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Solid Cauchy transform on the weighted poly-Bergman spaces

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Abstract. In the present paper, we deal with the weighted solid Cauchy transform C_s^μ (from inside the unit disc into the complement of its closure) acting on the weighted true poly-Bergman spaces in the unit disc introduced and studied by Ramazanov and Vasilevski. Mainly, we are concerned with the concrete description of its range and its null space. We also give the closed expression of their reproducing kernels. To this end, we begin by studying the basic properties of C_s^μ such as boundedness for appropriate probability measures. The main tool is an explicit expression of its action on the so-called disc polynomials which form an orthogonal basis of the considered weighted true poly-Bergman spaces.

1. Introduction and statement of main results

Let Ω be a bounded domain in the complex plane. Denote by $b\Omega$ its boundary and by $\overline{\Omega}^c$ the complement of its closure, $\overline{\Omega}^c := \mathbb{C} \setminus \overline{\Omega}$. Associated with a given measure μ in Ω, we define the weighted solid Cauchy transform to be the integral operator

$$C_s^\mu f(z) := \frac{1}{\pi}\int_\Omega \frac{f(\xi)}{z-\xi} d\mu(\xi), \quad z \in \overline{\Omega}^c, \tag{1}$$

which can be seen as a specific adjoint of the classical weighted Cauchy transform on $b\Omega$ [4, p. 89]. The importance of this operator (and their variants) lies in the fact that its kernel function is the fundamental solution of the $\overline{\partial}$ operator. Moreover, it is closely connected to the Green's function for Dirichlet Laplacian in Ω which is used to the inverse moment problem, see e.g. [4, 11, 24].

The study of local and global properties of C_s^μ including the description of its range, when acting on the different standard spaces of analytic functions has been raised and investigated by many authors [5, 23, 28, 29]. This problem has been solved by Napalkov and Yulmukhametov [28] for the Bergman space $A^2(\Omega)$ of analytic functions for Jordan domains, and later by Merenkov [25, 26] for a large class of domains including integrable Jordan domains and those bounded by a Jordan curve $b\Omega$ with $area(b\Omega) = 0$. It is shown in [25] that the restriction of C_s^μ to $A^2(\Omega)$ is an injective continuous operator from $A^2(\Omega)$ into the

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special Bergman-Sobolev space $B_1^2(\overline{\Omega}^c)$, defined as the space of holomorphic functions in $\overline{\Omega}^c$ vanishing at infinity and whose derivatives belong to $A^2(\overline{\Omega}^c)$,

$$B_1^2(\overline{\Omega}^c) = \{f \text{ holomorphic in } \overline{\Omega}^c, f' \in A(\overline{\Omega}^c), f(\infty) = 0\}.$$

Such characterization remains valid for Ω being a quasidisc. It has been used in [18] to investigate the action of the Laplace transform on Bergman spaces.

The present paper treats a similar problem and is concerned with the action of the solid Cauchy transform on specific subspaces of polyanalytic functions of order n (n-analytic). Such functions are a natural generalization of the holomorphic functions, and have found interesting applications in mathematical physics, signal processing, time–frequency analysis and wavelet theory [1, 3, 14, 15, 17, 21, 27, 34]. More precisely, they are solutions of the generalized Cauchy–Riemann equation on the unit disc $\mathbb{D}$,

$$\partial_{\bar{z}}^n f = \frac{\partial^n f}{\partial \bar{z}^n} = 0, \ \partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Mainly, we deal with the concrete description of the basic properties of the weighted solid Cauchy transform $C_s^{\omega_\alpha}$, $\omega_\alpha(|\xi|^2) := (1-|\xi|^2)^\alpha$, acting on the so-called weighted true poly-Bergman spaces $\mathcal{A}_n^{2,\gamma}(\mathbb{D})$ in the unit disc $\Omega = \mathbb{D}$, which are a specific generalization of the classical weighted Bergman space to the polyanalytic setting. To this end, we start by providing sufficient conditions on the given measures for C_s^{ω} to be a bounded operator for the corresponding Hilbert spaces of square integrable functions. Moreover, we show that $\mathcal{A}_n^{2,\gamma}(\mathbb{D})$ form an orthogonal sequence of reproducing kernel Hilbert subspaces in the underlying Hilbert space $L^{2,\gamma}(\mathbb{D}) := L^2\left(\mathbb{D}, (1-|z|^2)^\gamma dxdy\right)$. An orthogonal basis of $\mathcal{A}_n^{2,\gamma}(\mathbb{D})$ is proved to be given by the so-called disc polynomials defined by

$$\mathcal{R}_{m,n}^{\gamma}(z,\bar{z}) = \frac{z^m \bar{z}^n}{|z|^{2(m\wedge n)}} P_{m\wedge n}^{(\gamma,|m-n|)}(2|z|^2 - 1) \tag{2}$$

for varying $m = 0, 1, 2, \cdots$, where $m \wedge n = \min(m,n)$ and $P_m^{(\alpha,\beta)}(x)$ denotes the real Jacobi polynomials normalized so that $P_m^{(\alpha,\beta)}(1) = 1$.

Our next main task is the determination of the null space as well as the range of the restriction of $C_s^{\omega_\alpha}$ to the true weighted poly-Bergman spaces $\mathcal{A}_n^{2,\gamma}(\mathbb{D})$. We show that its image is a finite dimensional vector space contained in the one spanned by z^{n-m+1}, $m = 0, 1, 2, \cdots, n$. Its precise dimension depends on the quantization of $\gamma - \alpha$ and the order of the polyanalyticity. More precisely, making use of the explicit action of $C_s^{\omega_\alpha}$ on the disc polynomials $\mathcal{R}_{m,n}^{\gamma}$ we establish the following.

Theorem 1.1. *Let $\gamma > -1$ and $\alpha > (\gamma-1)/2$. Then $C_s^{\omega_\alpha}(\mathcal{A}_n^{2,\gamma}(\mathbb{D}))$ is a finite dimensional vector space of dimension $N = \dim(C_s^{\omega_\alpha}(\mathcal{A}_n^{2,\gamma}(\mathbb{D}))) = \min(n, \alpha-\gamma) + 1$ when $\gamma - \alpha \in \mathbb{Z}_0^-$ and $N = n+1$ otherwise.*

Moreover, we obtain the following

Corollary 1.2. *Under the conditions of Theorem 1.1, the null space of the restriction of $C_c{}^{\omega_\alpha}$ to $\mathcal{A}_n^{2,\gamma}(\mathbb{D})$ is spanned by the disc polynomials $\mathcal{R}_{m,n}^{\gamma}$ with $m \geq \min(n, \alpha-\gamma) + 1$.*

Corollary 1.3. *The spaces $C_s^{\omega_\alpha}(\mathcal{A}_n^{2,\gamma}(\mathbb{D}))_n$ form an increasing sequence of spaces.*

This work is outlined as follows. Section 2 is devoted to discuss the boundedness of the weighted solid Cauchy transform C_s^{μ} for specific weight functions. In Section 3, we present a brief review for the disc polynomials $\mathcal{R}_{m,n}^{\gamma}$. The basic properties of the true weighted poly-Bergman space $\mathcal{A}_n^{2,\gamma}(\mathbb{D})$, defined by Ramazanov and by Vasilevski, are presented and complemented in Section 4. In Section 5, we consider the explicit action of $C_s^{\omega_\alpha}$ on $\mathcal{A}_n^{2,\gamma}(\mathbb{D})$ in order to characterize its range as well as its null space. The proofs of Theorem 1.1 and their corollaries are also presented in this section.

