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# Abelian theorems for the index ${}_2F_1$-transform over distributions of compact support and generalized functions

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**Abstract.** The goal of this paper is to derive new Abelian theorems for the index ${}_2F_1$-transform over distributions of compact support and over certain spaces of generalized functions. From these results one also obtains Abelian theorems for the conventional index ${}_2F_1$-transform.

## 1. Introduction and preliminaries

The index ${}_2F_1$-transform of a suitable complex-valued function $f$ is given by

$$F(\tau) = \int_0^\infty f(t)\,{}_2F_1\left(\mu + \frac{1}{2} + i\tau, \mu + \frac{1}{2} - i\tau; \mu + 1; -t\right) t^\alpha dt, \quad \tau > 0, \tag{1.1}$$

where ${}_2F_1(\mu+\frac{1}{2}+i\tau, \mu+\frac{1}{2}-i\tau; \mu+1; -t)$ is the Gauss hypergeometric function, $\mu$ and $\alpha$ are complex parameters with $\Re(\mu) > -1/2$.

The Gauss hypergeometric function [3, p. 57] is defined for $|z| < 1$ as

$${}_2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

$$(\lambda)_n := \lambda(\lambda + 1) \cdots (\lambda + n - 1),\ n = 1, 2 \ldots (\lambda)_0 := 1.$$

For $|z| \geq 1$ is defined as its analytic continuation [16, p. 431] as

$${}_2F_1(a, b; c; z) := \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - tz)^{-a} dt,$$

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$\Re(c) > \Re(b) > 0;\ |\arg(1-z)| < \pi.$

The Gauss hypergeometric function satisfies the following differential equation [3, p. 56]

$$z(1-z)\frac{d^2w}{dz^2} + [c-(a+b+1)z]\frac{dw}{dz} - abw = 0,$$

where

$$w = w(z) = {}_2F_1(a,b;c;z).$$

The integral transform (1.1) was first mentioned in [25] as a particular case of a more general integral transform with the Meijer $G$-function as the kernel.

In a series of papers Hayek, González and Negrín have considered several properties of the index ${}_2F_1$-transform both from a classical point of view and spaces of generalized functions (cf. [5], [6], [7], [9], [10], [11]). Moreover this transform has been cited in [2], [26] and [27].

Abelian theorems have been studied in several works (see [4], [6], [13] and [20]), for certain index transforms. For more details of index transforms see [18], [19] and [26], amongst others.

Abelian theorems for distributional transforms were first established by Zemanian in [28], (see also [1], [4], [6], [15], [21], [22], [23], and [24]).

Now, we consider the differential operator

$$A_t = t^{\alpha-\mu}(t+1)^{\mu}D_t t^{\mu+1}(t+1)^{\mu+1}D_t t^{-\alpha}. \tag{1.2}$$

From [8, (2.3), p. 658] one has that

$$A_t\, {}_2F_1\left(\mu+\frac{1}{2}+i\tau, \mu+\frac{1}{2}-i\tau; \mu+1; -t\right)t^{\alpha}$$
$$= -\left[\left(\mu+\frac{1}{2}\right)^2+\tau^2\right]{}_2F_1\left(\mu+\frac{1}{2}+i\tau, \mu+\frac{1}{2}-i\tau; \mu+1; -t\right)t^{\alpha}. \tag{1.3}$$

Next, from [3, (7), p. 122 and (6), p. 155], we obtain

$${}_2F_1\left(\mu+\frac{1}{2}+i\tau, \mu+\frac{1}{2}-i\tau; \mu+1; -t\right)t^{\alpha} =$$
$$= \frac{\Gamma(\mu+1)t^{\alpha}}{\sqrt{\pi}\Gamma(\mu+\frac{1}{2})}\int_0^{\pi}\left(1+2t+2\sqrt{t(t+1)}\cos\xi\right)^{-\mu-1/2-i\tau}(\sin\xi)^{2\mu}d\xi, \tag{1.4}$$

which is valid for

$$t>0,\ \tau>0,\ \Re(\mu)>-1/2,\ \alpha\in\mathbb{C}.$$

Observe that one has

$$\sin\xi \geq 0, \quad \xi\in[0,\pi],$$
$$1+2\sqrt{t+2t(t+1)}\cos\xi \geq 0, \quad t>0,\ \xi\in[0,\pi],$$

and hence, for $\Re(\mu) > -1/2$, it follows from (1.4) that

$$\left|{}_2F_1\left(\mu+\frac{1}{2}+i\tau, \mu+\frac{1}{2}-i\tau; \mu+1; -t\right)t^{\alpha}\right|$$
$$\leq \frac{\left|\Gamma(\mu+1)\right| t^{\Re(\alpha)}}{\sqrt{\pi}\left|\Gamma\left(\mu+\frac{1}{2}\right)\right|}\int_0^{\pi}\left(1+2t+2\sqrt{t(t+1)}\cos\xi\right)^{-\Re(\mu)-\frac{1}{2}}(\sin\xi)^{2\Re(\mu)}\,d\xi$$

