Filomat 37:3 (2023), 957–969
https://doi.org/10.2298/FIL2303957B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Common solutions for a monotone variational inequality problem and an infinite family of inverse strongly monotone non-self operators

Nguyen Buong[a]

[a] *Institute of Theoretical and Applied Research, Hanoi, 100000, Faculty of Information Technology, Duy Tan University, Da Nang, 550000, Vietnam. Vietnam Academy of Science and Technology, Institute of Information Technology, 18, Hoang Quoc Viet, Hanoi, Vietnam*

Abstract. In this paper, we introduce regularization methods for finding a point, being not only a solution for a monotone variational inequality problem but also a common zero for an infinite family of inverse strongly monotone non-self operators of a closed convex subset in a real Hilbert space. In these methods, only a finite number of the operators is used at each iteration step. Applications to the problem of common fixed point for an infinite family of strictly pseudo-contractive non-self operators and the split feasibility and fixed point problems are considered. As a particular case, a regularization extragradient iterative method without prior knowledge of operator norms for solving the split feasibility problem (SFP) is obtained. Numerical examples are given for illustration.

1. Introduction and preliminaries

Let H be a real Hilbert space with an inner product and a norm denoted by the symbols $\langle ., . \rangle$ and $\|.\|$, respectively, and let K be a closed convex subset in H. We denote the metric projection of H onto K by P_K. An operator A of K into H is called monotone if $\langle Ax - Ay, x - y \rangle \geq 0$ for all $x, y \in K$. If $\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2$ for some positive real number α, then it is called an α-strongly monotone non-self operator. If $\langle Ax - Ay, x - y \rangle \geq \lambda \|Ax - Ay\|^2$, then it is said to be a λ-inverse strongly monotone non-self operator.

The variational inequality problem is to find $p \in K$ such that

$$\langle Ap, p - x \rangle \leq 0, \ \forall x \in K. \tag{1}$$

The set of solutions of the variational inequality problem is denoted by $VI(K, A)$.

Let $\{A_i\}_{i=1}^{\infty}$ be an infinite family of λ_i-inverse strongly monotone non-self operators of K such that $\inf_{i \geq 1} \lambda_i > 0$.

The problem considered in this paper is to find a point

$$p_* \in \Gamma := VI(K, A_0) \cap S, \tag{2}$$

2020 *Mathematics Subject Classification*. Primary 47H17; Secondary 47H09, 49J30.

Keywords. Browder-Tikhonov regularization; fixed points; extragradient method.

Received: 17 March 2021; Accepted: 31 October 2021

Communicated by Adrian Petrusel

This research is supported by Vietnam National Foundation for Science and Technology Development (NAFOSTED) (Grant N.101.02.2021.11).

Email address: nguyenbuong@dtu.edu.vn, nbuong@ioit.ac.vn (Nguyen Buong)

assumed to be non-empty, where A_0 is a monotone, non-self and L_0-Lipschitz continuous operator of K, i.e., $\|A_0x - A_0y\| \le L_0\|x - y\|$ for all $x, y \in K$ and L_0 is a positive constant, $S = \cap_{i=1}^{\infty} S_i$ and $S_i = \{x \in K : A_i(x) = 0\}$. Problem (2) and its particular cases were investigated in [1],[2],[7],[8],[15]-[17],[19],[21]-[33],[35]-[40] and references therein.

For solving (1) with $VI(K, A) \neq \emptyset$, when A is a strongly monotone and L-Lipschitz continuous operator, Polyak [28] introduced a projection-gradient iterative method,

$$x^{k+1} = P_K(x^k - \beta A x^k),\ x^1 \in K$$

and $k \ge 1$. He proved global convergence of the method under condition $\beta \in (0, 1/L)$. When the operator A of K is L-Lipschitz-continuous monotone in the finite-dimensional Euclidean space $\mathbb{E}^n$, Korpelevich [23] suggested the following so-called extragradient method,

$$\begin{aligned} x^1 &= x \in K, \\ \overline{x}^k &= P_K(x^k - \beta A x^k), \\ x^{k+1} &= P_K(x^k - \beta A \overline{x}^k). \end{aligned}$$

She showed that the sequences $\{x^k\}$ and $\{\overline{x}^k\}$ generated by this process converge to the same point $p \in VI(K, A)$. Next, Antipin and Vasiliev [1] proposed a regularization variant of the extragradient method by

$$\begin{aligned} x^1 &= x \in K, \\ y^k &= P_K(x^k - \beta_k(Ax^k + \alpha_k x^k)), \\ x^{k+1} &= P_K(x^k - \beta_k(Ay^k + \alpha_k y^k)), \end{aligned} \tag{3}$$

where $\alpha_k > 0, \beta_k > 0, k \ge 1$. They proved the following theorem.

Theorem 1.1. *Let K be a nonempty closed convex subset of $\mathbb{E}^n$, let A be an L-Lipschitz continuous monotone operator of K such that $VI(K, A) \neq \emptyset$ and let $\{\alpha_k\}, \{\beta_k\}$ be chosen such that*

$$\alpha_k > 0,\ \beta_k > 0,\ \lim_{k\to\infty} \alpha_k = 0,\ \sup_{k\ge 1} \beta_k < 1/L,\ \sum_{k=1}^{\infty} \beta_k \alpha_k = +\infty,$$

and
(b)′ $\lim_{k\to\infty} \tilde{\alpha}_k/(\beta_k\alpha_k) = 0$, where $\tilde{\alpha}_k = (\alpha_{k-1}/\alpha_k) - 1$.
Then, the sequence $\{x^k\}$, generated by (3), converges to p solving (1), as $k \to \infty$.

Obviously, any λ-inverse strongly monotone operator A is L-Lipschitz continuous with $L = 1/\lambda$. Put $T := I - A$. It is easy to see that A is λ-inverse strongly monotone if and only if T is λ-strictly pseudo-contractive, i.e.,

$$\langle Tx - Ty, x - y\rangle \le \|x - y\|^2 - \lambda\|(I - T)x - (I - T)y\|^2,$$

that is equivalent to

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \lambda'\|(I - T)x - (I - T)y\|^2,$$

where λ' is some positive constant. Thus, the class of strictly pseudo-contractive operators contains the class of nonexpansive ones, that is, $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in K$, as a subclass. It is well known [30] that if T is a nonexpansive operator of K, then $A = I - T$ is (1/2)-inverse strongly monotone. Therefore, the problem of finding a point of $VI(K, A_0) \cap S_A$, where S_A denotes the set of zeros for A, is equivalent to that of $VI(K, A_0) \cap \mathrm{Fix}(T)$ where $\mathrm{Fix}(T) := \{x \in K : Tx = x\}$.

Motivated by the idea of the extragradient method, Nadezhkina and Takahashi [25] introduced an iterative process for finding a point $p \in VI(K, A) \cap \mathrm{Fix}(T)$ when A is an L-Lipschitz continuous monotone operator and T is a nonexpansive operator of K in H. They proved the following result.