2. Boundedness of C_s^μ

In this section, we consider the weight functions A and B on $(0,1)$ and $(1,+\infty)$, respectively, that we extend as usual to measures on the unit disc $\mathbb{D}$ and its complement $\overline{\mathbb{D}}^c$ by considering $A(|z|^2)d\lambda(z)$ and $B(|z|^2)d\lambda(z)$, respectively. Here $d\lambda(z) = dxdy$ for $z = x+iy$ with $x, y \in \mathbb{R}$ being the standard Lebesgue measure. We denote by

$$L^2(\mathbb{D}, A) := L^2(\mathbb{D}, A(|z|^2)d\lambda)$$

and

$$L^2(\overline{\mathbb{D}}^c, B) := L^2(\overline{\mathbb{D}}^c, B(|z|^2)d\lambda)$$

the corresponding Hilbert spaces of all square integrable complex-valued functions with respect to the prescribed measures. We denote by $\langle\cdot,\cdot\rangle_A$ and $\langle\cdot,\cdot\rangle_B$ the associated scalar products and by $\|\cdot\|_A$ and $\|\cdot\|_B$ the induced norms, respectively.

Now, let ω be a weight function on the segment $(0,1)$ with finite moment

$$\gamma_n^\omega := \int_0^1 t^n\omega(t)dt \leq \gamma_0^\omega, \quad n = 0,1,2,\cdots. \tag{3}$$

The associated weighted solid Cauchy transform is given through

$$[C_s^\omega(f)](z) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{f(\xi)}{z-\xi}\omega(|\xi|^2)d\lambda(\xi), \quad z \in \overline{\mathbb{D}}^c. \tag{4}$$

In the sequel, we consider the action of C_s^ω on $L^2(\mathbb{D}, A)$ with possible values in $L^2(\overline{\mathbb{D}}^c, B)$, and provide sufficient conditions on ω, A and B for C_s^ω to be a bounded operator from $L^2(\mathbb{D}, A)$ into $L^2(\overline{\mathbb{D}}^c, B)$. Namely, we assume that

$$V_{\omega^2/A} := \int_0^1 \frac{\omega^2(t)}{A(t)}dt < +\infty \tag{5}$$

and

$$W_B := \int_0^1 \frac{B(1/t^2)}{t(1-t)^2}dt < +\infty, \tag{6}$$

which are sufficient conditions for the boundedness of the solid Cauchy transform. Concrete examples will be considered in Remark 2.2 below.

Proposition 2.1. *Under* (5) *and* (6), *the transform C_s^ω is a well defined bounded operator from $L^2(\mathbb{D}, A)$ into the Hilbert space $L^2(\overline{\mathbb{D}}^c, B)$.*

Proof. Let $f \in L^2(\mathbb{D}, A)$. Using Cauchy Schwarz inequality we get

$$\left|C_s^\omega(f)(z)\right| \leq \frac{1}{\pi}\left(\int_{\mathbb{D}} \frac{\omega^2(|w|^2)}{|w-z|^2A(|w|^2)}d\lambda(w)\right)^{1/2}\left\|f\right\|_A.$$

Keeping in mind and the fact that $|w-z|^{-2} \leq (|z|-1)^{-2}$ for varying $w \in \mathbb{D}$ and $z \in \overline{\mathbb{D}}^c$, the use of the polar coordinates $w = re^{i\theta}$, the fact $\int_0^{2\pi} e^{i(n-m)\theta}d\theta = 2\pi\delta_{m,n}$ and the change of variable $t = r^2$ yield

$$\begin{aligned}\left|C_s^\omega(f)(z)\right| &\leq \frac{1}{\pi}\left(\int_0^1\int_0^{2\pi} \frac{\omega^2(r^2)}{(|z|-1)^2A(r^2)}rdrd\theta\right)^{1/2}\left\|f\right\|_A \\ &\leq \frac{1}{\pi(|z|-1)}\left(\pi\int_0^1 \frac{\omega^2(t)}{A(t)}dt\right)^{1/2}\left\|f\right\|_A = \frac{\sqrt{V_{\omega^2/A}}}{\sqrt{\pi}(|z|-1)}\left\|f\right\|_A,\end{aligned} \tag{7}$$

which is finite by means of (5). Therefore, it follows

$$\begin{aligned}\left\|C_s^{\omega}(f)\right\|_B^2 &\leq 2V_{\omega^2/A}\left(\int_1^{\infty}\frac{B(r^2)}{(r-1)^2}rdr\right)\left\|f\right\|_A^2\\ &\leq 2V_{\omega^2/A}\left(\int_0^{1}\frac{B(1/t^2)}{t(1-t)^2}dt\right)\left\|f\right\|_A^2 = 2V_{\omega^2/A}W_B\left\|f\right\|_A^2.\end{aligned}$$

Under the assumptions (5) and (6), this proves the boundedness of the solid Cauchy transform C_s^{ω} viewed as operator from $L^2(\mathbb{D},A)$ into $L^2(\overline{\mathbb{D}}^c,B)$. □

Remark 2.2. *A precise estimate for the norm operator of the solid weighted Cauchy transform C_s^{ω} can be given for explicit weight functions satisfying assumptions* (5) *and* (6). *For example when considering $A(t)=\omega(t)=\omega_{\gamma}(t)=(1-t)^{\gamma}$ and $B(t)=B_{a,b}(t):=t^{-a}(t-1)^{-b}$ with $\gamma>-1>b>-a$, the evaluation of the integrals giving $V_{\omega^2/A}$ and W_B yields the following estimate*

$$\left\|C_s^{\omega}\right\|^2 \leq \frac{2^{1-b}\Gamma(2a+2b)\Gamma(-b-1)}{\pi(\gamma+1)\Gamma(2a+b-1)}. \tag{8}$$

Remark 2.3. *For $A(t)=\omega_{\gamma}(t)$ and $\omega(t)=\omega_{\alpha}(t)$ the corresponding quantity $V_{\omega_{\alpha}^2/\omega_{\gamma}}$ is finite if and only if $\gamma<1+2\alpha$.*

The concrete description of the basic properties of C_s^{ω} makes appeal to basis' property of $\mathcal{R}_{m,n}^{\gamma}$ in the underlying space.

