Filomat 37:30 (2023), 10133–10144
https://doi.org/10.2298/FIL2330133A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Simpson's and Newton's type inequalities in multiplicative fractional calculus

Muhammad Aamir Ali[a]

[a] *School of Mathematical Sciences, Nanjing Normal University, China*

Abstract. In this paper, we prove two multiplicative fractional integral identities involving multiplicative differentiable functions. Then, with the help of newly established identities, we establish multiplicative fractional versions of Simpson's and Newton's formulas type inequalities for differentiable multiplicative convex functions. It is also shown that the newly proved inequalities are extensions of some existing inequalities within the literature.

1. Introduction

Between1967 and 1970, Grossman and Katz, created the 1st non-Newtonian calculation system, called geometric calculation. Over the next few years they had created an infinite family of non-Newtonian calculi, thus modifying the classical calculus introduced by Newton and Leibniz in the 17th century each of which differed markedly from the classical calculus of Newton and Leibniz known today as the non-Newtonian calculus or the multiplicative calculus, where the ordinary product and ratio are used respectively as the sum and exponential difference over the domain of positive real numbers see [17]. This calculation is useful for dealing with exponentially varying functions. It is worth noting that the complete mathematical description of multiplicative calculus was given by Bashirov et al. [9].

Since the applications of multiplicative calculus are relatively limited than the calculus of Newton and Leibniz. Therefore a well-developed tool with a wider scope has already been made, and the question of whether it is fair to design a new tool with a limited scope arises. The solution is comparable to why mathematicians use a polar coordinate system when a rectangular coordinate system better describes points on a plane exists. We believe that the mathematical instrument of multiplicative calculus can be particularly helpful for the study of economics and finance.

Assume for motivation's sake that by depositing \$$a$, one will receive \$$b$ after a year. The original number then fluctuates b/a times. How frequently does it change each month? Assume that the change over a month is p times for this. The total then becomes $b = ap^{12}$ for a year. The formula for computing p is now $p = (b/a)^{\frac{1}{12}}$. Assuming that deposits fluctuate daily, hourly, minutely, secondarily, etc. and that the function

2020 *Mathematics Subject Classification*. 26D10, 26A51, 26D15.
Keywords. Simpson's and Newton's inequalities; Multiplicative calculus; Convex and multiplicative convex functions.
Received: 31 March 2023; Accepted: 19 June 2023
Communicated by Dragan S. Djordjević
This work is partially supported by National Natural Science Foundation of China (No. 11971241).
Email address: mahr.muhammad.aamir@gmail.com (Muhammad Aamir Ali)

Υ indicating its value at various time points are the formula

$$\lim_{h\to 0}\left(\frac{\Upsilon(\sigma+h)}{h}\right)^{\frac{1}{h}}. \tag{1}$$

The above formula shows how the value of $\Upsilon(\sigma)$ varies at moment σ. For the comparison of definition (1), the definition of derivative is

$$\Upsilon'(\sigma)=\lim_{h\to 0}\frac{\Upsilon(\sigma+h)-\Upsilon(\sigma)}{h}. \tag{2}$$

We observe that the difference in (2) is replaced by division and the division by h is replaced by the raising to the reciprocal power $1/h$. The limit (1) is called multiplicative derivative.

One of the main motivations behind multiplicative calculus is its ability to model non-linear systems that involve growth or decay. Traditional calculus uses linear approximations to solve complex problems, but multiplicative calculus uses nonlinear approximations which enables it to capture the complex interactions that occur in nonlinear systems. Another motivation for multiplicative calculus is its ability to model problems that involve positive and negative numbers, such as those in finance or economics. Traditional calculus does not account for negative numbers, which can lead to errors in calculations. Furthermore, multiplicative calculus provides a more natural way to model exponential growth and decay, which is a common phenomenon in various fields such as physics, biology, and economics. In summary, the motivation behind the development of multiplicative calculus is its ability to model nonlinear systems, account for positive and negative numbers, and provide a more natural way to model exponential growth and decay.

The Hermite-Hadamard inequality, named after Charles Hermite and Jacques Hadamard and commonly known as Hadamard's inequality, says that if a function $\Upsilon:[\omega,\varrho]\to\mathbb{R}$ is convex, the following double inequality holds:

$$\Upsilon\left(\frac{\omega+\varrho}{2}\right)\leq\frac{1}{\varrho-\omega}\int_{\omega}^{\varrho}\Upsilon(\sigma)\,d\sigma\leq\frac{\Upsilon(\omega)+\Upsilon(\varrho)}{2}. \tag{3}$$

If Υ is a concave mapping, the above inequality holds in the opposite direction. The inequality (3) can be proved using the Jensen inequality. There has been much research done in the direction of Hermite-Hadamard for different kinds of convexities. For example, in [8, 10, 14, 19], the authors established some inequalities linked with the midpoint and trapezoid formulas of numerical integration for convex functions. A study conducted in reference [13] established Simpson's inequalities for general convex functions through the use of fractional integrals. In [4], Ali et al. used Riemann-Liouville fractional integrals and established different variants of Newton's inequalities for differentiable convex functions.

In [7], Alomari et al. proved a new inequality to find the error bounds for Simpson's 1/3 formula which is stated as:

Theorem 1.1. *Let $\Upsilon:[\omega,\varrho]\to\mathbb{R}$ be a differentiable functions over (ω,ϱ). If $|\Upsilon'|$ is a convex function, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{6}\left[\Upsilon(\omega)+4\Upsilon\left(\frac{\omega+\varrho}{2}\right)+\Upsilon(\varrho)\right]-\frac{1}{\varrho-\omega}\int_{\omega}^{\varrho}\Upsilon(\sigma)\,d\sigma\right| \\ \leq\ &\frac{5(\varrho-\omega)}{72}\left[|\Upsilon'(\omega)|+|\Upsilon'(\varrho)|\right].\end{aligned} \tag{4}$$

Very recently, Ali et al. [3] proved the Hermite–Hadamard type inequality in the framework of multiplicative calculus and stated as:

Theorem 1.2. *Let $\Upsilon:[\omega,\varrho]\to\mathbb{R}^+$ be a multiplicatively convex function, then the following inequality holds:*

$$\Upsilon\left(\frac{\omega+\varrho}{2}\right)\leq\left(\int_{\omega}^{\varrho}(\Upsilon(\sigma))^{d\sigma}\right)^{\frac{1}{\varrho-\omega}}\leq\sqrt{\Upsilon(\omega)\Upsilon(\varrho)}. \tag{5}$$

In [12], Chasreechai et al. gave the multiplicative version of the inequality (4) and stated as:

Theorem 1.3. *Let $\Upsilon : [\omega, \varrho] \to \mathbb{R}^+$ be a multiplicative differentiable functions over (ω, ϱ). If Υ^* is a multiplicative convex function, then the following inequality holds:*

$$\left| \frac{\left[\Upsilon(\omega) \left(\Upsilon\left(\frac{\omega+\varrho}{2}\right)\right)^4 \Upsilon(\varrho) \right]^{\frac{1}{6}}}{\left(\int_\omega^\varrho (\Upsilon(\sigma))^{d\sigma} \right)^{\frac{1}{\varrho-\omega}}} \right| \leq \left[\Upsilon^*(\omega) \Upsilon^*(\varrho) \right]^{\frac{5(\varrho-\omega)}{72}}. \tag{6}$$

After the work of Ali et al. [3], many researchers started work in this direction and proved different variants of integral inequalities in the setting of multiplicative calculus. For example, the Hermite–Hadamard type inequalities for general multiplicatively convex functions were proved in [2] and Özcan used the multiplicatively preinvexity and established Hermite–Hadamard type inequalities in [21]. For multiplicatively s-convex and multiplicatively s-preinvex functions, the Hermite–Hadamard type inequalities were found in [22, 23] and for h-preinvex functions proved in [24]. Ali et al. [5] established some Ostrowski's and Simpson's type inequalities for multiplicatively convex functions and give their applications. Budak and Özcelik [11] used multiplicative fractional integrals and established Hermite–Hadamard-type inequalities. In [15], Fu et al. introduced multiplicative tempered fractional integrals and established some new fractional Hermite–Hadamard type inequalities for multiplicatively convex functions. Ali et al. [6] introduced the notions of multiplicative interval-valued integral and established some new Hermite–Hadamard type inequalities for interval-valued multiplicatively convex functions.

Inspired by the ongoing studies, we derived some new inequalities of Simpson's and Newton's type for multiplicative convex functions and these inequalities can help to find the error bounds of multiplicative numerical integration formulas. Since multiplicative calculus is modern calculus with a lot of applications in banking and finance, therefore the study about multiplicative calculus is valuable.

2. Multiplicative Calculus and related inequalities

In this section, we recall some concepts of multiplicative calculus and some inequalities. It is understood that multiplicative calculus only deal with the positive functions.

Definition 2.1. *[16, 18] Let $\Upsilon \in L_1[\omega, \varrho]$. The Riemann-Liouville fractional integrals $J_{\omega+}^{\alpha}\Upsilon$ and $J_{\varrho-}^{\alpha}\Upsilon$ of order $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$ with $\omega \geq 0$ are defined as follows:*

$$J_{\omega+}^{\alpha}\Upsilon(\sigma) = \frac{1}{\Gamma(\omega)} \int_\omega^\sigma (\sigma - \eta)^{\alpha-1} \Upsilon(\eta)\, d\eta,\ \sigma > \omega,$$

and

$$J_{\varrho-}^{\alpha}\Upsilon(\sigma) = \frac{1}{\Gamma(\alpha)} \int_\sigma^\varrho (\eta - \sigma)^{\alpha-1} \Upsilon(\eta)\, d\eta,\ \sigma < \varrho,$$

respectively, where Γ is the well-known Gamma function.

Definition 2.2. *[16, 18] The Riemann-Liouville fractional derivatives $D_{\omega+}^{\alpha}\Upsilon$ (left) and $D_{\varrho-}^{\alpha}\Upsilon$ (right) of order $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$ with $\omega \geq 0$ are defined as follows:*

$$\begin{aligned} D_{\omega+}^{\alpha}\Upsilon(\sigma) &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{d\sigma^n} \int_\omega^\sigma (\sigma-\eta)^{n-\alpha-1} \Upsilon(\eta)\, d\eta,\ \sigma > \omega \\ D_{\varrho-}^{\alpha}\Upsilon(\sigma) &= \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{d\sigma^n} \int_\sigma^\varrho (\eta-\sigma)^{n-\alpha-1} \Upsilon(\eta)\, d\eta,\ \sigma < \varrho. \end{aligned}$$

Here $n = [Re(\sigma) + 1]$, where $[\sigma]$ means the greatest integer less than or equal to σ.

In 2013, Sarikaya et al. proved the following fractional Hermite-Hamdard type inequality for the first time:

Theorem 2.3. *[25] For a positive convex function $\Upsilon : I \subset \mathbb{R} \to \mathbb{R}$ with $\Upsilon \in L_1[\omega, \varrho]$ and $0 \leq \omega < \varrho$, the following inequality holds:*

$$\Upsilon\left(\frac{\omega+\varrho}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(\varrho-\omega)^{\alpha}}\left[J_{\omega+}^{\alpha}\Upsilon(\varrho) + J_{\varrho-}^{\alpha}\Upsilon(\omega)\right] \leq \frac{\Upsilon(\omega)+\Upsilon(\varrho)}{2}. \tag{7}$$

Definition 2.4. *[9] Let $\Upsilon : \mathbb{R} \to \mathbb{R}^{+}$ be a positive function. The multiplicative derivative of the function Υ is given by*

$$\frac{d^{*}\Upsilon}{d\eta}(\eta) = \Upsilon^{*}(\eta) = \lim_{h\to 0}\left(\frac{\Upsilon(\eta+h)}{\Upsilon(\eta)}\right)^{1/h}.$$

If Υ has positive values and is differentiable at η, then Υ^{*} exists and the relation between Υ^{*} and ordinary derivative Υ' is as follows:

$$\Upsilon^{*}(\eta) = e^{\left[\log\Upsilon(\eta)\right]'} = e^{\frac{\Upsilon'(\eta)}{\Upsilon(\eta)}}.$$

If, additionally, the second derivative of Υ at η exists, then by an easy substitution, we obtain

$$\Upsilon^{**}(\eta) = e^{\left[\log\circ\Upsilon^{*}(\eta)\right]'} = e^{\left[\log\Upsilon(\eta)\right]''}.$$

Here $(\ln\Upsilon)''(\eta)$ exists because $\Upsilon''(\eta)$ exist. Repeating this procedure n times, we conclude that if Υ is a positive function and its nth derivative at η exists, then $\Upsilon^{*(n)}(\eta)$ exists and

$$\Upsilon^{*(n)}(\eta) = e^{\left(\log\Upsilon\right)^{(n)}(\eta)}, \quad n = 1, 2, \cdots.$$

We also recall that the concept of the * integral called multiplicative integral is denoted by $\int_{\omega}^{\varrho}(\Upsilon(\sigma))^{d\sigma}$ which introduced by Bashirov et al. in [9]. While the sum of the terms of product is used in the definition of a classical Riemann integral of Υ on $[\omega, \varrho]$, the product of terms raised to a power is used in the definition multiplicative integral of Υ on $[\omega, \varrho]$.
There is the following relation between Riemann integral and multiplicative integral [9]:

Proposition 2.5. *[9] If Υ is Riemann integrable on $[\omega, \varrho]$, then Υ is multiplicative integrable on $[\omega, \varrho]$ and*

$$\int_{\omega}^{\varrho}(\Upsilon(\sigma))^{d\sigma} = e^{\int_{\omega}^{\varrho}\log(\Upsilon(\sigma))d\sigma}.$$

For more details and properties of multiplicative calculus, one can consult [9].