Theorem 1.2. *Let K be a nonempty closed convex subset of a real Hilbert space H, let A be an L-Lipschitz continuous monotone non-self operator of K and let T be a nonexpansive operator of K into itself such that $VI(K, A) \cap Fix(T) \neq \emptyset$. Let $\{x^k\}$ and $\{y^k\}$ be sequences generated by*

$$\begin{aligned} x^1 &= x \in K, \\ y^k &= P_K(x^k - \beta_k A x^k), \\ x^{k+1} &= \alpha_k x^k + (1-\alpha_k) T P_K(x^k - \beta_k A y^k), \end{aligned} \tag{4}$$

for every $k = 1, 2, \ldots$, where $\{\beta_k\} \subset [a, b]$ for some $a, b \in (0, 1/L)$ and $\{\alpha_k\} \subset (c, d)$ for some $c, d \in (0, 1)$. Then, the sequences $\{x^k\}$ and $\{y^k\}$ converge weakly to the same point $z \in VI(K, A) \cap Fix(T)$, where $z = \lim_{k\to\infty} P_{VI(K,A)\cap Fix(T)} x^k$.

Further, Zeng and Yao [38] obtained strong convergence for method (4) by replacing the term $\alpha_k x^k$ in (4) by $\alpha_k x^1$ with some new condition on α_k provided $\lim_{k\to\infty} \|x^k - x^{k+1}\| = 0$. The last condition was overcomed in [27] and [36] by replacing the expression of x^{k+1} in (4), respectively, by

$$x^{k+1} = \alpha_k u + \theta_k x^k + \delta_k T P_K(x^k - \beta_k A y^k)$$

and

$$x^{k+1} = \alpha_k f(x^k) + (1-\alpha_k) T P_K(x^k - \beta_k A y^k),$$

where the parameters $\alpha_k, \theta_k, \delta_k, \beta_k$ have some properties, u is some point in H and f is a contraction, i.e., an α-Lipschitz continuous self operator of K with some fixed $\alpha \in [0, 1)$. See, also [26] and [39]. In the case that A is λ-inverse strongly monotone, hence $(1/\lambda)$-Lipschitz continuous monotone, several improved modifications of (4) were proposed in [15], [16] and [32]. If $A_{N+i} = A_N$ for all $i \geq 1$, we have a finite family of strictly pseudo-contractive operators $T_i := I - A_i$ with $1 \leq i \leq N$. To solve (2) in this case, the author [2] proposed the regularization methods of Browder-Tikhonov and extragradient types. Next, Ceng et al. [7] introduced two weakly convergent methods of extragradient type.

In this paper, motivated by the results in [2], for solving (2), we first construct a Browder-Tikhonov regularization solution u^k, that is

$$u^k \in K : \langle A^k u^k + \alpha_k(u^k - x^+), u^k - x\rangle \leq 0 \quad \forall x \in K, \tag{5}$$

where

$$A^k = A_0 + \alpha_k^\mu \sum_{i=1}^k (\gamma_i/\gamma^k) A_i,$$

α_k is a regularization parameter, x^+ is a guess point in H, $\mu \in (0, 1)$ is a fixed number and $\gamma^k = \sum_{i=1}^k \gamma_i$ with $\gamma_i > 0$ satisfying some condition formulated below. Next, we consider the regularization extragradient method,

$$\begin{aligned} x^1 &= x \in K, \\ y^k &= P_K(x^k - \beta_k(A^k x^k + \alpha_k(x^k - x^+))), \\ x^{k+1} &= P_K(x^k - \beta_k(A^k y^k + \alpha_k(y^k - x^+))), \end{aligned} \tag{6}$$

and will prove strong convergence of $\{x^k\}$, defined by (6), under the following assumptions:

(a) $\alpha_k \in (0, 1)$, $\alpha_{k-1} > \alpha_k$, $\lim_{k\to\infty} \alpha_k = 0$, $\sup_{k\geq 1} \beta_k < 1/L_0$ and $\sum_{k=1}^\infty \beta_k \alpha_k = \infty$;

(b) $\lim_{k\to\infty} \tilde{\alpha}_k/(\beta_k \alpha_k^{2-\mu}) = 0$;

(c) $\gamma_i > 0$ for all $i \geq 1$ such that $\sum_{i=1}^\infty \gamma_i = 1$ and $\lim_{k\to\infty} \gamma_k/(\beta_k \alpha_k^{2-\mu}) = 0$.

The following lemmas will be used in the proof of our results.

Lemma 1.3. *[34] Let $\{a_k\}$, $\{b_k\}$ and $\{c_k\}$ be the sequences of positive real numbers satisfying the conditions:*

(i) $a_{k+1} \le (1 - b_k)a_k + c_k, b_k < 1;$

(ii) $\sum_{k=1}^{\infty} b_k = +\infty, \quad \lim_{k\to+\infty}(c_k/b_k) = 0.$

Then, $\lim_{k\to+\infty} a_n = 0$.

Lemma 1.4. *[40] Assume that C is a closed convex subset of a real Hilbert space H and T is a non-self operator of C. If T is a λ-strict pseudo-contractive, then $I - T$ is demiclosed at zero. That is, if $\{u^k\}$ is a sequence in C such that $\{u^k\}$ converges weakly to a point $\tilde{x}$ and $\{(I - T)u^k\}$ converges strongly to 0, then $(I - T)\tilde{x} = 0$.*

Lemma 1.5. *[2] Let K be a closed convex subset in a Hilbert space H, let A be γ-inverse strongly monotone of K and let K_1 be a closed convex subset of K such that $K_1 \cap S_A \neq \emptyset$. Then, $VI(K_1, A) = S_A \cap K_1$.*

The rest of the paper is organized as follows. In Section 2, we present the theoretical results and show their particular cases. Applications to the problem of common fixed point for an infinite family of strictly pseudo-contractive non-self operators and the split feasibility and fixed point problems with illustrated numerical examples are given in Section 3.

2. Main results

We have the following results.

Theorem 2.1. *Let A_i, for each $i \ge 0$, be a non-self operator of a closed convex subset K in a real Hilbert space H such that A_0 be L_0-Lipschitz continuous monotone and the rest A_i be λ_i-inverse strongly-monotone with $\lambda = \inf_{i\ge1} \lambda_i > 0$. Assume that $\gamma_i > 0$ for all $i \ge 1$ such that $\sum_{i=1}^{\infty} \gamma_i = 1$. Then, we have:*

(i) For each $\alpha_k > 0$, problem (5) has a unique solution u^k;

(ii) If $\Gamma \neq \emptyset$, then $\lim_{k\to\infty} u^k = p_* \in \Gamma$, having the property:

$$\|p_* - x^+\| \le \|p - x^+\| \quad \forall p \in \Gamma; \tag{7}$$

(iii)

$$\|u^k - u^{k-1}\| \le d_k = \left[\left(\tilde{\alpha}_k + \frac{2\gamma_k}{\gamma^k}\right)\frac{1}{\alpha_k^{1-\mu}} + \tilde{\alpha}_k\right](M_1 + \|x^+\|), \tag{8}$$

where M_1 is some positive constant.

