setminus -intC(x^*). \qquad (2)$$

From (1), (2) we have

$$F(h(x^*, y)) \nsubseteq -intC(x^*) \Longrightarrow x^* \in T(y).$$

Hence $T(y)$ *is closed, for all* $y \in K$.
Now we prove that the mapping $y \longrightarrow T(y)$ *is a KKM− mapping. Suppose to the contrary there exists a finite subset* $\{y_1, y_2, ..., y_n\}$ *of D such that*

$$Co\{y_1, y_2, ..., y_n\} \nsubseteq \cup_{i=1}^n T(y_i).$$

Hence there exists $z \in Co\{y_1, y_2, ..., y_n\}$ *such that*

$$z = \sum_{i=1}^n \lambda_i y_i, \sum_{i=1}^n \lambda_i = 1, \lambda_i \geq 0, z \notin T(y_i), \forall i = 1, 2, ..., n.$$

Therefore

$$F(h(z), y_i) \subseteq -intC(z) \qquad (3),$$

therefore by assumption c(ii), we get

$$G(h(z), y_i) - F(h(z), y_i) \subseteq -intC(z) \qquad (4).$$

Moreover from (3) and (4), for each $i = 1, 2, ..., n$, *we have*

$$\begin{aligned} G(h(z), y_i) &\subseteq G(h(z), y_i) + 0 \\ &\subseteq G(h(z), y_i) + F(h(z), y_i) - F(h(z), y_i) \\ &\subseteq -intC(z) \end{aligned}$$

(Note that $-intC(z) + -intC(z) \subseteq -intC(z)$.) Therefore, we conclude that

$$y_i \in \{y \in K : G(h(z), y_i) \subseteq -intc(z)\} \quad \forall i = 1, 2, ..., n.$$

By assumption c(iii), we get

$$G(h(z), z = \sum_{i=1}^{n} \lambda_i y_i) \subseteq -intC(z),$$

which contradicts c(i). Hence T is a $KKM-$mapping.
Since D is compact and $T(y)$ is a closed subset of D, and T is a $KKM-$mapping. Hence by Lemma1.3, we have

$$\cap_{y\in D} T(y) \neq \emptyset.$$

Now, we show that

$$\cap_{y\in K} T(y) \neq \emptyset.$$

Otherwise

$$\cap_{y\in K} T(y) = (\cap_{y\in D} T(y)) \cap (\cap_{y\in K\setminus D} T(y)) = \emptyset.$$

Thus $\cap_{y\in D} T(y) \subseteq \cup_{y\in K\setminus D}(T(y))^c$. Also, it is obvious that $\cap_{y\in D} T(y) \subseteq D$ nd so $\cap_{y\in D} T(y)$ is compact.
(Note that $T(y)$ is closed for each $y \in D$ and D is compact).
Hence there exist $y_1^{'}, y_2^{'}, ..., y_1^{'} \in K \setminus D$ such hat

$$\cap_{y\in D} T(y) \subseteq \cup_{i=1}^{n}(T(y_i^{'}))^c,$$

which gives that

$$(\cap_{y\in D} T(y)) \cap (\cap_{i=1}^{n}(T(y_i^{'}))) = \cap_{y\in D\cup\{y_1^{'}, y_2^{'}, ..., y_n^{'}\}} T(y) = \emptyset.$$

Thus

$$\cap_{y\in Co(D\cup\{y_1^{'}, y_2^{'}, ..., y_n^{'}\})} T(y) \subseteq \cap_{y\in D\cup\{y_1^{'}, y_2^{'}, ..., y_n^{'}\}} T(y). \quad (5)$$