3. Disc polynomials

The disc polynomials $\mathcal{R}_{m,n}^{\gamma}(z,\bar{z})$, $\gamma>-1$, we deal with are those defined through (2). For the hypergeometric representation in terms of the Gauss-hypergeometric function ${}_2F_1$ one can refer to e.g. [35, p. 137], [7, p. 535], or also [2, 8]. The explicit expression reads [2, 7, 22]

$$\mathcal{R}_{m,n}^{\gamma}(z,\bar{z}) = m!n!\sum_{j=0}^{m\wedge n}\frac{(-1)^j(1-z\bar{z})^j}{j!(\gamma+1)_j}\frac{z^{m-j}}{(m-j)!}\frac{\bar{z}^{n-j}}{(n-j)!} \tag{9}$$

for varying $m,n=0,1,2,\cdots$. Here, $(a)_0=1$ and $(a)_k=a(a+1)\cdots(a+k-1)$ for $k=1,2,\cdots$ denotes the Pochhammer symbol. Clearly, they are polynomials of the two conjugate complex variables $z=x+iy\in\mathbb{D}$ and $\bar{z}=x-iy$, $x,y\in\mathbb{R}$, of degree m and n, respectively. The proposed definition agrees with the ones suggested by Koornwinder [22], Dunkl [7] and Wünsche [35]. The limit case of $\gamma=-1$ leads to the so-called scattering polynomials that have emerged in the context of wave propagation in layered media [13], while for $\gamma=0$, they turn out to be related to the radial Zernike polynomials $R_k^{\nu}(x)$, introduced by Zernike [37] in his framework on optical problems involving telescopes and microscopes, and playing an important role in expressing the wavefront data in optical tests and in the study of diffraction problems [38]. More exactly, we have

$$\mathcal{R}_{m,n}^{0}(z,\bar{z}) = (m+n)!e^{i(n-m)\arg z}R_{m+n}^{n-m}(\sqrt{z\bar{z}}),\ m\leq n.$$

The orthogonality property with respect to the weighted (probability) measure

$$d\mu_{\gamma}(z) = (1-|z|^2)^{\gamma}dxdy,\ z=x+iy\in\mathbb{D}, \tag{10}$$

reads

$$\int_{\mathbb{D}}\mathcal{R}_{m,n}^{\gamma}(z,\bar{z})\overline{\mathcal{R}_{j,k}^{\gamma}(z,\bar{z})}d\mu_{\gamma}(z) = d_{m,n}^{\gamma}\delta_{m,j}\delta_{n,k}, \tag{11}$$

and follows by a straightforward computation from its analog for the Jacobi polynomials. The involved constant $d^{\gamma}_{m,n}$ is explicitly given by

$$d^{\gamma}_{m,n} = \frac{\pi m!n!}{(\gamma+1+m+n)(\gamma+1)_m(\gamma+1)_n}. \tag{12}$$

For ulterior use, mainly for proving the convergence of the relevant series, we recall the following estimate $|\mathcal{R}^{\gamma}_{m,n}(z,\bar{z})| \leq 1$. It holds true for every nonnegative integers m and n, real $\gamma > -1$ and $z \in \mathbb{D}$ (see e.g. [7, 19]).

The proof of our main results relies essentially in the fact that $\mathcal{R}^{\gamma}_{m,n}(z,\bar{z})$ form an orthogonal basis of the Hilbert space $L^{2,\gamma}(\mathbb{D})$ as quoted in [7, 19, 35]. Here, for $\gamma > -1$, $L^{2,\gamma}(\mathbb{D})$ denotes the Hilbert space of complex-valued functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}$ endowed with the norm induced from the scalar product

$$\langle f, g\rangle_{\gamma} := \int_{\mathbb{D}} f(z)\overline{g(z)}(1-|z|^2)^{\gamma} dxdy, \quad z = x+iy \in \mathbb{D}.$$

Proposition 3.1 ([7, 19, 35]). *The disc polynomials $\mathcal{R}^{\gamma}_{m,n}(z,\bar{z})$ form a complete orthogonal system in the Hilbert space $L^{2,\gamma}(\mathbb{D})$.*

From the explicit expression of $\mathcal{R}^{\gamma}_{m,n}$ in (9), it is clear that they are a good class of polyanalytic functions of finite order in the unit disc. In fact, n-analyticity (polyanalytic of order n) is characterized by those functions of the form

$$f(z,\bar{z}) = \sum_{j=0}^{n-1} \bar{z}^j \varphi_j(z),$$

with φ_j being holomorphic functions in $\mathbb{D}$. This is equivalent to be uniquely expressed as [6]

$$f(z,\bar{z}) = P(z,\bar{z}) + \sum_{j=0}^{n-1} (1-|z|^2)^j \psi_j(z),$$

where $P(z,\bar{z})$ is a polynomial of degree $n-1$ in z and degree at most $n-1$ in $\bar{z}$, and ψ_j are holomorphic in $\mathbb{D}$.

In the next section, we will explore the crucial role played by these polynomials in describing the so-called weighted true poly-Bergman spaces.

4. Weighted true poly-Bergman spaces

A weighted poly-Bergman space in $\mathbb{D}$ is defined as a specific generalization of the weighted Bergman space to the polyanalytic setting [3]. The extension of the classical Bergman space $\mathcal{A}^2(\mathbb{D}) := L^2(\mathbb{D}, dxdy) \cap \mathcal{H}ol(\mathbb{D})$ to the context of polyanalytic functions was proposed by Koshelev [20], who proved that the set $\mathcal{H}^2_n(\mathbb{D})$ of $(n+1)$-analytic complex-valued functions in $\mathbb{D}$ belonging to $L^2(\mathbb{D}, dxdy)$ is a reproducing kernel Hilbert space for which the functions

$$e_{m,p}(z) := \frac{\sqrt{m+p+1}}{\sqrt{\pi}(m+p)!} \frac{\partial^{m+p}}{\partial z^p \partial \bar{z}^m} \left((|z|^2-1)^{m+p}\right), \tag{13}$$

for varying $p = 0, 1, 2, \cdots, n$ and $m = 0, 1, 2, \cdots$, form a complete orthonormal polynomial system. Subsequently, the so-called true poly-Bergman spaces are the particular subspaces