Proof. (i) Since $A^k + \alpha_k(I - x^+)$ is an α_k-strongly monotone operator from K into H, then, by [29], (5) possesses a unique solution u^k for each $k \ge 1$.

(ii) First, we prove that $\{u^k\}$ is bounded. Take a point $p \in \Gamma$. Then, we have immediately that $p \in VI(K, A_0) \cap S^k$, where $S^k = \cap_{i=1}^{k} S_i$, and hence,

$$\langle A^k p, p - x\rangle \le 0 \quad \forall x \in K. \tag{9}$$

Next, taking $x = p$ in (5) with $x = u^k$ in (9), adding the results and using the monotonicity of A^k, we get the inequality $\langle u^k - x^+, u^k - p\rangle \le 0 \quad \forall p \in \Gamma$. Consequently,

$$\|u^k - x^+\| \le \|p - x^+\| \quad \forall p \in \Gamma. \tag{10}$$

Therefore, $\{u^k\}$ is bounded. Since $\|A_i x - A_i y\| \le (1/\lambda_i)\|x - y\| \le (1/\lambda)\|x - y\|$ for any $x, y \in K$ and for all $i \ge 1$, the double sequence $\{A_i u^k\}$ is bounded. It means that there exists a positive constant M_1 such that $\|u^k\|, \|A_i u^k\| \le M_1$ for all $k, i \ge 1$.

Now, we prove that $\{u^k\}$ converges strongly to p_*, satisfying (7). For this purpose, we first prove that

$$\lim_{k\to\infty} \|A_i u^k\| = 0 \ \forall i \geq 1. \tag{11}$$

It is easy to see that from (5) and the monotonicities of A_0 and A_i it follows that, for any $p \in \Gamma$,

$$\begin{aligned}
\alpha_k^\mu \sum_{i=1}^k (\lambda_i\gamma_i/\gamma^k)\|A_i u^k\| &\leq \alpha_k^\mu \sum_{i=1}^k (\gamma_i/\gamma^k)\langle A_i u^k - A_i p, u^k - p\rangle \\
&= \alpha_k^\mu \sum_{i=1}^k (\gamma_i/\gamma^k)\langle A_i u^k, u^k - p\rangle \\
&\leq \langle A_0 u^k, p - u^k\rangle + \alpha_k\langle u^k - x^+, p - u^k\rangle \\
&\leq \langle A_0 u^k - A_0 p, p - u^k\rangle + \alpha_k\langle p - x^+, p - u^k\rangle \\
&\leq \alpha_k\langle p - x^+, p - u^k\rangle,
\end{aligned}$$

because $-\langle A_0 p, p - u^k\rangle \geq 0$, that is followed from $p \in VI(K, A_0)$. Thus, we have

$$\|A_i u^k\| \leq (\gamma^k/(\lambda_i\gamma_i))\alpha_k^{1-\mu}\|p - x^+\|(\|p\| + M_1).$$

Since $1 > \mu$, $\alpha_k \to 0$ and $\gamma^k \to 1$, tending $k \to \infty$ in the last inequality we get (11). Further, put $T_i = I - A_i$. It is clear that $p \in S_i$ if and only if $p \in \mathrm{Fix}(T_i)$. Since A_i is λ_i-inverse strongly monotone, the operator T_i is λ_i-strictly pseudocontrative. Noting (11),

$$\lim_{k\to\infty} \|(I - T_i)u^k\| = 0 \ \forall i \geq 1. \tag{12}$$

Now, from (5) and the monotonicity of A_0 we can write that

$$\begin{aligned}
\langle A_0 x, u^k - x\rangle &\leq \langle A_0 u^k, u^k - x\rangle \\
&\leq \alpha_k^\mu \sum_{i=1}^k (\gamma_i/\gamma^k)\langle A_i u^k, x - u^k\rangle + \alpha_k\langle u^k - x^+, x - u^k\rangle \\
&\leq \Big[\alpha_k^\mu M_1 + \alpha_k(M_1 + \|x^+\|)\Big](\|x\| + M_1) \ \forall x \in K.
\end{aligned} \tag{13}$$

Since $\{u^k\}$ is bounded, there exists a subsequence $\{u^n\}$ of the sequence $\{u^k\}$, $u^n = u^{k_n}$, such that $\{u^n\}$ converges weakly to some point $\overline{p} \in H$ as $n \to \infty$. As the sequence $\{u^n\} \subset K$, $\overline{p} \in K$. Next, replacing k in (13) by n and tending $n \to \infty$, we get the inequality $\langle A_0 x, \overline{p} - x\rangle \leq 0 \ \forall x \in K$ that is equivalent to $\langle A_0\overline{p}, \overline{p} - x\rangle \leq 0 \ \forall x \in K$, i.e., $\overline{p} \in VI(K, A_0)$. Moreover, by virtue of Lemma 1.4 and (12) with k replaced by n, we obtain that $\overline{p} \in \mathrm{Fix}(T_i) = S_i$ for each $i \geq 1$. It means that $\overline{p} \in S$. So, $\overline{p} \in \Gamma$. Using the weak convergence of $\{u^n\}$ and inequality (10) with k replaced by n, we get (7) where p_* is changed by $\overline{p}$. It is easy to see that any weak cluster point of $\{u^k\}$ has the property as $\overline{p}$ does. Moreover, we know that the point p_* in (7) is uniquely defined. Thus, all sequence $\{u^k\}$ converges weakly to p_* as $k \to \infty$. Next, from the weak convergence of $\{u^k\}$ to p_* and (10), we have that $\|u^k - x^+\| \to \|p_* - x^+\|$. Using the property of H, we get the strong convergence of $\{u^k\}$ to p_* as $k \to \infty$.

(iii) Now, we estimate the value $\|u^k - u^{k-1}\|$. Clearly, from (5), it follows that

$$\langle A^k u^k - A^{k-1}u^{k-1} + \alpha_k(u^k - x^+) - \alpha_{k-1}(u^{k-1} - x^+), u^{k-1} - u^k\rangle \geq 0,$$

which together the monotonicity of A^k implies that

$$\begin{aligned}
\alpha_k\|u^k - u^{k-1}\|^2 &\leq \langle A^{k-1}u^{k-1} - A^k u^{k-1}, u^k - u^{k-1}\rangle \\
&\quad + (\alpha_{k-1} - \alpha_k)\langle u^{k-1} - x^+, u^k - x^{k-1}\rangle \\
&\leq \Big[\|A^k u^{k-1} - A^{k-1}u^{k-1}\| \\
&\quad + (\alpha_{k-1} - \alpha_k)\|u^{k-1} - x^+\|\Big]\|u^k - u^{k-1}\|,
\end{aligned} \tag{14}$$