Now, if we consider $B = Co(D \cup \{y_1^{'}, y_2^{'}, ..., y_n^{'}\})$, B is compact and convex and the mapping $T : B \longrightarrow 2^B$ is a $KKM-$mapping. Hence by Lemma1.3, $\cap_{y\in B} T(y) \neq \emptyset$.
By (5), we get

$$\emptyset \neq \cap_{y\in B} T(y) \subseteq \cap_{y\in D\cup\{y_1^{'}, y_2^{'}, ..., y_1^{'}\}} T(y) = \emptyset,$$

which is a contraction. Hence $\cap_{y\in K} T(y) \neq \emptyset$.
Thus there exists $x^* \in K$ such that

$$F(h(x^*), y) \nsubseteq -intC(x^*), \forall y \in K.$$

Now, we show that the solution set of GIVEP, which is equals to the set $\cap_{y\in k} T(y)$, is compact.
If the condition (d) is satisfied, to see this, we show that $\cap_{y\in k} T(y) \subseteq B$.
Otherwise, there exists $x_0 \in \cap_{y\in k} T(y)$ such that $x_0 \in K\setminus B$. By condition (d) $\exists y_0 \in B$ such that $F(h(x_0), y_0) \subseteq -intc(x_0)$.
Thus $x_0 \notin T(y_0)$, which is a contradiction. Thus $\cap_{y\in K} T(y)$ is compact, This completes the proof. □

Theorem 2.1 improves conditions (2) and (3) of the following theorem (Theorem 4.1 in [13]) and moreover it is set-valued version of it.

Theorem 2.2. *(Theorem 4.1 in [13])) Let K be a nonempty, closed, convex subset of X, $g : K \to K$ and $f : K\times K \to Y$ be a vector valued bifunction. Suppose that the following assumptions hold:*

1. *$f(g(x), x) \notin -intC(x) and C(g(x)) \subset C(x), \forall x \in K$;*
2. *$x \to f(x, y)$ and g are continuous $\forall x \in K$;*
3. *$f(x, y)$ is $C-$ convex with respect to y for all $x \in K$;*
4. *$W : K \to 2^Y$ is a point-to-set mapping such that $W(z) = Y\setminus(-intC(z))$, for all $z \in K$, has closed graph in $K \times Y$;*

5. *there exists a nonempty, compact, convex subset D of K, such that for all $x \in K\backslash D$, $\exists y$ D such that $f(g(x), y)$- int$C(x)$.*

Then there exists $x \in K$ such that $f(g(x), y) \notin -intC(x)$, for all $y \in K$.

Remark 2.3. 1. *One can show that if $G(x, y)$ is $Q-$ functions with respect to y, then condition (iii) in Theorem 2.1 holds.*

2. *One can consider Theorem as an improvement of Theorem 3.1 in [16] by relaxing condition (d) and continuity of mapping g and continuity of mapping $f(., y)$ with respect to x.*
   Moreover in the paper we replace $L(X, Y)$ by a general topological vector space.
3. *The proof of Theorem 3.1 in [16] contains some gaps, for instance:*
   Line 5 of the proof of it, how can the authors, deduce

$$F(h(f_\alpha), g) \subseteq W(f_\alpha) = Y \setminus -intC(f_\alpha),$$

   from $F(h(f_\alpha), g) \nsubseteq -intCf(\alpha)$.
   Also, one can ask, what is the meaning of $F(h(f_\alpha), g) \longrightarrow F(h(f), g)$, where $F(h(f_\alpha), g)$ and $F(h(f), g)$ are sets? furthermore, how can the authors in line 9 from u.s.c. of W concluded $F(h(f), g) \subseteq w(f)$
4. *We note that if h is continuous and f is continuous with respect to the first variable then the mapping $x \longrightarrow f(h(x), y)$ is u.s.c. and so when K is compact condition (a) holds while the following simple exmple, which shows that the convers does not hold in general, proves that neither h nor F is continuous but the mapping $x \longrightarrow f(h(x), y)$is u.s.c., moreover, in the example, if we take $K = [0, 1]$, then F, g and h are satisfy all assumtions of Theorem 3.4 and so the solution set of IVEP is nonempty and compact. But the example can not fulfill all the conditions of Theorem 3.1 in [Filomat 2019]. Hence Theorem 3.1 extends Theorem 3.1 in [Filomat 2019].*

$$h(x) = \begin{cases} -1, & if x \in Q \cap [-1, 1], \\ 0, & if x \in Q^c \cap [-1, 1], \end{cases}$$

$$F(x, y) = \begin{cases} \{-1\}, & if x \in Q \cap [-1, 1], \\ \{0\}, & if x \in Q^c \cap [-1, 1], \end{cases}$$