Theorem 2.6. *[9] Let $\Upsilon : [\omega, \varrho] \to \mathbb{R}$ be multiplicative differentiable, let $g : [\omega, \varrho] \to \mathbb{R}$ be differentiable so the function Υ^{g} is multiplicative integrable. Then*

$$\int_{\omega}^{\varrho}\left(\Upsilon^{*}(\sigma)^{g(\sigma)}\right)^{d\sigma} = \frac{\Upsilon(\varrho)^{g(\varrho)}}{\Upsilon(\omega)^{g(\omega)}} \cdot \frac{1}{\int_{\omega}^{\varrho}\left(\Upsilon(\sigma)^{g'(\sigma)}\right)^{d\sigma}}.$$

Definition 2.7. *[1] The multiplicative Riemann-Liouville fractional integrals ${}_{\omega+}J_{*}^{\alpha}\Upsilon$ (left) and ${}_{*}J_{\varrho-}^{\alpha}\Upsilon$ (right) of order $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$ with $\omega \geq 0$ are defined as follows:*

$$\begin{aligned} {}_{\omega+}J_{*}^{\alpha}\Upsilon(\sigma) &= e^{\left(J_{\omega+}^{\alpha}(\ln\circ\Upsilon)\right)(\sigma)}, \\ {}_{*}J_{\varrho-}^{\alpha}\Upsilon(\sigma) &= e^{\left(J_{\varrho-}^{\alpha}(\ln\circ\Upsilon)\right)(\sigma)}. \end{aligned}$$

Definition 2.8. *[1] The Riemann-Liouville fractional derivatives ${}_{\omega+}D_{*}^{\alpha}\Upsilon$ (left) and ${}_{*}D_{\varrho-}^{\alpha}\Upsilon$ (right) of order $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$ with $\omega \geq 0$ are defined as follows:*

$$\begin{aligned} {}_{\omega+}D_{*}^{\alpha}\Upsilon(\sigma) &= e^{\left(D_{\omega+}^{\alpha}(\ln\circ\Upsilon)\right)(\sigma)}, \\ {}_{*}D_{\varrho-}^{\alpha}\Upsilon(\sigma) &= e^{\left(D_{\varrho-}^{\alpha}(\ln\circ\Upsilon)\right)(\sigma)}. \end{aligned}$$

For more details about the multiplicative fractional calculus, one can consulty [1].

Theorem 2.9. *[11] For a positive multiplicative convex function $\Upsilon : [\omega, \varrho] \to \mathbb{R}$, the following inequality holds for multiplicative Riemann-Liouville fractional integrals:*

$$\Upsilon\left(\frac{\omega+\varrho}{2}\right) \leq \left[{}_{\omega+}J_{*}^{\alpha}\Upsilon(\varrho) \cdot {}_{*}J_{\varrho-}^{\alpha}\Upsilon(\omega)\right]^{\frac{\Gamma(\alpha+1)}{2(\varrho-\omega)^{\alpha}}} \leq G(\Upsilon(\omega), \Upsilon(\varrho)).$$

For our main results we need to follow the definitions.

Definition 2.10. *[20] A non-empty set K is said to be convex, if for every $\omega, \varrho \in K$ we have*

$$\omega + \eta(\varrho - \omega) \in K, \ \forall \eta \in [0, 1].$$

Definition 2.11. *[20] A function Υ is said to be convex function on set K, if*

$$\Upsilon(\eta\sigma + (1-\eta)y) \leq \eta\Upsilon(\sigma) + (1-\eta)\Upsilon(y), \ \forall \eta \in [0, 1].$$

Definition 2.12. *[20] A function Υ is said to be log or multiplicatively convex function on set K, if*

$$\Upsilon(\eta\sigma + (1-\eta)y) \leq [\Upsilon(\sigma)]^{\eta} . [\Upsilon(y)]^{1-\eta}, \forall \eta \in [0, 1].$$

3. Multiplicative Integral Identities

In this section, two multiplicative fractional integral equalities are derived which plays an important role in the main results.

Lemma 3.1. *Let $\Upsilon : [\omega, \varrho] \to \mathbb{R}$ be multiplicative differentiable function over (ω, ϱ). If Υ^{*} is integrable functions, then following equality holds:*

$$\begin{aligned} &\frac{\left[\Upsilon(\omega)\left(\Upsilon\left(\frac{\omega+\varrho}{2}\right)\right)^{4}\Upsilon(\varrho)\right]^{\frac{1}{6}}}{\left[{}_{*}J_{\omega+}^{\alpha}\Upsilon(\sigma) \cdot {}_{\varrho-}J_{*}^{\alpha}\Upsilon(\sigma)\right]^{\frac{\Gamma(\alpha+1)}{2(\varrho-\omega)^{\alpha}}}} \qquad (8) \\ &= \left(\int_{0}^{\frac{1}{2}}\left(\left[\Upsilon^{*}(\eta\varrho+(1-\eta)\omega)\right]^{\eta^{\alpha}-\frac{1}{6}}\right)^{d\eta}\right)^{\frac{\varrho-\omega}{2}} \times \left(\int_{0}^{\frac{1}{2}}\left(\left[\Upsilon^{*}(\eta\omega+(1-\eta)\varrho)\right]^{\frac{1}{6}-\eta^{\alpha}}\right)^{d\eta}\right)^{\frac{\varrho-\omega}{2}} \\ &\quad \times \left(\int_{\frac{1}{2}}^{1}\left(\left[\Upsilon^{*}(\eta\varrho+(1-\eta)\omega)\right]^{\eta^{\alpha}-\frac{5}{6}}\right)^{d\eta}\right)^{\frac{\varrho-\omega}{2}} \times \left(\int_{\frac{1}{2}}^{1}\left(\left[\Upsilon^{*}(\eta\omega+(1-\eta)\varrho)\right]^{\frac{5}{6}-\eta^{\alpha}}\right)^{d\eta}\right)^{\frac{\varrho-\omega}{2}}. \end{aligned}$$