where

$$
\begin{aligned}
\|A^k u^{k-1} - A^{k-1}u^{k-1}\| &= \left\| \Big(A_0 + \frac{\alpha_k^\mu}{\gamma^k} \sum_{i=1}^{k} \gamma_i A_i \Big) u^{k-1} - \Big(A_0 + \frac{\alpha_{k-1}^\mu}{\gamma^{k-1}} \sum_{i=1}^{k-1} \gamma_i A_i \Big) u^{k-1} \right\| \\
&= \left\| \frac{\alpha_k^\mu}{\gamma^k} \sum_{i=1}^{k-1} \gamma_i A_i u^{k-1} - \frac{\alpha_{k-1}^\mu}{\gamma^{k-1}} \sum_{i=1}^{k-1} \gamma_i A_i u^{k-1} + \frac{\alpha_k^\mu}{\gamma^k} \gamma_k A_k u^{k-1} \right\| \\
&\le \left| \frac{\alpha_k^\mu}{\gamma^k} - \frac{\alpha_{k-1}^\mu}{\gamma^{k-1}} \gamma^{k-1} \right| M_1 + \frac{\alpha_k^\mu}{\gamma^k} \gamma_k M_1 \\
&= \left| \frac{\alpha_k^\mu}{\gamma^k} \gamma^{k-1} - \frac{\alpha_{k-1}^\mu}{\gamma^k} \gamma^k \right| M_1 + \frac{\alpha_k^\mu}{\gamma^k} \gamma_k M_1 \\
&= \left| \frac{\alpha_k^\mu \gamma^k - \alpha_{k-1}^\mu \gamma^k - \alpha_k^\mu \gamma_k}{\gamma^k} \right| + \frac{\alpha_k^\mu}{\gamma^k} \gamma_k M_1 \\
&\le |\alpha_k^\mu - \alpha_{k-1}^\mu| + 2 \frac{\alpha_k^\mu}{\gamma^k} \gamma_k M_1 .
\end{aligned} \tag{15}
$$

Further, from assumption (a) it follows that

$$
|\alpha_k^\mu - \alpha_{k-1}^\mu| = \alpha_{k-1}^\mu - \alpha_k^\mu = \left(\left(\frac{\alpha_{k-1}}{\alpha_k} \right)^\mu - 1 \right) \alpha_k^\mu \le \tilde{\alpha}_k \alpha_k^\mu ,
$$

which together with (14) and (15) implies (8). This completes the proof. □

Remark 1. Obviously, if $\{u^k\}$ converges strongly to some point $\tilde{u}$, where u^k is the solution of (5), and $\alpha_k \to 0$ as $k \to +\infty$, then $VI(K, A_0) \cap S \neq \emptyset$.

Theorem 2.2. *Let A_i be as in Theorem 2.1 for each $i \ge 0$. Assume that there hold assumptions (a)-(c). Then, the sequence $\{x^k\}$, defined by (6), converges strongly to the point p_* in (7) as $k \to +\infty$.*

Proof. Since $\langle P_K(v) - v, x - P_K(v) \rangle \ge 0$ for any $x \in K$ and $v \in H$, from (6) we have the following inequalities:

$$
\begin{aligned}
\langle y^k - x^k + \beta_k [A^k x^k + \alpha_k (x^k - x^+)], x - y^k \rangle \ge 0 \quad \forall x \in K, \\
\langle x^{k+1} - x^k + \beta_k [A^k y^k + \alpha_k (y^k - x^+)], x - x^{k+1} \rangle \ge 0 \quad \forall x \in K.
\end{aligned}
$$

By replacing $x = x^{k+1}$ and $x = u^k$, in the first and second inequalities above, respectively, and adding the results, we obtain

$$
\begin{aligned}
&\langle y^k - x^k, x^{k+1} - y^k \rangle + \langle x^{k+1} - x^k, u^k - x^{k+1} \rangle + \\
&\beta_k \langle A^k x^k + \alpha_k (x^k - x^+), x^{k+1} - y^k \rangle + \\
&\beta_k \langle A^k y^k + \alpha_k (y^k - x^+), u^k - x^{k+1} \rangle \ge 0.
\end{aligned} \tag{16}
$$

Next, from

$$
\begin{aligned}
&2\Big[\langle y^k - x^k, x^{k+1} - y^k \rangle + \langle x^{k+1} - x^k, u^k - x^{k+1} \rangle \Big] = \\
&\qquad \|x^k - u^k\|^2 - \|x^k - y^k\|^2 - \|x^{k+1} - u^k\|^2 - \|x^{k+1} - y^k\|^2
\end{aligned}
$$

and (16) it follows that

$$
\begin{aligned}
\|x^{k+1} - u^k\|^2 \le\ & \|x^k - u^k\|^2 - \|x^k - y^k\|^2 - \|x^{k+1} - y^k\|^2 \\
& + 2\beta_k\langle A^k x^k - A^k y^k, x^{k+1} - y^k\rangle \\
& + 2\beta_k\langle A^k y^k - A^k u^k, u^k - y^k\rangle \\
& + 2\beta_k\langle A^k u^k + \alpha_k(u^k - x^+), u^k - y^k\rangle \\
& + 2\alpha_k\beta_k\Big[-\langle u^k - x^+, u^k - y^k\rangle + \langle x^k - x^+, x^{k+1} - y^k\rangle \\
& + \langle y^k - x^+, u^k - x^{k+1}\rangle\Big].
\end{aligned}
\tag{17}
$$

By using (17), the properties of A_i with $i \ge 0$ and the inequality $2|ab| \le \varepsilon a^2 + b^2/\varepsilon$ with $\varepsilon = 1/2$ we have the following estimates:

$$
\begin{aligned}
2\langle A^k x^k - A^k y^k, x^{k+1} - y^k\rangle &\le 2\Big[L_0 + \alpha_k^\mu \sum_{i=1}^{k}(\gamma_i/\gamma^k\lambda_i)\Big]\|x^k - y^k\|\|x^{k+1} - y^k\| \\
&\le \Big(L_0 + (\alpha_k^\mu/\lambda)(\|x^k - y^k\|^2 + \|x^{k+1} - y^k\|^2\Big), \\
\langle A^k y^k - A^k u^k, u^k - y^k\rangle &\le 0, \\
\langle A^k u^k + \alpha_k(u^k - x^+), u^k - y^k\rangle &\le 0, \\
2\beta_k\alpha_k\Big[-\langle u^k - x^+, u^k - y^k\rangle + \langle x^k - x^+, x^{k+1} - y^k\rangle &+ \langle y^k - x^+, u^k - x^{k+1}\rangle\Big] = \\
2\beta_k\alpha_k\Big[-\|y^k - u^k\|^2 + \langle x^k - y^k, x^{k+1} - y^k\rangle\Big] &= \\
\beta_k\alpha_k\Big[-\|(y^k - x^k) + (x^k - u^k)\|^2 + \langle x^k - y^k, x^{k+1} - y^k\rangle\Big] &= \\
-2\langle y^k - x^k, x^k - u^k\rangle + \langle x^k - y^k, x^{k+1} - y^k\rangle\Big] & \\
&\le \beta_k\alpha_k\Big[-\|y^k - x^k\|^2 - \|x^k - u^k\|^2 \\
&+ \frac{1}{2}\|x^k - u^k\|^2 + 2\|x^k - y^k\|^2 + \frac{1}{2}\|x^k - y^k\|^2 + \frac{1}{2}\|x^{k+1} - y^k\|^2\Big] \\
&\le -\beta_k\alpha_k\|x^k - u^k\|^2 + 3\beta_k\alpha_k\|x^k - y^k\|^2 + \beta_k\alpha_k\|x^{k+1} - y^k\|^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|x^{k+1} - u^k\|^2 \le\ & (1 - \beta_k\alpha_k)\|x^k - u^k\|^2 \\
& + \Big[-1 + \beta_k(L_0 + (\alpha_k^\mu/\lambda)) + 3\beta_k\alpha_k\Big]\|x^k - y^k\|^2 \\
& + \Big[-1 + \beta_k(L_0 + (\alpha_k^\mu/\lambda)) + \beta_k\alpha_k\Big]\|x^{k+1} - y^k\|^2.
\end{aligned}
$$