$$\mathcal{A}^2_n(\mathbb{D}) = \left\{f : f(z) = \partial^n_z \left((1-z\bar{z})^n F(z)\right), F \in \mathcal{A}(\mathbb{D}), f \in \mathcal{H}^2_n(\mathbb{D})\right\},$$

considered by Ramazanov in [30]. They give rise to the piecewise decomposition

$$\mathcal{H}^2_n(\mathbb{D}) = \mathcal{A}^2_0(\mathbb{D}) \oplus \mathcal{A}^2_1(\mathbb{D}) \oplus \ldots \oplus \mathcal{A}^2_n(\mathbb{D}),$$

so that $\mathcal{A}_0^2(\mathbb{D}) = \mathcal{A}^2(\mathbb{D}) = \mathcal{H}_0^2$. The involved operator $D_n f := \partial_z^n((1-|z|^2)^n u(z))$ defines a bounded and boundedly invertible operator from $\mathcal{A}^2(\mathbb{D})$ onto $\mathcal{A}_n^2(\mathbb{D})$ (see e.g. [30, Theorem 1]). For more details on these spaces, one can refer to [20, 30, 32, 33].

The weighted version $\mathcal{A}^{2,\gamma}(\mathbb{D})$ of the Bergman space is defined as the closed subspace of the holomorphic functions belonging to the Hilbert space $L^{2,\gamma}(\mathbb{D}) := L^2\left(\mathbb{D}, d\mu_\gamma\right)$. It is a reproducing kernel Hilbert space with kernel given by

$$K^\gamma(z,w) = \frac{\gamma+1}{\pi\,(1-z\overline{w})^{\gamma+2}}. \tag{14}$$

Its orthogonal in $L^{2,\gamma}(\mathbb{D})$ makes appeal to the n-th weighted poly-Bergman space $H_n^{2,\gamma}(\mathbb{D})$ in [9, 16, 31, 33] defined as the space of complex-valued functions $f = \mathbb{D} \longrightarrow \mathbb{C}$ that are square integrable on $\mathbb{D}$ with respect to a given radial weight function $\omega_\gamma(|\zeta|^2)$ and satisfying the generalized Cauchy–Riemann equation $\partial_{\overline{z}}^{n+1} f = 0$. Thus,

$$H_n^{2,\gamma}(\mathbb{D}) := \ker(\partial_{\overline{z}}^{n+1}) \cap L^{2,\gamma}(\mathbb{D}).$$

It should be mentioned here that for $\gamma = 0$ and $\omega_0 = 1$, the spaces $\mathcal{A}_n^{2,0}(\mathbb{D})$ reduces further to the conventional unweighted poly-Bergman spaces $\mathcal{A}_n^2(\mathbb{D})$.

The disc polynomials discussed in Section 3 are crucial in providing concrete description of $H_n^{2,\gamma}(\mathbb{D})$. The next result is a minor variant of the one obtained by Ramazanov in [31], and shows in addition that the introduced spaces can equivalently be defined by mean of the disc polynomials $\mathcal{R}_{m,n}^\gamma$.

Theorem 4.1. *The weighted poly-Bergman spaces $H_n^{2,\gamma}(\mathbb{D})$ are closed subspaces of $L^{2,\gamma}(\mathbb{D})$. Moreover, they coincide with those spanned by the disc polynomials $\mathcal{R}_{m,k}^\gamma$ for varying $m = 0, 1, 2, \cdots$ and $k = 0, 1, \cdots, n$.*

Proof. We begin by proving that each weighted poly-Bergman space $H_n^{2,\gamma}(\mathbb{D})$ is a closed subspace of $L^{2,\gamma}(\mathbb{D})$ and spanned by the disc polynomials $\mathcal{R}_{m,k}^\gamma$ for varying $m = 0, 1, 2, \cdots$ and $k = 0, 1, \cdots, n$. To this end, notice first that any $f \in L^{2,\gamma}(\mathbb{D})$ can be expanded as

$$f(z) = \sum_{m=0}^{\infty}\sum_{j=0}^{\infty} a_{m,j}\mathcal{R}_{m,j}^\gamma(z,\overline{z}) \tag{15}$$

for some complex-valued constants $a_{m,n}$ satisfying the growth condition

$$\left\|f\right\|_\gamma^2 = \sum_{m=0}^{\infty}\sum_{j=0}^{\infty} d_{m,j}^\gamma |a_{m,j}|^2 < +\infty.$$

The series in (15) is absolutely and uniformly convergent on compact sets of $\mathbb{D}$. Therefore, by straightforward computation, we arrive at

$$\partial_{\overline{z}}^{k+1} f(z) = n! \sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{(\alpha+m+1)_\ell}{\ell!(\gamma+m+k+2)_\ell} a_{m,\ell+k+1}\mathcal{R}_{m,\ell}^{\gamma+k+1}(z,\overline{z}).$$

This follows since [2, 22, 35]

$$\partial_{\overline{z}}^k \mathcal{R}_{m,j}^\gamma = \varepsilon_{j-k}\frac{n!(\gamma+m+1)_k(\gamma+1)_{m+j-k}}{(j-k)!(\gamma+1)_{m+j}}\mathcal{R}_{m,j-k}^{\gamma+k},$$

where for every integer ℓ we have set

$$\varepsilon_\ell = \begin{cases} 1 & \text{if} \quad \ell \geq 0, \\ 0 & \text{if} \quad \ell < 0. \end{cases} \tag{16}$$

Subsequently, the constants $a_{m,\ell+k+1}$ are closely connected to the Fourier coefficients of $\partial_{\overline{z}}^{k+1} f \in L^{2,\gamma+k+1}(\mathbb{D})$ (see e.g. [19]). More precisely, we have

$$a_{m,\ell+k+1} = \frac{\ell!(\gamma+m+k+2)_\ell}{n!(\alpha+m+1)_\ell d_{m,l}^{\gamma+k+1}} \left\langle \partial_{\overline{z}}^{k+1} f, \mathcal{R}_{m,\ell}^{\gamma+k+1} \right\rangle_{\gamma+k+1}.$$

Thus, $f \in H_n^{2,\gamma}(\mathbb{D})$ is equivalent to $a_{m,\ell+k+1} = 0$ for any nonnegative integers m, ℓ. This infers

$$f(z) = \sum_{m=0}^{\infty} \sum_{j=0}^{n} a_{m,j} \mathcal{R}_{m,j}^{\gamma}(z, \overline{z}) \tag{17}$$

in $L^{2,\gamma}(\mathbb{D})$ and hence

$$H_n^{2,\gamma}(\mathbb{D}) = \overline{Span\{\mathcal{R}_{m,j}^{\gamma}, m = 0, 1, \cdots, j = 0, 1, \cdots, n\}}^{L^{2,\gamma}(\mathbb{D})},$$

which is clearly a closed subspace of $L^{2,\gamma}(\mathbb{D})$. This completes the proof. $\square$