Proof. From the basic rules of multiplicative integration by parts and multiplicative fractional integrals, we have

$$\begin{aligned} I_1 &= \left(\int_0^{\frac{1}{2}} \left(\left[\Upsilon^*\left(\eta\varrho+(1-\eta)\omega\right)\right]^{\eta^\alpha-\frac{1}{6}}\right)^{d\eta}\right)^{\frac{\varrho-\omega}{2}} \quad (9)\\ &= e^{\frac{\varrho-\omega}{2}\int_0^{\frac{1}{2}}\left(\eta^\alpha-\frac{1}{6}\right)(\ln\circ\Upsilon)'\left(\eta\varrho+(1-\eta)\omega\right)d\eta}\\ &= e^{\frac{1}{2}\left(\frac{1}{2^\alpha}-\frac{1}{6}\right)\ln\Upsilon\left(\frac{\omega+\varrho}{2}\right)+\frac{1}{12}\ln\Upsilon(\omega)-\frac{\alpha}{2}\int_0^{\frac{1}{2}}\eta^{\alpha-1}\ln\Upsilon\left(\eta\varrho+(1-\eta)\omega\right)}, \end{aligned}$$

and

$$\begin{aligned} I_2 &= \left(\int_{\frac{1}{2}}^{1} \left(\left[\Upsilon^*\left(\eta\varrho+(1-\eta)\omega\right)\right]^{\eta^\alpha-\frac{5}{6}}\right)^{d\eta}\right)^{\frac{\varrho-\omega}{2}} \quad (10)\\ &= e^{\frac{1}{12}\ln\Upsilon(\varrho)+\frac{1}{2}\left(\frac{5}{6}-\frac{1}{2^\alpha}\right)\ln\Upsilon\left(\frac{\omega+\varrho}{2}\right)-\frac{\alpha}{2}\int_{\frac{1}{2}}^{1}\eta^{\alpha-1}\ln\Upsilon\left(\eta\varrho+(1-\eta)\omega\right)}. \end{aligned}$$

From (9) and (10), we have

$$I_1\times I_2 = e^{\frac{1}{12}\ln\Upsilon(\omega)+\frac{1}{12}\ln\Upsilon(\varrho)+\frac{1}{3}\ln\Upsilon\left(\frac{\omega+\varrho}{2}\right)-\frac{\Gamma(\alpha+1)}{2(\varrho-\omega)^\alpha}J_{\omega+}^{\alpha}\ln\Upsilon(\sigma)}. \quad (11)$$

Similarly, we have

$$\begin{aligned} I_3\times I_4 &= \left(\int_0^{\frac{1}{2}} \left(\left[\Upsilon^*\left(\eta\omega+(1-\eta)\varrho\right)\right]^{\frac{1}{6}-\eta^\alpha}\right)^{d\eta}\right)^{\frac{\varrho-\omega}{2}} \times\left(\int_{\frac{1}{2}}^{1} \left(\left[\Upsilon^*\left(\eta\omega+(1-\eta)\varrho\right)\right]^{\frac{5}{6}-\eta^\alpha}\right)^{d\eta}\right)^{\frac{\varrho-\omega}{2}} \quad (12)\\ &= e^{\frac{1}{12}\ln\Upsilon(\omega)+\frac{1}{12}\ln\Upsilon(\varrho)+\frac{1}{3}\ln\Upsilon\left(\frac{\omega+\varrho}{2}\right)-\frac{\Gamma(\alpha+1)}{2(\varrho-\omega)^\alpha}J_{\varrho-}^{\alpha}\ln\Upsilon(\sigma)}. \end{aligned}$$

Thus, from (11) and (12), we have

$$\begin{aligned} &\left(\int_0^{\frac{1}{2}} \left(\left[\Upsilon^*\left(\eta\varrho+(1-\eta)\omega\right)\right]^{\eta^\alpha-\frac{1}{6}}\right)^{d\eta}\right)^{\frac{\varrho-\omega}{2}} \times\left(\int_{\frac{1}{2}}^{1} \left(\left[\Upsilon^*\left(\eta\varrho+(1-\eta)\omega\right)\right]^{\eta^\alpha-\frac{5}{6}}\right)^{d\eta}\right)^{\frac{\varrho-\omega}{2}}\\ &\times\left(\int_0^{\frac{1}{2}} \left(\left[\Upsilon^*\left(\eta\omega+(1-\eta)\varrho\right)\right]^{\frac{1}{6}-\eta^\alpha}\right)^{d\eta}\right)^{\frac{\varrho-\omega}{2}} \times\left(\int_{\frac{1}{2}}^{1} \left(\left[\Upsilon^*\left(\eta\omega+(1-\eta)\varrho\right)\right]^{\frac{5}{6}-\eta^\alpha}\right)^{d\eta}\right)^{\frac{\varrho-\omega}{2}}\\ =&\ \frac{\left[\Upsilon(\omega)\left(\Upsilon\left(\frac{\omega+\varrho}{2}\right)\right)^4\Upsilon(\varrho)\right]^{\frac{1}{6}}}{\left[{}_*J_{\omega+}^{\alpha}\Upsilon(\sigma)\cdot{}_{\varrho-}J_*^{\alpha}\Upsilon(\sigma)\right]^{\frac{\Gamma(\alpha+1)}{2(\varrho-\omega)^\alpha}}} \end{aligned}$$

and the proof is completed. □

Lemma 3.2. *Let $\Upsilon:[\omega,\varrho]\to\mathbb{R}$ be a multiplicative differentiable functions over (ω,ϱ). If Υ^* is integrable functions, then following equality holds:*

$$\begin{aligned} &\frac{\left[\Upsilon(\omega)\left[\Upsilon\left(\frac{\omega+2\varrho}{3}\right)\right]^3\left[\Upsilon\left(\frac{2\omega+\varrho}{3}\right)\right]^3\Upsilon(\varrho)\right]^{\frac{1}{8}}}{\left[{}_*J_{\omega+}^{\alpha}\Upsilon(\sigma)\cdot{}_{\varrho-}J_*^{\alpha}\Upsilon(\sigma)\right]^{\frac{\Gamma(\alpha+1)}{2(\varrho-\omega)^\alpha}}}\\ =&\ \left(\int_0^{\frac{1}{3}}\left[\left(\Upsilon^*\left(\eta\varrho+(1-\eta)\omega\right)\right)^{\eta^\alpha-\frac{1}{8}}\right]^{d\eta}\right)^{\frac{\varrho-\omega}{2}} \times\left(\int_0^{\frac{1}{3}}\left[\left(\Upsilon^*\left(\eta\omega+(1-\eta)\varrho\right)\right)^{\frac{1}{8}-\eta^\alpha}\right]^{d\eta}\right)^{\frac{\varrho-\omega}{2}} \end{aligned}$$

$$\times\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left[\left(\Upsilon^{*}\left(\eta\varrho+(1-\eta)\omega\right)\right)^{\eta^{\alpha}-\frac{1}{2}}\right]^{d\eta}\right)^{\frac{\varrho-\omega}{2}}\times\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left[\left(\Upsilon^{*}\left(\eta\omega+(1-\eta)\varrho\right)\right)^{\frac{1}{2}-\eta^{\alpha}}\right]^{d\eta}\right)^{\frac{\varrho-\omega}{2}}$$
$$\times\left(\int_{\frac{2}{3}}^{1}\left[\left(\Upsilon^{*}\left(\eta\varrho+(1-\eta)\omega\right)\right)^{\eta^{\alpha}-\frac{7}{8}}\right]^{d\eta}\right)^{\frac{\varrho-\omega}{2}}\times\left(\int_{\frac{2}{3}}^{1}\left[\left(\Upsilon^{*}\left(\eta\omega+(1-\eta)\varrho\right)\right)^{\frac{7}{8}-\eta^{\alpha}}\right]^{d\eta}\right)^{\frac{\varrho-\omega}{2}}.$$

Proof. This lemma can be proved like the previous lemma. □

4. Multiplicative Fractional Simpson's Inequalities

In this section, the multiplicative fractional version of Simpson's inequalities for multiplicative convex functions are established.