Since $\alpha_k \to 0, \mu > 0$ and $\beta_k L_0 < 1$, there exists an integer k_0 such that for $k \ge k_0$ two last terms in the above inequality are negative. Hence,

$$\|x^{k+1} - u^k\|^2 \le (1 - \beta_k\alpha_k)\|x^k - u^k\|^2 \quad \forall k \ge k_0.$$

Set $a_k = \|x^k - u^k\|$. Then, $a_k \le \|x^k - u^{k-1}\| + \|u^{k-1} - u^k\|$, and hence, we have

$$\|x^{k+1} - u^k\| \le (1 - \beta_k\alpha_k)^{1/2}\|x^k - u^{k-1}\| + \|u^{k-1} - u^k\| \quad \forall k \ge k_0.$$

Thus, applying the inequality $(a+b)^2 \leq (1+\varepsilon)(a^2+b^2/\varepsilon)(\varepsilon > 0)$, $\varepsilon = \beta_k\alpha_k/2$, we obtain

$$\begin{aligned}\|x^{k+1}-u^k\|^2 &\leq (1+(\beta_k\alpha_k/2))(1-\beta_k\alpha_k)\|x^k-u^k\|^2 + d_k^2\frac{2}{\beta_k\alpha_k}(1+(\beta_k\alpha_k/2))\\ &\leq \Big(1-\frac{1}{2}\beta_k\alpha_k-\frac{1}{2}(\beta_k\alpha_k)^2\Big)\|x^k-u^k\|^2\\ &\quad+\Big(\frac{d_k}{\beta_k\alpha_k}\Big)^2 2\beta_k\alpha_k(1+\beta_k\alpha_k/2).\end{aligned}$$

Set

$$\begin{aligned}b_k &= \frac{1}{2}\beta_k\alpha_k(1+\beta_k\alpha_k)\\ c_k &= \Big(\frac{d_k}{\beta_k\alpha_k}\Big)^2 2\beta_k\alpha_k(1+\beta_k\alpha_k/2).\end{aligned}$$

Next, by virtue of (8) and assumptions (a), (b) with (c),

$$\lim_{k\to\infty} d_k/(\beta_k\alpha_k) = 0.$$

Hence by lemma (1.3), $\lim_{\to+\infty}\|x^k-u^{k-1}\|^2 = 0$. This fact together with

$$\|x^k-u^k\| \leq \|x^k-u^{k-1}\| + \|u^{k-1}-u^k\|$$

and $d_k, \|u^{k-1}-u^k\| \to 0$ implies that $\lim_{k\to+\infty}\|x^k-u^k\| = 0$. The proof is completed. □

Remark 2. In the case that operator A_0 is also λ_0-inverse strongly monotone, instead of A^k in (5) and (6), we can use the operator $\overline{A}^k$, defined by

$$\overline{A}^k = \sum_{i=0}^{k}(\gamma_i/\gamma^k)A_i,$$

where $\gamma^k = \gamma_0+\gamma_1+\cdots+\gamma_k$ and γ_i has the property
(c)′ $\gamma_i > 0$ for all $i \geq 0$ such that $\sum_{i=0}^{\infty}\gamma_i = 1$ and $\lim_{k\to\infty}\gamma_k/(\beta_k\alpha_k^2) = 0$.
Then, repeating the proof processes for Theorems 2.1 and 2.2 and noting that

$$\begin{aligned}2\langle\overline{A}^k x^k - \overline{A}^k y^k, x^{k+1}-y^k\rangle &\leq 2\sum_{i=1}^{k}(\gamma_i/\gamma^k\lambda_i)\|x^k-y^k\|\|x^{k+1}-y^k\|\\ &\leq (1/\lambda)(\|x^k-y^k\|^2+\|x^{k+1}-y^k\|^2\Big),\end{aligned}$$

and

$$\begin{aligned}\|x^{k+1}-u^k\|^2 &\leq (1-\beta_k\alpha_k)\|x^k-u^k\|^2\\ &\quad+\Big[-1+(\beta_k/\lambda)+3\beta_k\alpha_k\Big]\|x^k-y^k\|^2\\ &\quad+\Big[-1+(\beta_k/\lambda)+\beta_k\alpha_k\Big]\|x^{k+1}-y^k\|^2,\end{aligned}$$

we can obtain the following results.

Theorem 2.3. *Let A_i for each $i \geq 0$ be a λ_i-inverse strongly-monotone non-self operator of a closed convex subset K in a real Hilbert space H with $\lambda = \inf_{i\geq 0}\lambda_i > 0$. Then, we have:*

(i) For each $\alpha_k > 0$, the variational inequality problem

$$\langle \overline{A}^k u^k + \alpha_k(u^k - x^+), u^k - x\rangle \le 0 \quad \forall x \in K,$$

has a unique solution u^k;

(ii) If $\Gamma \neq \emptyset$, then $\lim_{k\to\infty} u^k = p_* \in \Gamma$ satisfying (7);

(iii)

$$\|x^k - x^{k-1}\| \le \overline{d}_k := \frac{2\gamma_k}{\alpha_k \gamma^k} M_3 + \tilde{\alpha}_k(M_3 + \|x^+\|),$$

where M_3 is some positive constant.

Theorem 2.4. *Let A_i be as in Theorem 2.3 for each $i \ge 0$. Assume that there hold assumptions (a) with $\sup_{k\ge0}\beta_k < \lambda$, (b)′ and (c)′. Then, the sequence $\{x^k\}$ defined by*

$$\begin{aligned} x^1 &= x \in K,\\ y^k &= P_K(x^k - \beta_k(\overline{A}^k x^k + \alpha_k(x^k - x^+))),\\ x^{k+1} &= P_K(x^k - \beta_k(\overline{A}^k y^k + \alpha_k(y^k - x^+))), \end{aligned}$$

converges strongly to the element p_* in (7) as $k \to +\infty$.