$$= \frac{\left|\Gamma(\mu+1)\right| t^{\Re(\alpha)}}{\sqrt{\pi}\left|\Gamma\left(\mu+\frac{1}{2}\right)\right|}\int_0^{\pi}\left(1+2t+2\sqrt{t(t+1)}\cos\xi\right)^{-\Re(\mu)-\frac{1}{2}}(\sin\xi)^{2\Re(\mu)}\,d\xi$$

$$= \frac{\left|\Gamma(\mu+1)\right|\Gamma(\Re(\mu)+\frac{1}{2})}{\sqrt{\pi}\left|\Gamma\left(\mu+\frac{1}{2}\right)\right|\Gamma(\Re(\mu)+1)}\,{}_2F_1\left(\Re(\mu)+\frac{1}{2},\Re(\mu)+\frac{1}{2};\Re(\mu)+1;-t\right)t^{\Re(\alpha)}. \tag{1.5}$$

Also, from [3, (7), p. 122] and [14, p.171, Entry (12.08) and p. 172, Entry (12.20)], for $\Re(\mu) > -1/2$ we have

$${}_2F_1\left(\Re(\mu)+\frac{1}{2},\Re(\mu)+\frac{1}{2};\Re(\mu)+1;-t\right)t^{\Re(\alpha)} = O\left(t^{\Re(\alpha)}\right),\quad t\to 0^+, \tag{1.6}$$

$${}_2F_1\left(\Re(\mu)+\frac{1}{2},\Re(\mu)+\frac{1}{2};\Re(\mu)+1;-t\right)t^{\Re(\alpha)} = O\left(t^{\Re(\alpha)-\Re(\mu)-\frac{1}{2}}\ln t\right),\quad t\to +\infty. \tag{1.7}$$

## 2. Abelian theorems for the distributional index ${}_2F_1$-transform

The space $\mathcal{E}((0,\infty))$ is defined as the vector space of all infinitely differentiable complex-valued functions $\phi$ defined on $(0,\infty)$. This space equipped with the locally convex topology arising from the family of seminorms

$$\rho_{k,K}(\phi) = \sup_{t\in K}\left|D_t^k\phi(t)\right|$$

for all $k \in \mathbb{N}\cup\{0\}$, all compact sets $K \subset (0,\infty)$, and with $D_t^k$ denoting the $k-$th derivative with respect to the variable $t$, becomes a Fréchet space. As usual, we denote by $\mathcal{E}'((0,\infty))$ the dual of the space $\mathcal{E}((0,\infty))$.

The generalized index ${}_2F_1$-transform of $f \in \mathcal{E}'((0,\infty))$ was defined by the kernel method in [8] by means of

$$F(\tau) = \left\langle f(t), {}_2F_1\left(\mu+\frac{1}{2}+i\tau,\mu+\frac{1}{2}-i\tau;\mu+1;-t\right)t^{\alpha}\right\rangle,\quad \tau>0, \tag{2.1}$$

where $\mu$ and $\alpha$ are complex parameters with $\Re(\mu) > -1/2$.

In this section we establish Abelian theorems for the index ${}_2F_1$-transform (2.1). Previously we prove some results.

The following Lemma was showed in [8, Lemma 2.1, p. 659]

**Lemma 2.1.** *For each compact $K \subset (0,\infty)$ and $k \in \mathbb{N}\cup\{0\}$ let $\gamma_{k,K}$ be the seminorm defined by*

$$\gamma_{k,K}(\phi) = \sup_{t\in K}\left|A_t^k\phi(t)\right|,\quad \phi\in\mathcal{E}'((0,\infty)),$$

*where $A_t$ is the operator given by (1.2). Then, $\{\gamma_{k,K}\}$ gives rise to a topology on $\mathcal{E}'((0,\infty))$ which coincides with is usual topology.*

Now, by using the above Lemma 2.1 we obtain the following result

**Lemma 2.2.** *Set $\Re(\mu) > -1/2$ and $\alpha \in \mathbb{C}$. Let $f$ be in $\mathcal{E}'((0,\infty))$, and let $F$ be defined by (2.1). Then there exist a constant $M > 0$ and a nonnegative integer $p$, all depending on $f$, such that*

$$|F(\tau)| \le M\max_{0\le k\le p}\left[\left(|\mu|+\frac{1}{2}\right)^2+\tau^2\right]^k,\quad \forall\tau>0. \tag{2.2}$$