Theorem 4.1. *If Lemma 3.1 holds and Υ^* is multiplicative convex function, then we have the following inequality for $\lambda=\left(\frac{1}{6}\right)^{\frac{1}{\alpha}}$ and $\mu=\left(\frac{5}{6}\right)^{\frac{1}{\alpha}}$:*

$$\left|\frac{\left[\Upsilon(\omega)\left(\Upsilon\left(\frac{\omega+\varrho}{2}\right)\right)^{4}\Upsilon(\varrho)\right]^{\frac{1}{6}}}{\left[{}_{*}J_{\omega+}^{\alpha}\Upsilon(\sigma)\cdot{}_{\varrho-}J_{*}^{\alpha}\Upsilon(\sigma)\right]^{\frac{\Gamma(\alpha+1)}{2(\varrho-\omega)^{\alpha}}}}\right| \leq \left[\Upsilon^{*}(\omega)\Upsilon^{*}(\varrho)\right]^{\frac{\left(A_{1}(\lambda)+A_{3}(\lambda)+B_{1}(\mu)+B_{3}(\mu)\right)(\varrho-\omega)}{2}},$$

where

$$\begin{aligned}
A_{1}(\lambda) &= \int_{0}^{\frac{1}{2}}\eta\left|\eta^{\alpha}-\frac{1}{6}\right|d\eta=\frac{\lambda^{2}}{6}-\frac{2\lambda^{\alpha+2}}{\alpha+2}+\frac{1}{2^{\alpha+2}(\alpha+2)}-\frac{1}{48},\\
A_{2}(\lambda) &= \int_{0}^{\frac{1}{2}}\left|\eta^{\alpha}-\frac{1}{6}\right|d\eta=\frac{\lambda}{3}-\frac{2\lambda^{\alpha+1}}{\alpha+1}+\frac{1}{2^{\alpha+1}(\alpha+1)}-\frac{1}{2},\\
A_{3}(\lambda) &= \int_{0}^{\frac{1}{2}}(1-\eta)\left|\eta^{\alpha}-\frac{1}{6}\right|d\eta=A_{2}(\lambda)-A_{1}(\lambda),\\
B_{1}(\mu) &= \int_{\frac{1}{2}}^{1}\eta\left|\eta^{\alpha}-\frac{5}{6}\right|d\eta=\frac{5\mu^{2}}{6}-\frac{2\mu^{\alpha+2}}{\alpha+2}-\frac{25}{48}+\frac{1}{2^{\alpha+2}(\alpha+2)}+\frac{1}{\alpha+2},\\
B_{2}(\mu) &= \int_{\frac{1}{2}}^{1}\left|\eta^{\alpha}-\frac{5}{6}\right|d\eta=\frac{5}{3}\mu-\frac{2\mu^{\alpha+1}}{\alpha+1}-\frac{15}{12}+\frac{1}{2^{\alpha+1}(\alpha+1)}+\frac{1}{\alpha+1},\\
B_{3}(\mu) &= \int_{\frac{1}{2}}^{1}(1-\eta)\left|\eta^{\alpha}-\frac{5}{6}\right|d\eta=B_{2}(\mu)-B_{1}(\mu).
\end{aligned}$$

Proof. Taking modulus of (8) and multiplicative convexity of Υ^*, we have

$$\begin{aligned}
&\left|\frac{\left[\Upsilon(\omega)\left(\Upsilon\left(\frac{\omega+\varrho}{2}\right)\right)^{4}\Upsilon(\varrho)\right]^{\frac{1}{6}}}{\left[{}_{*}J_{\omega+}^{\alpha}\Upsilon(\sigma)\cdot{}_{\varrho-}J_{*}^{\alpha}\Upsilon(\sigma)\right]^{\frac{\Gamma(\alpha+1)}{2(\varrho-\omega)^{\alpha}}}}\right|\\
\leq\ & \exp\left[\frac{\varrho-\omega}{2}\left(\int_{0}^{\frac{1}{2}}\left|\eta^{\alpha}-\frac{1}{6}\right|\ln\Upsilon^{*}\left(\eta\varrho+(1-\eta)\omega\right)d\eta\right)\right.\\
&+\frac{\varrho-\omega}{2}\left(\int_{0}^{\frac{1}{2}}\left|\eta^{\alpha}-\frac{1}{6}\right|\ln\Upsilon^{*}\left(\eta\omega+(1-\eta)\varrho\right)d\eta\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\varrho-\omega}{2}\left(\int_{\frac{1}{2}}^{1}\left|\eta^{\alpha}-\frac{5}{6}\right|\ln\Upsilon^{*}\left(\eta\varrho+(1-\eta)\,\omega\right)d\eta\right)\\
&+\frac{\varrho-\omega}{2}\left(\int_{\frac{1}{2}}^{1}\left|\eta^{\alpha}-\frac{5}{6}\right|\ln\Upsilon^{*}\left(\eta\omega+(1-\eta)\,\varrho\right)d\eta\right)\Bigg]\\
\leq\ & \exp\left[\frac{\varrho-\omega}{2}\left(\int_{0}^{\frac{1}{2}}\left|\eta^{\alpha}-\frac{1}{6}\right|\left(\eta\ln\Upsilon^{*}(\varrho)+(1-\eta)\ln\Upsilon^{*}(\omega)\right)d\eta\right)\right.\\
&+\frac{\varrho-\omega}{2}\left(\int_{0}^{\frac{1}{2}}\left|\eta^{\alpha}-\frac{1}{6}\right|\left(\eta\ln\Upsilon^{*}(\omega)+(1-\eta)\ln\Upsilon^{*}(\varrho)\right)d\eta\right)\\
&+\frac{\varrho-\omega}{2}\left(\int_{\frac{1}{2}}^{1}\left|\eta^{\alpha}-\frac{5}{6}\right|\left(\eta\ln\Upsilon^{*}(\varrho)+(1-\eta)\ln\Upsilon^{*}(\omega)\right)d\eta\right)\\
&+\frac{\varrho-\omega}{2}\left(\int_{\frac{1}{2}}^{1}\left|\eta^{\alpha}-\frac{5}{6}\right|\left(\eta\ln\Upsilon^{*}(\omega)+(1-\eta)\ln\Upsilon^{*}(\varrho)\right)d\eta\right)\\
=\ & \exp\left[\frac{\varrho-\omega}{2}\left(A_{1}(\lambda)\ln\Upsilon^{*}(\varrho)+(A_{3}(\lambda))\ln\Upsilon^{*}(\omega)\right)\right.\\
&+\frac{\varrho-\omega}{2}\left(A_{1}(\lambda)\ln\Upsilon^{*}(\omega)+(A_{3}(\lambda))\ln\Upsilon^{*}(\varrho)\right)\\
&+\frac{\varrho-\omega}{2}\left(B_{1}(\mu)\ln\Upsilon^{*}(\varrho)+(B_{3}(\mu))\ln\Upsilon^{*}(\omega)\right)\\
&\left.+\frac{\varrho-\omega}{2}\left(B_{1}(\mu)\ln\Upsilon^{*}(\omega)+(B_{3}(\mu))\ln\Upsilon^{*}(\varrho)\right)\right]\\
=\ & \left[\Upsilon^{*}(\omega)\Upsilon^{*}(\varrho)\right]^{\frac{\left(A_{1}(\lambda)+A_{3}(\lambda)+B_{1}(\mu)+B_{3}(\mu)\right)(\varrho-\omega)}{2}}.
\end{aligned}
$$