We can formulate similar results as Theorems 2.1–(2.4, when $A_{N+i} = A_N$ for all $i \ge 1$, by replacing A^k and $\overline{A}^k$, by the operators $A_0 + \alpha_k^\mu \sum_{i=1}^N \gamma_i' A_i$ with $\gamma_i' > 0$, for $1 \le i \le N$ and $\sum_{i=0}^N \gamma_i' A_i$ with the same properties for γ_i', respectively. For example, we have the following result.

Theorem 2.5. *Let A_i be as in Theorem 2.3 for each $i = 0, 1, \cdots, N$ with some positive integer N and let $\gamma_i > 0$ for all $i = 0, 1, \cdots, N$. Assume that there hold assumptions (a) with $\sup_{k\ge0}\beta_k < \lambda$ and (b)′. Then, the sequence $\{x^k\}$, defined by*

$$\begin{aligned} x^1 &= x \in K,\\ y^k &= P_K\Big(x^k - \beta_k\Big(Sx^k + \alpha_k(x^k - x^+)\Big)\Big),\\ x^{k+1} &= P_K\Big(x^k - \beta_k\Big(Sy^k + \alpha_k(y^k - x^+)\Big)\Big), \end{aligned}$$

where $S = \sum_{i=0}^N \gamma_i A_i$, converges strongly to the element p_* in (7) as $k \to +\infty$.

Remark 3. Examples of sequences, having all properties (a)-(c) and (b)′-(c)′, are: $\gamma_i = 1/(i(i+1))$, $\beta_k = 1/(k+1)^a$ and $\alpha_k = 1/(k+1)^b$, where $0 < b < a$ and $a + (2-\mu)b < 1$.

Remark 4. Note that problem (2), when A_0 is pseudomonotone, L_0-Lipschitz continuous and sequentially weakly continuous on K and $S = S_1 \cap S_2$, where $A_i = I - T_i$ for $i = 1, 2$ and one of $\{T_1, T_2\}$ is asymptotically nonexpansive and the other is pseudocontractive, is considered very recently by Ceng et al. [9]. To solve it, they introduced two new iterative algorithms with linear-search process, that are different from our methods, presented in this paper.

3. Applications and numerical examples

3.1. The fixed point problem for infinite families of strictly pseudo-contractive operators

It ie well known in [40] that if T is a λ-strictly pseudo-contractive operator of a closed convex subset K in H, then $T' := \tilde{\lambda}I + (1-\tilde{\lambda})T$, where $\tilde{\lambda} \in [\lambda, 1)$ is a fixed number, is nonexpansive with $\mathrm{Fix}(T) = \mathrm{Fix}(T')$. Therefore, in order to find a common fixed point for an infinite family of λ_i-strictly pseudo-contractive

Table 1: Computational results by method (18)

k	x_1^{k+1}	x_2^{k+1}	$e^{k+1} = \|z^{k+1} - p_*\|$
05	-0.0129142584	0.9999166075	0.0129145277
10	-0.0006517104	0.9999997876	0.0006517104
15	-0.0000716226	0.9999999974	0.0000716226
20	-0.0000121665	0.9999999999	0.0000121665

operator T_i for all $i \geq 0$, ones usually have introduced iterative methods for finding a common fixed point for an infinite family of nonexpansive mappings $\{T_i'\}_{i\geq 0}$ where $T_i' = \tilde{\lambda}_i I + (1-\tilde{\lambda}_i)T_i$ with $\tilde{\lambda}_i \in [\lambda_i, 1)$, by using W_k, V_k or S_k-mappings, where

$$S_k = \sum_{i=0}^{k} (\gamma_i/\gamma^k)T_i' \quad \text{or} \quad S_k = \sum_{i=0}^{k} (\gamma_i/\gamma^k)(\alpha_i' I + (1-\alpha_i')T_i'),$$

with some conditions on α_i'. See, for example, [3]-[5],[18],[20] and references therein. Using the results in the previous section, we show that the transformation process from T_i to T_i' is not necessary. Indeed, put $A_i := I - T_i$. By Lemma 1.5, $p \in S_{A_0}$ if and only if $p \in VI(K, A_0)$. Then, by Theorem 2.4, we have the result.

Theorem 3.1. *Let K be a closed convex subset in a real Hilbert space H and let $\{T_i\}_{i\geq 0}$ be an infinite family of λ_i-strictly pseudo-contractive non-self operators of K such that the common fixed point set $\Gamma := \cap_{i\geq 0} Fix(T_i) \neq \emptyset$. Assume that there hold assumptions (a) with $\sup_{k\geq 0} \beta_k < \lambda$, (b)′ and (c)′. Then, the sequence $\{x^k\}$ defined by*

$$\begin{aligned} x^1 &= x \in K, \\ y^k &= P_K\Big(x^k - \beta_k\Big(\tilde{A}^k x^k + \alpha_k(x^k - x^+)\Big)\Big), \\ x^{k+1} &= P_K\Big(x^k - \beta_k\Big(\tilde{A}^k y^k + \alpha_k(y^k - x^+)\Big)\Big), \end{aligned} \tag{18}$$

converges strongly to the element p_* in (7) as $k \to +\infty$, where $\tilde{A}^k = I - \sum_{i=0}^{k} (\gamma_i/\gamma^k)T_i$, $\gamma^k = \gamma_0 + \gamma_1 + \cdots + \gamma_k$ and x^+ is a guess point in H.

For computations, we consider a concrete example, where $K = \{x = x \in \mathbb{E}^2 : x_1^2 + x_2^2 \leq 1\}$ and $T_i = P_{C_i}$, where $C_i = \{x \in \mathbb{E}^2 : (1/(i+1))x_1 - x_2 \leq 0\}$ and $\mathbb{E}^2$ is the Euclidian space, whose inner product and norm are defined by $\langle x, y\rangle = x_1y_1 + x_2y_2$ and $\|x\|^2 = x_1^2 + x_2^2$, respectively. It is well known that T_i is nonexpansive for all $i \geq 0$, and hence, strictly pseudo-contractive in $\mathbb{E}^2$. $\Gamma = \{x \in \mathbb{E}^2 : x_2 \geq 0, 2x_2 \geq x_1 \geq 0\} \cap K$ is the set of common fixed points for $\{T_i\}_{i\geq 0}$. Taking $x^+ = (0; 3)$, we have $p_* = (0; 1)$. Using method (18) with $\gamma_i = 1/((i+1)(i+2)), \beta_k = 1/(k+1)^{1/2}$, $\alpha_k = 1/(k+1)^{1/4}$ and the starting point $z^1 = (-0.9; -0.3)$, we obtain the following table of numerical results, Table 1.