Theorem 2.4. *Let K be a non-empty convex subset of X, $h : K \longrightarrow K$ be a mapping and $F_i : K \times K \longrightarrow 2^Y$ be two set-valued mappings for $i = 1, 2$. Let $Q := \cap_{x \in K}(-C(x))$, such that int$Q \neq \emptyset$. Suppose that the following assumptions hold:*

(a) The set-valued mappings $y \longrightarrow F_2(h(x), y)$ and $x \longrightarrow F_1(y, h(x))$ are u.s.c. with compact and open values respectively, for all $x, y \in K$;

(b) The mapping $x \longrightarrow Y \setminus -intC(x)$ is u.s.c.;

(c) $F_1(x, y)$ is $(-Q)-$ function with respect to x and $F_2(x, y)$ is $(+Q)-$ function with respect to y;

(d) $0_Y \in F_2(h(x), y) - F_1(x, h(x)), \forall x \in K$;

(e) $F_1(h(x), y) + F_1(h(x), y) \subseteq C(x), \forall x \in K$;

Then there exists $x^* \in K$ such that $F_1(h(x^*), y) + F_2(h(x^*), y) \nsubseteq -intC(x^*), \forall y \in K$.

Proof. *Let D be an arbitrary compact and convex subset of K.*
Define $T : D \longrightarrow 2^D$ by

$$T(y) = \{x \in D : F_2(h(x), y) - F_1(y, h(x)) \nsubseteq -intC(x)\}, \forall y \in D.$$

We show that T satisfies all the assumptions of Lemma1.3. We first prove that $T(y)$ is closed, for all $y \in K$. For this, let y be fixed and arbitrary and $\{x_\alpha\}$ be a net in $T(y)$ such that $x_\alpha \longrightarrow x^*$. Define the mapping $H_y : D \longrightarrow 2^Y$ by

$$H_y(x) = F_2(h(x), y) - F_1(y, h(x)).$$

It follows from $x_\alpha \in T(y)$ that $H_y(x_\alpha) \nsubseteq -intC(x_\alpha)$. Hence, for each α,

$$\exists z_\alpha \in H_y(x_\alpha) \quad s.t. \quad z_\alpha \in Y \setminus -intC(x_\alpha) =: W(x_\alpha),$$

It follows from Lemma1.5 and Proposition1.6, H_y is u.s.c. with compact values. Hence there exists subnet $\{z_{\alpha_\beta}\}$ and $z \in H_y(x^*)$ such that $z_{\alpha_\beta} \longrightarrow z$. Also $(x_{\alpha_\beta}, z_{\alpha_\beta}) \longrightarrow (x^*, z)$ and $(x_{\alpha_\beta}, z_{\alpha_\beta}) \in G_rW$. Lemma1.4 implies that $(x^*, z) \in G_rW$. That is $z \in W(x^*)$.
Thus $H_y(x^*) \nsubseteq -intC(x^*)$. Hence $T(y)$ is closed, for all $y \in K$.
Now we prove that the mapping $y \longrightarrow T(y)$ is a KKM− mapping.
Suppose to the contrary there exists a finite subset $\{z_1, z_2, ..., z_n\}$ of D such that

$$Co\{z_1, z_2, ..., z_n\} \nsubseteq \cup_{i=1}^n T(z_i).$$

Hence there exists $z \in Co\{z_1, z_2, ..., z_n\}$ such that

$$z = \sum_{i=1}^n \lambda_i z_i, \sum_{i=1}^n \lambda_i = 1, \lambda_i \geq 0, z \notin T(z_i), \forall i = 1, 2, ..., n.$$

Therefore

$$H_{z_i}(z) \subseteq -intC(z) \quad \forall i = 1, 2, ..., n,$$

By assumptions (c) and (d), it holds that

$$\begin{aligned} 0_Y &\in F_2(h(z), z) - F_1(z, h(z)) \\ &\subseteq \sum_{i=1}^n \lambda_i F_2(h(z), z_i) + Q - \sum_{i=1}^n \lambda_i F_1(z_i, h(z)) + Q \\ &= \sum_{i=1}^n \lambda_i (F_2(h(z), z_i) - F_1(z_i, h(z))) + Q + Q. \end{aligned}$$