Thus, the proof is completed. □

Remark 4.2. *For $\alpha=1$ in Theorem 4.1, the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{\left[\Upsilon(\omega)\left(\Upsilon\left(\frac{\omega+\varrho}{2}\right)\right)^{4}\Upsilon(\varrho)\right]^{\frac{1}{6}}}{\left(\int_{\omega}^{\varrho}(\Upsilon(\sigma))^{d\sigma}\right)^{\frac{1}{\varrho-\omega}}}\right|\\
\leq\ & \left[\Upsilon^{*}(\omega)\Upsilon^{*}(\varrho)\right]^{\frac{5(\varrho-\omega)}{72}}.
\end{aligned}
$$

This inequality is proved by Chasreechai et al. in [12].

Theorem 4.3. *If Lemma 3.1 holds and $(\ln(\Upsilon^{*}))^{q}$, $q>1$ is convex function, then we have the following inequality:*

$$
\begin{aligned}
&\left|\frac{\left[\Upsilon(\omega)\left(\Upsilon\left(\frac{\omega+\varrho}{2}\right)\right)^{4}\Upsilon(\varrho)\right]^{\frac{1}{6}}}{\left[{}_{*}J_{\omega+}^{\alpha}\Upsilon(\sigma)\cdot{}_{\varrho-}J_{*}^{\alpha}\Upsilon(\sigma)\right]^{\frac{\Gamma(\alpha+1)}{2(\varrho-\omega)^{\alpha}}}}\right|\\
\leq\ & \left[\sqrt{\Upsilon^{*}(\omega)\Upsilon^{*}(\varrho)}\right]^{\frac{\varrho-\omega}{2}\left[\Psi_{1}+\Psi_{2}\right]},
\end{aligned}
$$

where $\Psi_{1}=\left(\int_{0}^{\frac{1}{2}}\left|\eta^{\alpha}-\frac{1}{6}\right|^{p}d\eta\right)^{\frac{1}{p}}$, $\Psi_{2}=\left(\int_{\frac{1}{2}}^{1}\left|\eta^{\alpha}-\frac{5}{6}\right|^{p}d\eta\right)^{\frac{1}{p}}$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From (8) and Hölder inequality, we have

$$
\begin{aligned}
&\left|\frac{\left[\Upsilon(\omega)\left(\Upsilon\left(\frac{\omega+\varrho}{2}\right)\right)^{4}\Upsilon(\varrho)\right]^{\frac{1}{6}}}{\left[{}_{*}J_{\omega+}^{\alpha}\Upsilon(\sigma)\cdot{}_{\varrho-}J_{*}^{\alpha}\Upsilon(\sigma)\right]^{\frac{\Gamma(\alpha+1)}{2(\varrho-\omega)^{\alpha}}}}\right| \\
\leq\ & \exp\left[\frac{\varrho-\omega}{2}\left(\int_{0}^{\frac{1}{2}}\left|\eta^{\alpha}-\frac{1}{6}\right|\ln\Upsilon^{*}(\eta\varrho+(1-\eta)\omega)\,d\eta\right)\right. \\
&+\frac{\varrho-\omega}{2}\left(\int_{0}^{\frac{1}{2}}\left|\eta^{\alpha}-\frac{1}{6}\right|\ln\Upsilon^{*}(\eta\omega+(1-\eta)\varrho)\,d\eta\right) \\
&+\frac{\varrho-\omega}{2}\left(\int_{\frac{1}{2}}^{1}\left|\eta^{\alpha}-\frac{5}{6}\right|\ln\Upsilon^{*}(\eta\varrho+(1-\eta)\omega)\,d\eta\right) \\
&\left.+\frac{\varrho-\omega}{2}\left(\int_{\frac{1}{2}}^{1}\left|\eta^{\alpha}-\frac{5}{6}\right|\ln\Upsilon^{*}(\eta\omega+(1-\eta)\varrho)\,d\eta\right)\right] \\
\leq\ & \exp\left[\frac{\varrho-\omega}{2}\left(\int_{0}^{\frac{1}{2}}\left|\eta^{\alpha}-\frac{1}{6}\right|^{p}d\eta\right)^{\frac{1}{p}}\left\{\left(\int_{0}^{\frac{1}{2}}\left|\ln\Upsilon^{*}(\eta\varrho+(1-\eta)\omega)\right|^{q}d\eta\right)^{\frac{1}{q}}\right.\right. \\
&\left.+\left(\int_{0}^{\frac{1}{2}}\left|\ln\Upsilon^{*}(\eta\omega+(1-\eta)\varrho)\right|^{q}d\eta\right)^{\frac{1}{q}}\right\} \\
&+\frac{\varrho-\omega}{2}\left(\int_{\frac{1}{2}}^{1}\left|\eta^{\alpha}-\frac{5}{6}\right|^{p}d\eta\right)^{\frac{1}{p}}\left\{\left(\int_{\frac{1}{2}}^{1}\left|\ln\Upsilon^{*}(\eta\varrho+(1-\eta)\omega)\right|^{q}d\eta\right)^{\frac{1}{q}}\right. \\
&\left.\left.+\left(\int_{\frac{1}{2}}^{1}\left|\ln\Upsilon^{*}(\eta\omega+(1-\eta)\varrho)\right|^{q}d\eta\right)^{\frac{1}{q}}\right\}\right].
\end{aligned}
$$