3.2. *The split feasibility and fixed point problems*

We consider the problem of finding a common point of the solution set S_{SFP} for the SFP and the set Fix(T) of fixed points for a nonexpansive operator T in the setting of infinite-dimensional Hilbert spaces. The SFP is to find a point

$$p \in C \quad \text{such that} \quad Ap \in Q, \tag{19}$$

where C and Q are two closed convex subsets in two Hilbert spaces H_1 and H_2, respectively, and A is a bounded linear operator from H_1 to H_2 with inner products and norms denoted also by the symbols $\langle ., .\rangle$ and $\|.\|$, respectively. Problem (19) was first introduced by Censor and Elfving [10] for modeling inverse problems that arise from phase retrievals and in image reconstruction [6]. Recently, it can also be used to model the intensity-modulated radiation therapy [11]–[14].

Let T be a nonexpansive operator of C such that $\Gamma := S_{SFP} \cap \mathrm{Fix}(T) \neq \emptyset$. The problem of finding a point $p \in \Gamma$ has been studied in [8],[16],[17],[22] and [37] and references therein. Ceng et al. [8] introduced the iterative method,

$$\begin{aligned} x^1 &= x \in C,\\ y^k &= P_C(x^k - \beta_k(A_1x^k + \alpha_k x^k)),\\ x^{k+1} &= \tau_k x^k + (1-\tau_k)TP_C(x^k - \beta_k(A_1y^k + \alpha_k y^k)), \end{aligned} \tag{20}$$

that converges weakly to a point in Γ with some conditions on τ_k, β_k and α_k, one of which is that $\{\beta_k\} \subset [a, b]$ for some $a, b \in (0, 1/\|A\|^2)$ where $A_1 := A^*(I - P_Q)A$ and A^* is the adjoint of A. Yao et al. [37] obtained the strong convergence for method (20) with conditions:

(i) $\alpha_k \in (0, 1)$, $\lim_{k\to\infty} \alpha_k = 0$ and $\sum_{k=1}^{\infty} \alpha_k = \infty$;

(ii) $0 < \liminf_{k\to\infty} \tau_k \le \limsup_{k\to\infty} \tau_k < 1$;

(iii) $0 < \beta_k < 2/(\|A\|^2 + \alpha_k)$, $0 < \liminf_{k\to\infty} \beta_k \le \limsup_{k\to\infty} \beta_k < 2/\|A\|^2$ such that $\lim_{k\to\infty}(\beta_{k+1} - \beta_k) = 0$.

Deepho et al. [17], by combining (20) with the hybrid methods, proposed a strong convergent modification of (20). It is not difficult to verify that A_1 is an $(1/\|A\|^2)$-inverse strongly monotone operator of H_1. As spoken in Introduction, $A_0 := I - T$ is also an λ_0-inverse strongly monotone non-self operator of C with $\lambda_0 = 1/2$. Therefore, A_0 is 2-Lipschitz continuous and $p \in \mathrm{Fix}(T)$ if and only if $p \in S_{A_0} = VI(C, A_0)$, by Lemma 1.5. Using Theorem 2.5 with $N = 1$, we obtain the following result.

Theorem 3.2. *Let C and Q be two closed convex subsets in two real Hilbert spaces H_1 and H_2, respectively, let A be a bounded linear operator from H_1 to H_2 with adjoint A^* and let T be a nonexpansive non-self operator of C. Assume that there hold assumptions (a) with $\sup_{k\ge0} \beta_k < \lambda(= \min\{1/2; 1/\|A\|^2\}$ and (b)′. Then, the sequence $\{x^k\}$ defined by*

$$\begin{aligned} x^1 &= x \in C,\ B = \gamma A_0 + (1-\gamma)A_1,\\ y^k &= P_C\Big(x^k - \beta_k\Big(Bx^k + \alpha_k(x^k - x^+)\Big)\Big),\\ x^{k+1} &= P_C\Big(x^k - \beta_k\Big(By^k + \alpha_k(y^k - x^+)\Big)\Big), \end{aligned} \tag{21}$$

converges strongly to the element p_* in (7), as $k \to +\infty$, where x^+ is a guess point in H_1 and $\gamma \in (0, 1)$ is a fixed number.

Remark 5. From Theorem 2.2, it is easily to see that Theorem 3.2 has still value under conditions (a) with $\lambda = 1/2$ and (b), if the operator B is replaced by $B^k = A_0 + \alpha_k^{\mu} A_1$. Next, by taking $T = I$, the identity operator of H_1, we obtain the regularization extragradient method for the SFP,

$$\begin{aligned} x^1 &= x \in C,\\ y^k &= P_C\Big(x^k - \beta_k\Big(\alpha_k^{\mu} A_1 x^k + \alpha_k(x^k - x^+)\Big)\Big),\\ x^{k+1} &= P_C\Big(x^k - \beta_k\Big(\alpha_k^{\mu} A_1 y^k + \alpha_k(y^k - x^+)\Big)\Big), \end{aligned} \tag{22}$$

strong convergence is guaranteed by assumptions (a) with $\sup_{k\ge0} \beta_k < 1/2$ and (b). Clearly, the parameter β_k in method (22) can be chosen without prior knowledge of $\|A\|^2$ as (20) and its modifications need.

For computations, we take

$$C = \{(x_1; x_2) : (1/2)x_1 - x_2 \le 0\} \subset \mathbb{E}^2, \quad Q = \{(x_1, x_3, x_3) : x_1^2 + x_2^2 + x_3^2 \le 1\} \subset \mathbb{E}^3$$

and A is a matrix of order 3×2, whose elements $a_{11} = a_{22} = a_{31} = a_{32} = 1$ and the rest ones are zeros. It is not difficult to verify that $p_* = (0; 0)$ is the unique solution of (19) with the given data. Using method (22) with the same values for $\beta_k = 1/(k + 2)^{1/2}$, $\alpha_k = \beta_k^{1/2}$, $\mu = 1/2$ and the starting point $x^1 = (-2.0; -1.0)$, we get the following numerical table, Table 2.

Table 2: Computational results by method (22).

k	x_1^{k+1}	x_2^{k+1}	$e^{k+1} = \|z^{k+1} - p_*\|$
05	-0.0085032236	0.0208253661	0.0224944590
10	-0.0011621321	0.0028461944	0.0030743086
15	-0.0002820483	0.0006907684	0.0007461315
20	-0.0000918736	0.0002250090	0.0002430428

The numerical results above show fast convergence of the proposed methods.

Acknowledgments

The author is extremely grateful to the referees for their useful comments and suggestions which helped to improve this paper.
This work was supported by the Vietnam National Foundation for Science and Technology Development under grant No. 101.02-2021.11.

References

[1] A.S. Antipin, F.P. Vasiliev, Regularization extragradient method for solving variational inequlities, Vychisl. Metody i Program. **3** (2002), 237–244.

[2] Ng. Buong, Regularization extragradient method for Lipschitz continuous mappings and inverse strongly monotone mappings in Hilbert spaces, Zh. Vychisl. Mat. i Mat. Fiz. **48**(11) (2008), 1927–1935.

[3] Ng. Buong, Ng.Th.H. Phuong, Regularization methods for a class of variational inequalities in Banach spaces, Zh. Vychisl. Mat. i Mat. Fiz. **52**(11) (2012), 1487–1496.