By the assumption a, we obtain

$$\{0_Y\} \subseteq -intC(z) - C(z) - C(z) \subseteq -intC(z).$$

Hence $0_Y \in -intC(z)$, which contradicts $C(z) \cap -C(z) = \{0_Y\}$. Hence T is a KKM− mapping.
Since D is compact and $T(y)$ is a closed subset of D, and T is a KKM−mapping, by Lemma1.3, we have

$$\cap_{y \in D} T(y) \neq \emptyset.$$

Now, we show that

$$\cap_{y \in K} T(y) \neq \emptyset.$$

Otherwise

$$\cap_{y \in K} T(y) = (\cap_{y \in D} T(y)) \cap (\cap_{y \in K \setminus D} T(y)) = \emptyset.$$

Thus $\cap_{y \in D} T(y) \subseteq \cup_{y \in K \setminus D} (T(y))^c$. Also, it is obvious that $\cap_{y \in D} T(y) \subseteq D$ nd so $\cap_{y \in D} T(y)$ is compact.
(Note that $T(y)$ is closed for each $y \in D$ and D is compact).
Hence there exist $y'_1, y'_2, ..., y'_n \in K \setminus D$ such hat

$$\cap_{y \in D} T(y) \subseteq \cup_{i=1}^n (T(y'_i))^c.$$

Which gives that

$$(\cap_{y\in D}T(y)) \cap (\cap_{i=1}^{n}(T(y_i^{'}))) = \cap_{y\in D\cup\{y_1^{'},y_2^{'},...,y_n^{'}\}}T(y) = \emptyset.$$

Thus, we deduce that

$$\cap_{y\in Co(D\cup\{y_1^{'},y_2^{'},...,y_n^{'}\})}T(y) \subseteq \cap_{y\in D\cup\{y_1^{'},y_2^{'},...,y_n^{'}\}}T(y). \quad (5)$$

Now, if we consider $B = Co(D \cup \{y_1^{'}, y_2^{'}, ..., y_n^{'}\})$, *B is compact and convex and the mapping* $T : B \longrightarrow 2^B$ *is a KKM−mapping. Hence by Lemma1.3,* $\cap_{y\in B}T(y) \neq \emptyset$.
By (5), we find that

$$\emptyset \neq \cap_{y\in B}T(y) \subseteq \cap_{y\in D\cup\{y_1^{'},y_2^{'},...,y_1^{'}\}}T(y) = \emptyset,$$

which is a contraction. Hence $\cap_{y\in K}T(y) \neq \emptyset$.
Thus there exists $x^* \in K$ *such that*

$$F_2(h(x^*), y) - F_1(y, h(x^*)) \nsubseteq -intC(x^*), \forall y \in D. \quad (2)$$

By the assumption (e) and (2), we obtain

$$F_1(h(x^*), y) + F_2h(x^*), y) \nsubseteq -intC(x^*),$$

This completes the proof. □

Remark 2.5. *It seems Definition 2.4 of hemicontinuity given in [16] cannot be meaningful and the conclusion in page 3.3 line 12 is vague.*

The following theorem is a generalization of Theorem 3 in [1] from locally convex spaces to topological spaces and moreover reducing the domain of the set-valued mapping F from compact convex to convex and deleting conditions (ii), (iv)-(vii) of Theorem 3 in [1]. Further, it is implicit version of it.

Theorem 2.6. *Let K be a nonempty convex subset of Hausdorff topological vector space X and* $C : K \to 2^Y\backslash\emptyset$, *where Y is a topological vector space. The set-valued mapping* $F : K \times K \to 2^Y$, *and single-valued mapping* $g : K \to K$ *satisfying in the following conditions.*

1. $F(g(x), x) \cap C(g(x)) \neq \emptyset, \forall x \in K$,
2. $\{y \in K : F(x, y) \cap C(x) = \emptyset\}$ *is convex,* $\forall x \in K$,
3. $\{x \in K : F(g(x), y) \cap C(g(x)) \neq \emptyset\}$ *is closed,* $\forall x \in K$,
4. *there exist compact convex set D and compact set M of K such that*

$$\forall x \in K\backslash M, \exists y \in D, F(x, y) \cap C(x) = \emptyset.$$

Then there exists $x \in K$ *such that the set*

$$\{x \in K : F(x, y) \cap C(x) \neq \emptyset, \forall y \in K\},$$

is nonempty and compact.