Applying convexity of $(\ln(\Upsilon^{*}))^{q}$, we have

$$
\begin{aligned}
&\left|\frac{\left[\Upsilon(\omega)\left(\Upsilon\left(\frac{\omega+\varrho}{2}\right)\right)^{4}\Upsilon(\varrho)\right]^{\frac{1}{6}}}{\left[{}_{*}J_{\omega+}^{\alpha}\Upsilon(\sigma)\cdot{}_{\varrho-}J_{*}^{\alpha}\Upsilon(\sigma)\right]^{\frac{\Gamma(\alpha+1)}{2(\varrho-\omega)^{\alpha}}}}\right| \\
\leq\ & \exp\left[\frac{\varrho-\omega}{2}\left(\int_{0}^{\frac{1}{2}}\left|\eta^{\alpha}-\frac{1}{6}\right|^{p}d\eta\right)^{\frac{1}{p}}\left\{\left(\int_{0}^{\frac{1}{2}}\left(\eta(\ln\Upsilon^{*}(\varrho))^{q}+(1-\eta)(\ln\Upsilon^{*}(\omega))^{q}\right)d\eta\right)^{\frac{1}{q}}\right.\right. \\
&\left.+\left(\int_{0}^{\frac{1}{2}}\left(\eta(\ln\Upsilon^{*}(\omega))^{q}+(1-\eta)(\ln\Upsilon^{*}(\varrho))^{q}\right)d\eta\right)^{\frac{1}{q}}\right\} \\
&+\frac{\varrho-\omega}{2}\left(\int_{\frac{1}{2}}^{1}\left|\eta^{\alpha}-\frac{5}{6}\right|^{p}d\eta\right)^{\frac{1}{p}}\left\{\left(\int_{\frac{1}{2}}^{1}\left(\eta(\ln\Upsilon^{*}(\varrho))^{q}+(1-\eta)(\ln\Upsilon^{*}(\omega))^{q}\right)d\eta\right)^{\frac{1}{q}}\right. \\
&\left.\left.+\left(\int_{\frac{1}{2}}^{1}\left(\eta\ln\Upsilon^{*}(\omega)+(1-\eta)\ln\Upsilon^{*}(\varrho)\right)d\eta\right)^{\frac{1}{q}}\right\}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \exp\left[\frac{\varrho-\omega}{2}\left(\int_0^{\frac{1}{2}}\left|\eta^\alpha-\frac{1}{6}\right|^p d\eta\right)^{\frac{1}{p}}\left\{\left(\frac{(\ln\Upsilon^*(\varrho))^q}{8}+\frac{3(\ln\Upsilon^*(\omega))^q}{8}\right)^{\frac{1}{q}}\right.\right.\\
&\quad\left.+\left(\frac{(\ln\Upsilon^*(\omega))^q}{8}+\frac{3(\ln\Upsilon^*(\varrho))^q}{8}\right)^{\frac{1}{q}}\right\}\\
&\quad+\frac{\varrho-\omega}{2}\left(\int_{\frac{1}{2}}^1\left|\eta^\alpha-\frac{5}{6}\right|^p d\eta\right)^{\frac{1}{p}}\left\{\left(\frac{3(\ln\Upsilon^*(\varrho))^q}{8}+\frac{(\ln\Upsilon^*(\omega))^q}{8}\right)^{\frac{1}{q}}\right.\\
&\quad\left.\left.+\left(\frac{3(\ln\Upsilon^*(\omega))^q}{8}+\frac{(\ln\Upsilon^*(\varrho))^q}{8}\right)^{\frac{1}{q}}\right\}\right]\\
&= \left[\sqrt{\Upsilon^*(\omega)\Upsilon^*(\varrho)}\right]^{\frac{\varrho-\omega}{2}[\Psi_1+\Psi_2]}.
\end{aligned}
$$

Thus, the proof is completed. □

Remark 4.4. *For $\alpha = 1$ in Theorem 4.3, the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{\left[\Upsilon(\omega)\left(\Upsilon\left(\frac{\omega+\varrho}{2}\right)\right)^4\Upsilon(\varrho)\right]^{\frac{1}{6}}}{\left(\int_\omega^\varrho(\Upsilon(\sigma))^{d\sigma}\right)^{\frac{1}{\varrho-\omega}}}\right|\\
\leq\ &\left[\sqrt{\Upsilon^*(\omega)\Upsilon^*(\varrho)}\right]^{(\varrho-\omega)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)}.
\end{aligned}
$$

This inequality is proved by Chasreechai et al. in [12].

5. Multiplicative Fractional Newton's Inequalities

In this section, the multiplicative fractional version of Newton's inequalities for multiplicative convex functions are established with the help of Lemma 3.2.

Theorem 5.1. *If Lemma 3.2 holds and Υ^* is multiplicative convex function, then we have the following inequality for $\nu = \left(\frac{1}{8}\right)^{\frac{1}{\alpha}}$, $\xi = \left(\frac{1}{2}\right)^{\frac{1}{\alpha}}$ and $\kappa = \left(\frac{7}{8}\right)^{\frac{1}{\alpha}}$:*

$$
\begin{aligned}
&\left|\frac{\left[\Upsilon(\omega)\left[\Upsilon\left(\frac{\omega+2\varrho}{3}\right)\right]^3\left[\Upsilon\left(\frac{2\omega+\varrho}{3}\right)\right]^3\Upsilon(\varrho)\right]^{\frac{1}{8}}}{\left[{}_*J^\alpha_{\omega+}\Upsilon(\sigma)\cdot{}_{\varrho-}J^\alpha_*\Upsilon(\sigma)\right]^{\frac{\Gamma(\alpha+1)}{2(\varrho-\omega)^\alpha}}}\right|\\
\leq\ &\left[\Upsilon^*(\omega)\Upsilon^*(\varrho)\right]^{\frac{(C_1(\nu)+C_3(\nu)+D_1(\xi)+D_3(\xi)+E_1(\kappa)+E_3(\kappa))(\varrho-\omega)}{2}},
\end{aligned}
$$

where

$$
\begin{aligned}
C_1(\nu) &= \int_0^{\frac{1}{3}}\eta\left|\eta^\alpha-\frac{1}{8}\right|d\eta = \frac{\nu^2}{8}-\frac{2\nu^{\alpha+2}}{\alpha+2}+\frac{1}{3^{\alpha+2}(\alpha+2)}-\frac{1}{144},\\
C_2(\nu) &= \int_0^{\frac{1}{3}}\left|\eta^\alpha-\frac{1}{8}\right|d\eta = \frac{\nu}{4}-\frac{2\nu^{\alpha+1}}{\alpha+1}+\frac{1}{3^{\alpha+1}(\alpha+1)}-\frac{1}{24},\\
C_3(\nu) &= \int_0^{\frac{1}{3}}(1-\eta)\left|\eta^\alpha-\frac{1}{8}\right|d\eta = C_2(\nu)-C_1(\nu),
\end{aligned}
$$