[4] Ng. Buong, Ng.Th.H. Phuong, Strong convergence to solutions for a class of variational inequalities in Banach spaces by implicit iteration methods, J. Optim. Theory Appl. **159** (2013), 399–411.

[5] Ng. Buong, Ng.S. Ha, Ng.Th.Th. Thuy, A new explicit iteration method for a class of variational inequalities, Numer. Algorithms **72** (2016), 467–481.

[6] C. Byrne, A unified treatment of some iterative methods in signal processing and image reconstruction, Inverse Problems **20** (2004), 1003–1020.

[7] L.C.Ceng, A. Petrusel, C. Lee, M.M. Wong, Two extragradient methods for variational inequality problems and fixed point problems, Taiwanese J. Math. **13**, (2009), 607–632.

[8] L.C. Ceng, Q.H. Ansari, J.C. Yao, An extragradient method for solving split feasibility and fixed point problems, Computers & Math. with Appl. **64** (2012), 633–642.

[9] L.C. Ceng, A. Petrusel, X. Qin, J.C. Yao, Pseudomonotone variational inequalities and fixed points, Fixed Point Theory **22**(2) (2021), 543–558.

[10] Y. Censor, T. Elfving, A multiprojection algorithm using Bregman projections in a product spaces, Numer. Algorithms **8** (1994), 221–239.

[11] Y. Censor, T. Elfving, N. Knop, T. Bortfeld, The multiple-sets split feasibility problem and its applications for inverse problems, Inverse Problems **21** (2005), 2071–2084.

[12] Y. Censor, T. Bortfeld, B. Martin, A. Trofimov, A unified approach for inverse problems in intensity-modulated radiation therapy, Phys. Med. Biol. **51** (2006), 2353–2365.

[13] Y. Censor, A. Motova, A. Segal, Perturbed projections and subgradient projections for the multiple-sets split feasibility problem, J. Math. Anal. Appl. **327** (2007), 1244–1256.

[14] Y. Censor, A. Segal, Iterative projection methods in biomedical inverse problems, Mathematical Methods in Biomedical Imaging and Intensity-Modulated Radiation Therapy (IMRT) ed Y Censor, M Jiang and A K Louis (Edizioni della Normale, Pisa, Italy) (2008), 65–96.

[15] J. Chen, L. Zhang, T. Fan, Viscosity approximation methods for nonexpansive mappings and monotone mappings, J. Math. Anal. Appl. **334**(2) (2007), 1450–1461.

[16] J.Z. Chen, L.C. Ceng, Y.Q. Qiu, Z.R. Kong, Extra-gradient methods for solving split feasibility and fixed point problems, Fixed Point Theory and App. **2015** (2015), 192.

[17] J. Deepho, W. Kumam, P. Kumam, The hybrid extragradient method for the split feasibility and fixed point problems, Proceeding IMECS, **I** 2014, March 12–14.

[18] Sh. Iemoto, W. Takahashi, Strong convergence theorems by a hybrid steepest descent method for countable nonexpansive mappings in Hilbert spaces, Sci. Math. Jpn. Online, e-**2008** (2008), 555–570.

[19] H. Iiduka, W. Takahashi, Strong convergence theorems nonexpansive mappings and inverse strongly monotone mappings, Nonl. Anal. **61**(3) (2005), 341–350.

[20] M. Kikkawa, W. Takahashi, Strong convergence theorems by the viscosity approximation methods for a countable family of nonexpansive mappings, Taiwanese J. Math. **12**(3) (2008), 583–598.

[21] J.K. Kim, Ng. Buong, A new iterative method for equilibrium problems and fixed point problems for infinite family of nonself strictly pseudocontractive mapping, Fixed Point Theory and Appl. **2013** (2013), 286.

[22] Z.R. Kong, L.C. Ceng, C.F. Wen, Some modified etragradient methods for solving split feasibility and fixed point problems, Abstract and Appl. Anal. **2012** (2012), 32.

[23] G.M. Korpelevich, Extragradient method for finding sadle points and for other problems, Ekon. i Matem. Metody **12** (1976), 747–756.

[24] G. Marino, H.K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl. **329** (2007), 336–346.

[25] N. Nadezhkina, W. Takahashi, Weak convergence theorems by extragradient method for nonexpansive mappings and monotone mappings, J. Optim. Theory and Appl. **128**(1) (2006), 191–201.

[26] M.A. Noor, Y. Yao, R. Chen, Y.C. Liou, An iterative method for fixed point problems and variational inequality problems, Math. Commun. **12** (2007), 121–132.

[27] A. Petrusel, J.C. Yao, An extragradient iterative scheme by viscosity approximation methods for fixed point problems and variational inequality problems, Cent. Eur. J. Math. **7**(2) (2009), 335–347.

[28] P.T. Polyak, Introduction for Optimization, Optimization Software, New-York, (1987).

[29] I.P. Ryazantseva, On solving variational inequalities with monotone operators by method of regularization, Zh. Vychisl. Mat. i Mat. Fiziki **23** (1983), 479–483.

[30] W. Takahashi, M. Toyota, Weak convergence theorems for nonexpansive mappings and monotone mappings, J. Optim. Theory and Appl. **118**(2) (2003), 417–428.

[31] M.H. Takele, K. Reddy, Iterative method for approximating a common fixed point of infinite family of strictly pseudo-contractive mappings in Hilbert spaces, Intern. J. of Comput. and Appl. **12**(2) (2017), 293–303.

[32] H. Wang, Y. Song, An iteration scheme for nonexpansive mappings and variational inequalities, Bull. Korean Math. Soc. **48**(5) (2011), 991–1002.

[33] Sh. Wang, A general iterative method for obtaining an infinite family of strictly pseudo-contractions in Hilbert spaces, Appl. Math. Letters **24** (2011), 901–907.

[34] H.K. Xu, Iterative algorithms for nonlinear operators, J. Lond. Math. Soc. **66** (2002), 240–256.

[35] L.P. Yang, Modified Noor iterations for infinite family of strict pseudo-contractions, Bull. Iranian Math. Soc. **37**(1) (2011), 43–61.

[36] Y. Yao, Y.C. Liou, Y.C. Yao, An extragradient method for fixed point problems and variational inequality problems, J. Inequal. Appl. **2007** (2007), 12.

[37] Y. Yao, T.H. Kim, S. Chebbi, H.K. Xu, A modified extragradient method for the split feasibility and fixed point problems, J. Nonl. Convex Anal. **13** (2012), 383–396.

[38] L.C. Zeng, J.C. Yao, Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems, Taiwanese Math. J. **10**(5) (2006), 1293–1303.

[39] L.C. Zeng, J.C. Yao, Extragradient-like approximation method for variational inequality problems and fixed point problems, Appl. Math. Comput. **190**(1) (2007), 2005–2015.

[40] H. Zhou, Convergence theorems for k-strict pseudo-contractions in Hilbert spaces, Nonl. Anal. **69** (2008), 456–462.