The next result concerns the so-called the n-th true weighted poly-Bergman space defined in [31] and realizable as

$$\mathcal{A}_n^{2,\gamma}(\mathbb{D}) = H_n^{2,\gamma}(\mathbb{D}) \ominus H_{n-1}^{2,\gamma}(\mathbb{D}) \tag{18}$$

for $n \geq 1$ with $\mathcal{A}_0^{2,\gamma}(\mathbb{D}) = \mathcal{A}^{2,\gamma}(\mathbb{D})$. It also gives the explicit closed expression of its reproducing kernel $K_n^{\gamma}(z, w)$ in terms of the Gauss hypergeometric function

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| x \right) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j} \frac{x^j}{j!}.$$

Theorem 4.2. *The following assertions hold trues.*

(i) For a fixed nonnegative integer n, the family of functions $\mathcal{R}_{m,n}^{\gamma}$, for varying $m = 0, 1, 2, \cdots$, forms an orthogonal basis of $\mathcal{A}_n^{2,\gamma}(\mathbb{D})$.

(ii) The spaces $\mathcal{A}_n^{2,\gamma}(\mathbb{D})$ form an orthogonal sequence of reproducing kernel Hilbert spaces in $L^{2,\gamma}(\mathbb{D})$. Moreover, we have

$$L^{2,\gamma}(\mathbb{D}) = \bigoplus_{n=0}^{\infty} \mathcal{A}_n^{2,\gamma}(\mathbb{D}) \quad \text{and} \quad H_n^{2,\gamma}(\mathbb{D}) = \bigoplus_{k=0}^{n} \mathcal{A}_k^{2,\gamma}(\mathbb{D}).$$

(iii) The reproducing kernel of $\mathcal{A}_n^{2,\gamma}(\mathbb{D})$ is given by

$$\begin{aligned} K_n^{\gamma}(z, w) &= \frac{(\gamma+n+1)}{\pi n!(\gamma+1)_n} \left(1 - |z|^2\right)^{-\gamma} \left(1 - |w|^2\right)^{-\gamma} \\ &\quad \times \partial_z^n \partial_{\overline{w}}^n \left\{ \left(1 - |z|^2\right)^{\gamma+n} \left(1 - |w|^2\right)^{\gamma+n} {}_2F_1\left(\begin{matrix} \gamma+n+2, \gamma+1 \\ \gamma+n+1 \end{matrix} \middle| z\overline{w} \right) \right\}. \end{aligned} \tag{19}$$

Proof. As an immediate consequence of the discussion in the proof of Theorem 4.1, we claim that the orthogonal Hilbertian decompositions

$$L^{2,\gamma}(\mathbb{D}) = \bigoplus_{n=0}^{\infty} \mathcal{A}_n^{2,\gamma}(\mathbb{D}) \quad \text{and} \quad H_n^{2,\gamma}(\mathbb{D}) = \bigoplus_{k=0}^{n} \mathcal{A}_k^{2,\gamma}(\mathbb{D})$$

hold trues. In fact, we need only to prove that the functions $\mathcal{R}_{m,n}^{\gamma}$, $m = 0, 1, 2, \cdots$, constitute an orthogonal basis of $\mathcal{A}_n^{2,\gamma}(\mathbb{D}) = H_n^{2,\gamma}(\mathbb{D}) \ominus H_{n-1}^{2,\gamma}(\mathbb{D})$. Indeed, by considering

$$\mathcal{B}_n^{2,\gamma}(\mathbb{D}) := \overline{Span\left\{\mathcal{R}_{m,n}^{\gamma}, \ m = 0, 1, 2, \cdots\right\}}^{L^{2,\gamma}(\mathbb{D})},$$

it is clear that they form an orthogonal sequence of Hilbert spaces in $L^{2,\gamma}(\mathbb{D})$. The orthogonality follows from the orthogonality property for the disc polynomials. Moreover, we

$$H_n^{2,\gamma}(\mathbb{D}) = \bigoplus_{k=0}^{n} \mathcal{B}_k^{2,\gamma}(\mathbb{D})$$

follows due to Theorem 1.1 and the fact that $\mathcal{B}_n^{2,\gamma}(\mathbb{D}) \subset \mathcal{A}_n^{2,\gamma}(\mathbb{D})$. The latter one can be handled by induction. Hence $\mathcal{B}_n^{2,\gamma}(\mathbb{D}) = \mathcal{A}_n^{2,\gamma}(\mathbb{D})$. The fact that $\mathcal{A}_n^{2,\gamma}(\mathbb{D})$ is a reproducing kernel Hilbert space follows mainly using Riesz' representation theorem for the linear mapping $f \longmapsto f(z_0)$, for fixed z_0, being continuous. Subsequently, there exists an element $K_{n,z_0}^{\gamma} = K_n^{\gamma}(\cdot, z_0) \in \mathcal{A}_n^{2,\gamma}(\mathbb{D})$ such that $f(z) = \left\langle f, K_{n,z_0}^{\gamma} \right\rangle_{\gamma}$ for every $f \in \mathcal{A}_n^{2,\gamma}(\mathbb{D})$. Indeed, by expanding any $f \in \mathcal{A}_n^{2,\gamma}(\mathbb{D})$ as $f(z) = \displaystyle\sum_{m=0}^{\infty} a_m \mathcal{R}_{m,n}^{\gamma}(z,\overline{z})$ and next making use of the Cauchy Schwarz inequality, keeping in mind the fact that $|\mathcal{R}_{m,n}^{\gamma}(z,\overline{z})| \leq 1$, we get

$$|f(z)| \leq \sum_{m=0}^{\infty} |a_m| \leq c \left(\sum_{m=0}^{\infty} d_{m,n}^{\gamma} |a_m|^2 \right)^{1/2} = c \|f\|_{\gamma},$$

where the square of the involved nonnegative real number c is given by

$$c^2 := \sum_{m=0}^{\infty} \frac{1}{d_{m,n}^{\gamma}} = \frac{(\gamma+n+1)(\gamma+1)_n}{\pi n!} {}_2F_1\left(\begin{matrix} \gamma+n+2, \gamma+1 \\ \gamma+n+1 \end{matrix} \middle| 1 \right).$$

This completes our check of (*i*) and (*ii*).