Proof. Assume that H is an arbitrary convex subset of K. Define $G : H \to 2^K$ by

$$G(x) = \{x \in K : F(x, y) \cap C(x) \neq \emptyset\}, \forall x \in K.$$

We prove that the mapping G is a $KKM-$ mapping. Suppose to the contrary there exists a finite subset $\{y_1, y_2, ..., y_n\}$ of H such that

$$Co\{y_1, y_2, ..., y_n\} \nsubseteq \cup_{i=1}^{n}G(y_i).$$

Hence there exists $z \in Co\{y_1, y_2, ..., y_n\}$ such that

$$z = \sum_{i=1}^{n} \lambda_i y_i, \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0, z \notin G(y_i), \forall i = 1, 2, ..., n.$$

Therefore

$$F(g(z), y_i) \cap C(g(z)) = \emptyset.$$

Thus by assumption (2),we get $F(g(z), z) \cap C(g(z)) = \emptyset$, which is contracted by (1). Hence G is a KKM mapping and so the family $\{G(x)\}_{x \in H}$ has the finite intersection property. It follows from condition (4) that $\bigcap_{x \in D} G(x)$ is a closed subset of the compact set M. Consequently, $\bigcap_{x \in D} G(x) \neq \emptyset$. Now we claim that

$$\bigcap_{x \in K} G(x) \neq \emptyset.$$

Otherwise

$$\bigcap_{x \in K} G(x) = (\bigcap_{x \in D} G(x)) \bigcap (\bigcap_{x \in K \setminus D} G(x)) = \emptyset.$$

Hence $\bigcap_{x \in D} G(x) \subseteq \bigcup_{x \in K \setminus D} G^c(x)$ and since $\bigcap_{x \in D} G(x)$ is compact then there exist $x_1, ..., x_n$ of $K \setminus D$ such that $\bigcap_{x \in D} G(x) \subseteq \bigcup_{i=1}^{n} G^c(x_i)$. This mean that $\bigcap_{x \in D \cup \{x_1, x_2, ..., x_n\}} G(x) = \emptyset$. Hence

$$\bigcap_{x \in H = Co(D \cup \{x_1, x_2, ..., x_n\})} G(x) \subseteq \bigcap_{x \in D \cup \{x_1, x_2, ..., x_n\}} G(x) = \emptyset$$

which is contradiction with being KKM of G on $H = Co(D \cup \{x_1, x_2, ..., x_n\})$. Hence there exists $z \in K$ such $z \in \bigcap_{x \in K} G(x) = (\bigcap_{x \in D} G(x))$, and so

$$F(z, y) \cap C(z) \neq \emptyset, \forall y \in K.$$

The compactness of $\{z \in K : F(z, y) \cap C(z) \neq \emptyset, \forall y \in K\}$ directly follows from (4). This completes the proof. □

Remark 2.7. *It is easy to check that the result of Theorem 2.6 is still valid if one replaces the closedness of the set* $\{x \in K : F(g(x), y) \cap C(g(x)) \neq \emptyset\}$, *in condition (3) by the transfer closedness (that is, if* $z \notin G(y) = \{x \in K : F(g(x), y) \cap C(g(x)) \neq \emptyset\}$ *then there exists* $w \in K$ *such that* $z \notin \overline{G(w)}$, *the closure of* $G(w)$.

The next result is a direct consequence of Theorem 2.6 which is an improvement version of Corollary 2 in [1].

Corollary 2.8. *Let K be a nonempty convex subset of Hausdorff topological vector space X and P is a nonempty subset of Y. The vector valued mappings $F : K \times K \to Y$ and $g : K \to K$ satisfying the following conditions.*

1. $F(g(x), x) \in P$, $\forall x \in K$,
2. $\{y \in K : F(x, y) \notin P\}$ *is convex,* $\forall x \in K$,
3. $\{x \in K : F(g(x), y) \in P\}$ *is closed,* $\forall x \in K$,
4. *there exist compact convex set D and compact set M of K such that*

$$\forall x \in K \setminus M, \exists y \in D, F(x, y) \notin P.$$

Then there exists $x \in K$ such that the set

$$\{x \in K : F(x, y) \in P, \forall y \in K\},$$

is nonempty and compact.

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