$$\begin{aligned}
D_1(\xi) &= \int_{\frac{1}{3}}^{\frac{2}{3}} \eta \left|\eta^\alpha - \frac{1}{2}\right| d\eta = \frac{\xi^2}{2} - \frac{2\xi^{\alpha+2}}{\alpha+2} + \frac{1}{3^{\alpha+2}(\alpha+2)} + \frac{2^{\alpha+2}}{3^{\alpha+2}(\alpha+2)} - \frac{5}{36}, \\
D_2(\xi) &= \int_{\frac{1}{3}}^{\frac{2}{3}} \left|\eta^\alpha - \frac{1}{2}\right| d\eta = \xi - \frac{2\xi^{\alpha+1}}{\alpha+1} + \frac{1}{3^{\alpha+1}(\alpha+1)} + \frac{2^{\alpha+1}}{3^{\alpha+1}(\alpha+1)} - \frac{1}{2}, \\
D_3(\xi) &= \int_{\frac{1}{3}}^{\frac{2}{3}} (1-\eta) \left|\eta^\alpha - \frac{1}{2}\right| d\eta = D_2(\xi) - D_1(\xi), \\
E_1(\kappa) &= \int_{\frac{2}{3}}^{1} \eta \left|\eta^\alpha - \frac{7}{8}\right| d\eta = \frac{7\kappa^2}{8} - \frac{2\kappa^{\alpha+2}}{\alpha+2} + \frac{2^{\alpha+2}}{3^{\alpha+2}(\alpha+2)} + \frac{1}{\alpha+2} - \frac{91}{144}, \\
E_2(\kappa) &= \int_{\frac{2}{3}}^{1} \left|\eta^\alpha - \frac{7}{8}\right| d\eta = \frac{7\kappa}{4} - \frac{2\kappa^{\alpha+1}}{\alpha+1} + \frac{2^{\alpha+1}}{3^{\alpha+1}(\alpha+1)} + \frac{1}{\alpha+1} - \frac{35}{24}, \\
E_3(\kappa) &= \int_{\frac{2}{3}}^{1} (1-\eta) \left|\eta^\alpha - \frac{7}{8}\right| d\eta = E_2(\kappa) - E_1(\kappa).
\end{aligned}$$

Proof. One can obtain the required inequality by using the steps used in the proof of Theorem 4.1. □

Remark 5.2. *For $\alpha = 1$ in Theorem 5.1, we have the following inequality:*

$$\begin{aligned}
&\left|\frac{\left[\Upsilon(\omega)\left[\Upsilon\left(\frac{\omega+2\varrho}{3}\right)\right]^3\left[\Upsilon\left(\frac{2\omega+\varrho}{3}\right)\right]^3\Upsilon(\varrho)\right]^{\frac{1}{8}}}{\left(\int_\omega^\varrho (\Upsilon(\sigma))^{d\sigma}\right)^{\frac{1}{\varrho-\omega}}}\right| \\
\leq\ & \left[\Upsilon^*(\omega)\Upsilon^*(\varrho)\right]^{\frac{25(\varrho-\omega)}{576}}.
\end{aligned}$$

This inequality is proved by Chasreechai et al. in [12].

Theorem 5.3. *If Lemma 3.2 holds and $(\ln(\Upsilon^*))^q$, $q > 1$ is a convex function, then we have the following inequality:*

$$\begin{aligned}
&\left|\frac{\left[\Upsilon(\omega)\left[\Upsilon\left(\frac{\omega+2\varrho}{3}\right)\right]^3\left[\Upsilon\left(\frac{2\omega+\varrho}{3}\right)\right]^3\Upsilon(\varrho)\right]^{\frac{1}{8}}}{\left[{}_*J_{\omega+}^{\alpha}\Upsilon(\sigma)\cdot{}_{\varrho-}J_*^{\alpha}\Upsilon(\sigma)\right]^{\frac{\Gamma(\alpha+1)}{2(\varrho-\omega)^\alpha}}}\right| \\
\leq\ & \left[\sqrt[3]{\Upsilon^*(\omega)\Upsilon^*(\varrho)}\right]^{\frac{\varrho-\omega}{2}[\Psi_3+\Psi_4+\Psi_5]},
\end{aligned}$$

where $\Psi_3 = \left(\int_0^{\frac{1}{3}} \left|\eta^\alpha - \frac{1}{8}\right|^p d\eta\right)^{\frac{1}{p}}$, $\Psi_4 = \left(\int_{\frac{1}{3}}^{\frac{2}{3}} \left|\eta^\alpha - \frac{1}{2}\right|^p d\eta\right)^{\frac{1}{p}}$, $\Psi_5 = \left(\int_{\frac{2}{3}}^{1} \left|\eta^\alpha - \frac{7}{8}\right|^p d\eta\right)^{\frac{1}{p}}$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. One can obtain the required inequality by using the steps used in the proof of Theorem 4.3. □

Remark 5.4. *For $\alpha = 1$ in Theorem 5.3, we have the following inequality:*

$$\begin{aligned}
&\left|\frac{\left[\Upsilon(\omega)\left[\Upsilon\left(\frac{\omega+2\varrho}{3}\right)\right]^3\left[\Upsilon\left(\frac{2\omega+\varrho}{3}\right)\right]^3\Upsilon(\varrho)\right]^{\frac{1}{8}}}{\left(\int_\omega^\varrho (\Upsilon(\sigma))^{d\sigma}\right)^{\frac{1}{\varrho-\omega}}}\right| \\
\leq\ & \left[\Upsilon^*(\omega)\Upsilon^*(\varrho)\right]^{\frac{(\varrho-\omega)}{3}\left(\frac{1}{8^{p+1}(p+1)}+\frac{5^{p+1}}{24^{p+1}(p+1)}\right)^{\frac{1}{p}}+\frac{(\varrho-\omega)}{6}\left(\frac{2}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}}.
\end{aligned}$$

This inequality is proved by Chasreechai et al. in [12].

6. Conclusion

In this work, we have established multiplicative fractional variants of Simpson's and Newton's inequalities for multiplicative convex functions. Moreover, we have proved that the newly established inequalities are extensions of inequalities proved in [12]. After understanding the results of this paper, the upcoming researchers can obtain similar inequalities on coordinates and can use these results in finding the error bounds for multiplicative fractional numerical integration formulas.

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