To prove (*iii*), we recall that from the general theory of reproducing kernels for separable Hilbert spaces we know that the reproducing kernel function K_n^{γ} is uniquely determined by

$$K_n^{\gamma}(z,w) = \sum_{m=0}^{+\infty} \psi_m^{\gamma,n}(z) \overline{\psi_m^{\gamma,n}(w)}$$

for any complete orthonormal basis $\psi_m^{\gamma,n}$ of $\mathcal{A}_n^{2,\gamma}(\mathbb{D})$. Thus, the closed formula for $K_n^{\gamma}(z,w)$ involves successive derivatives of a special function ${}_2F_1$. In fact, by means of $\overline{\mathcal{R}_{m,n}^{\gamma}(w,\overline{w})} = \mathcal{R}_{m,n}^{\gamma}(\overline{w},w)$ as well as the observation that the disc polynomials can be realized as

$$\mathcal{R}_{m,n}^{\gamma}(z,\overline{z}) = \frac{(-1)^m}{(\gamma+1)_m} (1-|z|^2)^{-\gamma} \partial_z^n \left(z^m (1-|z|^2)^{\gamma+n} \right), \tag{20}$$

we get

$$\begin{aligned} K_n^{\gamma}(z,w) &= \sum_{m=0}^{+\infty} \frac{\mathcal{R}_{m,n}^{\gamma}(z,\overline{z}) \overline{\mathcal{R}_{m,n}^{\gamma}(w,\overline{w})}}{d_{m,n}^{\gamma}} \\ &= \frac{(1-|z|^2)^{-\gamma}(1-|w|^2)^{-\gamma}}{[(\gamma+1)_n]^2} \partial_z^n \partial_{\overline{w}}^n \left\{ (1-|z|^2)^{\gamma+n} (1-|w|^2)^{\gamma+n} N_n^{\gamma}(z,w) \right\} \end{aligned}$$

with

$$\begin{aligned} N_n^{\gamma}(z,w) &:= \sum_{m=0}^{+\infty} \frac{z^m \overline{w}^m}{d_{m,n}^{\gamma}} = \frac{(\gamma+1)_n}{\pi n!} \sum_{m=0}^{+\infty} (\gamma+m+n+1)(\gamma+1)_m \frac{z^m \overline{w}^m}{m!} \\ &= \frac{(\gamma+n+1)(\gamma+1)_n}{\pi n!} {}_2F_1\left(\begin{matrix} \gamma+n+2, \gamma+1 \\ \gamma+n+1 \end{matrix} \middle| z\overline{w} \right). \end{aligned}$$

The last equality follows making use of the fact that

$$(\gamma + m + n + 1) = \frac{(\gamma + n + 1)_m (\gamma + 1)_n}{(\gamma + 1)_m},$$

keeping in mind the definition of the hypergeometric function ${}_2F_1$. This completes the check of the closed expression in (19). □

Remark 4.3. *We have* $\mathcal{A}_n^{2,\gamma}(\mathbb{D}) = \overline{Span\left\{\mathcal{R}_{m,n}^{\gamma},\ m = 0, 1, 2, \cdots\right\}}^{L^{2,\gamma}(\mathbb{D})}$ *and the sequential characterization is given by*

$$\mathcal{A}_n^{2,\gamma}(\mathbb{D}) = \left\{ \sum_{m=0}^{+\infty} \alpha_m \mathcal{R}_{m,n}^{\gamma}(\xi, \overline{\xi}), \quad \sum_{m=0}^{\infty} \frac{m!}{(\gamma + 1 + m + n)(\gamma + 1)_m} |\alpha_n|^2 < +\infty \right\}. \tag{21}$$

Remark 4.4. *The spaces* $\mathcal{A}_n^{2,0}(\mathbb{D})$*, for* $\gamma = 0$*, are exactly the true poly-Bergman spaces in [30, 33], and the polynomials* $\mathcal{R}_{p,m}^0$ *reduce further to the* $e_{m,p}$ *in* (13)*. In this case, we recover the closed expression obtained in [20, Theorem 2] and [30, Theorem 3] for the reproducing kernel of* $H_n^{2,\gamma}(\mathbb{D})$ *and* $\mathcal{A}_n^{2,\gamma}(\mathbb{D})$*, respectively.*

Remark 4.5. *For* $n = 0$ *we recover the closed expression of the classical weighted Bergman space since*

$${}_2F_1\left(\begin{matrix} \gamma + 2, \gamma + 1 \\ \gamma + 1 \end{matrix} \middle| z\overline{w} \right) = (1 - z\overline{w})^{-\gamma - 2}.$$

Remark 4.6. *The orthogonal projection from* $L^{2,\gamma}(\mathbb{D})$ *onto* $\mathcal{A}_n^{2,\gamma}(\mathbb{D})$ *is given by the integral operator*

$$\mathcal{P}_n^{\gamma}(f)(\zeta) = \int_{\mathbb{D}} f(z) K_n^{\gamma}(z, \zeta) d\mu_{\gamma}(z),$$

while the one of $L^{2,\gamma}(\mathbb{D})$ *onto* $H_n^{2,\gamma}(\mathbb{D})$ *is given by*

$$\widetilde{\mathcal{P}_n^{\gamma}}(f)(\zeta) = \int_{\mathbb{D}} f(z) \left(\sum_{k=1}^{n} K_n^{\gamma}(z, \zeta) \right) d\mu_{\gamma}(z).$$

5. The range and null space of $C_s^{\omega_\alpha}$

In order to explore the basic properties of the weighted solid Cauchy transform

$$C_s^{\omega} f(z) := \frac{1}{\pi} \int_{\mathbb{D}} \frac{f(\xi)}{z - \xi} \omega(|\xi|^2) dx dy, \quad z \in \overline{\mathbb{D}}^c, \tag{22}$$

with respect to the specific weight function $\omega = \omega_\alpha$, $\omega_\alpha(|\xi|^2) := (1 - |\xi|^2)^\alpha$, and provide a concrete description of the null space as well as the range of its restriction to the true weighted poly-Bergman spaces, we give the explicit action of $C_s^{\omega_\alpha}$ on the disc polynomials $\mathcal{R}_{m,n}^{\gamma}$ and we specify the weight function $A(t) = \omega_\gamma(t) = (1 - t)^\gamma$. To this end, we begin by giving its action on the generic functions

$$e_{jk}^{\ell}(\xi, \overline{\xi}) = \xi^j \overline{\xi}^k (1 - |\xi|^2)^\ell$$

for nonnegative integers j, k, ℓ.

Lemma 5.1. *We have*

$$\begin{cases} \left[C_s^\omega(e^\ell_{k+m,k})\right](z) = 0 & \text{for} \quad m > 0, \\ \left[C_s^\omega(e^\ell_{k,k+m})\right](z) = \gamma^\omega_{k,s}\frac{1}{z^{m+1}} & \text{for} \quad m \geq 0, \end{cases} \tag{23}$$

where $\gamma^\omega_{k,s}$ *stands for*

$$\gamma^\omega_{k,s} = \int_0^1 t^k(1-t)^s\omega(t)dt. \tag{24}$$

Proof. Notice first that for any $z \in \overline{\mathbb{D}}^c$ and $\xi \in \mathbb{D}$ we have $\xi/z \in \mathbb{D}$. Then, by expanding the kernel function as power series in ξ, we obtain

$$\begin{aligned} \left[C_s^\omega(e^s_{jk})\right](z) &= \frac{1}{\pi}\sum_{l=0}^{+\infty}\frac{1}{z^{l+1}}\int_{\mathbb{D}} \xi^{j+l}(\overline{\xi})^k(1-|\xi|^2)^s\omega\left(|\xi|^2\right)d\lambda(\xi) \\ &= \sum_{l=0}^{+\infty}\frac{1}{z^{l+1}}\left(\int_0^1 t^k(1-t)^s\omega(t)dt\right)\delta_{j+l,k} \\ &= \varepsilon_{k-j}\frac{\gamma^\omega_{k,s}}{z^{k-j+1}}, \end{aligned} \tag{25}$$

which is equivalent to (23). The two last equations follow using the polar coordinates $\xi = re^{i\theta}$ and the change of variable $r^2 = t$ keeping in mind the definition of $\gamma^\omega_{k,s}$ and ε_ℓ given through (24) and (16), respectively. □

Accordingly, it is clear from Lemma 5.1 that the holomorphic functions e^s_{jk} belong to $\ker(C^\omega_s)$ for any s and any $j > k$. Moreover, we assert the following.

Proposition 5.2. *The function* $C^\omega_s(e^s_{jk})$ *belongs to* $L^2(\overline{\mathbb{D}}^c, B)$ *and its square norm is given by*

$$\left\|C^\omega_s(e^s_{jk})\right\|^2_B = \pi\varepsilon_{k-j}(\gamma^\omega_{k,s})^2\int_0^1 \frac{B(1/u)}{u^{k-j+1}}du.$$

Moreover, the system $\left(C^\omega_s(e^s_{jk})\right)_{j,s}$ *is orthogonal in* $L^2(\overline{\mathbb{D}}^c, B)$ *for every fixed* k.

Proof. For the proof we compute $\left\langle C^\omega_s(e^s_{jk}), C^\omega_s(e^r_{mn})^{\omega_\alpha}\right\rangle_B$ for arbitrary m, n, j, k. Indeed, by (25) we get

$$\begin{aligned} \left\langle C^\omega_s(e^s_{jk}), C^\omega_s(e^r_{mn})\right\rangle_B &= \varepsilon_{k-j}\varepsilon_{n-m}\gamma^\omega_{k,s}\gamma^\omega_{n,r}\int_{\overline{\mathbb{D}}^c}\frac{1}{z^{k-j+1}\overline{z}^{n-m+1}}B(|z|^2)d\lambda(z) \\ &= \pi\varepsilon_{k-j}\gamma^\omega_{k,s}\gamma^\omega_{n,r}\delta_{k-j,n-m}\int_1^{+\infty}\frac{1}{t^{k-j+1}}B(t)dt \\ &= \pi\varepsilon_{k-j}\gamma^\omega_{k,s}\gamma^\omega_{n,r}\delta_{k-j,n-m}\int_0^1 u^{k-j-1}B(1/u)du. \end{aligned} \tag{26}$$

This proves that $C^\omega_s(e^s_{jk}) \in L^2(\overline{\mathbb{D}}^c, B)$. The orthogonality of the system $\left(C^\omega_s(e^s_{jk})\right)_{j,s}$ in $L^2(\overline{\mathbb{D}}^c, B)$ (for fixed k) follows as particular case by taking $n = k$. The explicit expression of the norm is exactly (26) with $m = j$ and $n = k$. □

Remark 5.3. *In view of* (26) *it is clear that the family* $\left(C_s^\omega(e_{jk}^s)\right)_{j,k,s}$ *is not orthogonal.*

Using Lemma 5.1, we give the explicit action of $C_s^{\omega_\alpha}$ on the disc polynomials.

Proposition 5.4. *Let* $\gamma > -1$. *For every nonnegative integers* m, n, *there exists some constant* $c_{m,n}^{\gamma,\omega}$ *depending on* γ, ω, *m and n such that*

$$C_s^\omega(\mathcal{R}_{m,n}^\gamma)(z) = c_{m,n}^{\gamma,\omega}\frac{z^m}{z^{n+1}}.$$

For the weight function $\omega(t) = \omega_\alpha(t) = (1-t)^\alpha$, *the involved constant is given explicitly by*

$$c_{m,n}^{\gamma,\omega_\alpha} = \varepsilon_{n-m}\frac{(\gamma-\alpha)_m n!}{(\alpha+1)_{n+1}(\gamma+1)_m}.$$

Proof. By setting

$$a_{m,n}^{\gamma,j} = \frac{(-1)^j m! n!}{(\gamma+1)_j j!(m-j)!(n-j)!},$$

we can rewrite the disc polynomials (9) by means of the generic elements $e_{n-j,m-j}^j$ as

$$\mathcal{R}_{m,n}^\gamma(z,\bar{z}) = \sum_{j=0}^{m\wedge n} a_{m,n}^{\gamma,j} e_{m-j,n-j}^j(z,\bar{z}).$$

Accordingly, the linearity of the weighted Cauchy transform and Lemma 5.1 show that

$$C_s^\omega(\mathcal{R}_{m,n}^\gamma)(\xi) = \varepsilon_{n-m}\left(\sum_{j=0}^m a_{m,n}^{\gamma,j}\gamma_{n-j,j}^\omega\right)\frac{1}{z^{n-m+1}} = c_{m,n}^{\gamma,\omega}\frac{z^m}{z^{n+1}}.$$

For the explicit computation of the involved finite sum, that we denote by $S_{m,n}^{\gamma,\alpha}$, it should be noticed that the quantity $\gamma_{n-j,j}^{\omega_\alpha}$ reduces further to a beta function when restricting ourself to the weight function $\omega(t) := \omega_\alpha(t) = (1-t)^\alpha$. More exactly, we have

$$\gamma_{n-j,j}^{\omega_\alpha} = \int_0^1 t^{n-j}(1-t)^{\alpha+j}dt = \frac{(n-j)!(\alpha+1)_j}{(\alpha+1)_{n+1}}.$$

Therefore, it turns out to be the value of a Gauss hypergeometric function at 1. Indeed, we have

$$S_{m,n}^{\gamma,\alpha} = \frac{n!}{(\alpha+1)_{n+1}}\sum_{j=0}^m \frac{(-m)^j(\alpha+1)_j}{j!(\gamma+1)_j} = \frac{n!}{(\alpha+1)_{n+1}}\,{}_2F_1\left(\begin{matrix}-m, \alpha+1\\ \gamma+1\end{matrix}\,\middle|\,1\right).$$

Now, the use of the well-known Chu–Vandermonde identity,

$${}_2F_1\left(\begin{matrix}-m, b\\ c\end{matrix}\,\middle|\,1\right) = \frac{(c-b)_m}{(c)_m},$$

yields the explicit expression of the constant $c_{m,n}^{\gamma,\omega}$ for $\omega(t) = \omega_\alpha(t)$. Indeed,

$$c_{m,n}^{\gamma,\omega_\alpha} = \varepsilon_{n-m}\frac{(\gamma-\alpha)_m n!}{(\alpha+1)_{n+1}(\gamma+1)_m}.$$

□

Some immediate consequences readily follow.

Corollary 5.5. *The following assertions hold trues.*

1) *The range of $C_s^{\omega_\alpha}$ restricted to $\mathcal{A}_n^{2,\gamma}(\mathbb{D})$ is a finite dimensional vector space whose dimension does not exceed $n+1$.*

2) *$\mathcal{R}_{m,n}^{\gamma} \in \ker(C_s^{\omega_\alpha})$ for any $m > n$.*

3) *We have $C_s^{\omega_\alpha}(\mathcal{R}_{m,n}^{\gamma}) = const.C_s(e_{n,m}^{s})$ for fixed n and varying $m = 0, 1, 2, \cdots$. Moreover, they form an orthogonal system of holomorphic functions in $L^2(\overline{\mathbb{D}}^c, B)$.*

4) *For $\alpha = \gamma$, the involved constant is exactly zero for any $m > 0$, while for $m = 0$ the action reduces to*

$$C_s^{\omega_\alpha}(\mathcal{R}_{0,n}^{\gamma})(\xi) = \frac{n!}{(\gamma+1)_{n+1}} \frac{1}{z^{n+1}}.$$

With the background presented in this section we can prove our main result (Theorem 1.1) and therefore its corollaries.

Proof of Theorem 1.1:

Notice first that since we are placed in the case of $\omega = \omega_\alpha$, we have to assume that $\alpha > (\gamma-1)/2 > -1$ (by Remark 2.3) to guarantee the boundedness of $C_s^{\omega_\alpha}$, the finiteness of the weight function ω_γ and therefore the fact that the disc polynomials is an orthogonal basis for $L^{2,\gamma}(\mathbb{D})$.

According to the explicit expression of the constant $c_{m,n}^{\gamma,\omega_\alpha}$ in Proposition 5.4, it is clear that $C_s^{\omega_\alpha}(\mathcal{R}_{m,n}^{\gamma}) \neq 0$ if and only if $m \geq n$ and $(\gamma-\alpha)_m \neq 0$. In particular, $Span\{\mathcal{R}_{m,n}^{\gamma}, m < n\} \subset \ker(C_s^{\omega_\alpha}|_{\mathcal{A}_n^{2,\gamma}(\mathbb{D})})$. For the determination of $\dim(C_s^{\omega_\alpha}(\mathcal{A}_n^{2,\gamma}(\mathbb{D}))) \leq n+1$, two cases are to be distinguished $\gamma - \alpha \in \mathbb{Z}_0^- = \{0, -1, -2, \cdots\}$ and $\gamma - \alpha \notin \mathbb{Z}_0^-$.

If $\gamma - \alpha \notin \mathbb{Z}_0^-$, then $(\gamma-\alpha)_m$ is not zero for every nonnegative integer $m \geq 0$. Hence $C_s^{\omega_\alpha}(\mathcal{R}_{m,n}^{\gamma}) = 0$ if and only if $m > n$. Subsequently, the restriction of the solid weighted Cauchy transform $C_s^{\omega_\alpha}$ to $\mathcal{A}_n^{2,\gamma}(\mathbb{D})$ is spanned by

$$\frac{z^m}{z^n+1}, \quad m = 0, 1, 2, \cdots, n.$$

In this case the dimension of $C_s^{\omega_\alpha}(\mathcal{A}_n^{2,\gamma}(\mathbb{D}))$ not infected by the choice of the weight functions and is equal to $n+1$.

Now, for $\gamma - \alpha \in \mathbb{Z}_0^-$ it is not hard to rewrite the result of Proposition 5.4 as

$$C_s^{\omega_\alpha}(\mathcal{R}_{m,n}^{\gamma})(\xi) = \begin{cases} 0 & \text{if } m > n \\ \dfrac{(\gamma-\alpha)_m n!}{(\alpha+1)_{n+1}(\gamma+1)_m} \dfrac{z^m}{z^{n+1}} & \text{if } m \leq \min(n, \alpha-\gamma) \\ 0 & \text{if } \alpha - \gamma + 1 \leq m \leq n. \end{cases}$$

Therefore, $C_s^{\omega_\alpha}(\mathcal{R}_{m,n}^{\gamma}) \neq 0$ if and only if $m \leq \min(n, \alpha-\gamma)$. It follows

$$C_s^{\omega_\alpha}(\mathcal{A}_n^{2,\gamma}(\mathbb{D})) = Span\left\{\frac{z^m}{z^{n+1}}, 0 \leq m \leq \min(n, \alpha-\gamma)\right\}$$

and

$$\ker(C_s^{\omega_\alpha}|_{\mathcal{A}_n^{2,\gamma}(\mathbb{D})}) = Span\left\{\mathcal{R}_{m,n}^{\gamma}, m > \min(n, \alpha-\gamma)\right\}.$$

Thence the dimension of $C_s^{\omega_\alpha}(\mathcal{A}_n^{2,\gamma}(\mathbb{D}))$ is finite and given by $N = \min(n, \alpha-\gamma) + 1$. Moreover, it is clearly observed that the family of subspace $C_s^{\omega_\alpha}(\mathcal{A}_n^{2,\gamma}(\mathbb{D}))$, for varying $n = 0, 1, \cdots$, constitutes an increasing sequence. □

Remark 5.6. *For $\gamma = \alpha$, the ranges $C_s^{\omega_\alpha}(\mathcal{A}_n^{2,\gamma}(\mathbb{D}))_n$ are all of dimension one.*

Remark 5.7. *The case of $\gamma > \alpha > (\gamma - 1)/2 > -1$ is clearly contained in the case of $\gamma - \alpha \notin \mathbb{Z}_0^-$. While the case of $-1 < \gamma < \alpha$ depends on the quantization of $\gamma - \alpha$.*

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