*Proof.* Observe that ${}_2F_1\left(\mu+\frac{1}{2}+i\tau,\mu+\frac{1}{2}-i\tau;\mu+1;-t\right)t^{\alpha}$ is an eigenfunction of $A_t$, i.e.,

$$A_t\,{}_2F_1\left(\mu+\frac{1}{2}+i\tau,\mu+\frac{1}{2}-i\tau;\mu+1;-t\right)t^{\alpha}$$
$$=-\left[\left(\mu+\frac{1}{2}\right)^2+\tau^2\right]{}_2F_1\left(\mu+\frac{1}{2}+i\tau,\mu+\frac{1}{2}-i\tau;\mu+1;-t\right)t^{\alpha}. \tag{2.3}$$

According to Lemma 2.1, we may consider the space $\mathcal{E}((0,\infty))$ equipped with the topology arising from the family of seminorms $\gamma_{k,K}$. From [12, Proposition 2, p. 97], there exist $C>0$, a compact set $K\subset(0,\infty)$, and a nonnegative integer $p$, all depending on $f$, such that

$$\left|\left\langle f,\phi\right\rangle\right|\leq C\max_{0\leq k\leq p}\max_{t\in K}\left|A_t^k\phi(t)\right| \tag{2.4}$$

for all $\phi\in\mathcal{E}((0,\infty))$. In particular,

$$|F(\tau)|=\left|\left\langle f(t),{}_2F_1\left(\mu+\frac{1}{2}+i\tau,\mu+\frac{1}{2}-i\tau;\mu+1;-t\right)t^{\alpha}\right\rangle\right|$$
$$\leq C\max_{0\leq k\leq p}\max_{t\in K}\left|A_t^k{}_2F_1\left(\mu+\frac{1}{2}+i\tau,\mu+\frac{1}{2}-i\tau;\mu+1;-t\right)t^{\alpha}\right|$$
$$=C\max_{0\leq k\leq p}\max_{t\in K}\left|\left[\left(\mu+\frac{1}{2}\right)^2+\tau^2\right]^k{}_2F_1\left(\mu+\frac{1}{2}+i\tau,\mu+\frac{1}{2}-i\tau;\mu+1;-t\right)t^{\alpha}\right|. \tag{2.5}$$

From (1.5) it follows that, for $\Re(\mu)>-1/2$, (2.5) is bounded above by

$$C\max_{0\leq k\leq p}\max_{t\in K}\left\{\frac{\left|\Gamma(\mu+1)\right|\Gamma(\Re(\mu)+\frac{1}{2})}{\sqrt{\pi}\left|\Gamma\left(\mu+\frac{1}{2}\right)\right|\Gamma(\Re(\mu)+1)}\left[\left(|\mu|+\frac{1}{2}\right)^2+\tau^2\right]^k{}_2F_1\left(\Re(\mu)+\frac{1}{2},\Re(\mu)+\frac{1}{2};\Re(\mu)+1;-t\right)t^{\Re(\alpha)}\right\}$$
$$\leq M\max_{0\leq k\leq p}\left[\left(|\mu|+\frac{1}{2}\right)^2+\tau^2\right]^k. \tag{2.6}$$

for all $\tau>0$ and certain $M>0$, since $t$ ranges on the compact set $K\subset(0,\infty)$. □

The smallest integer $p$ which verifies the inequality (2.4) is defined as the order of the distribution $f$ (cf. [17, Théorème XXIV, p. 88]).

In the following statement we establish Abelian theorems for the distributional index ${}_2F_1$-transform (2.1).

**Theorem 2.3.** *(Abelian theorem) Set $\Re(\mu)>-1/2$ and $\alpha\in\mathbb{C}$. Let $f$ be a member of $\mathcal{E}'((0,\infty))$ of order $r\in\mathbb{N}\cup\{0\}$, and let $F$ be given by (2.1). Then*

*(i) for any $\gamma>0$ one has*

$$\lim_{\tau\to 0^+}\{\tau^{\gamma}F(\tau)\}=0,$$

*(ii) for any $\gamma>0$ one has*

$$\lim_{\tau\to+\infty}\{\tau^{-2r-\gamma}F(\tau)\}=0.$$

*Proof.* From Lemma 2.2 one obtains

$$|F(\tau)|\leq M\max_{0\leq k\leq r}\left[\left(|\mu|+\frac{1}{2}\right)^2+\tau^2\right]^k,\quad\forall\tau>0,$$

for some $M>0$, from which the conclusion follows. □

Next, let $f$ be a locally integrable function on $(0,\infty)$ and $f$ has compact support on $(0,\infty)$, then $f$ gives rise to a regular member $T_f$ of $\mathcal{E}'((0,\infty))$ of order $r=0$ by means of

$$\left\langle T_f,\phi\right\rangle = \int_0^\infty f(t)\phi(t)dt, \quad \forall\phi\in\mathcal{E}((0,\infty)).$$

Observe that

$$\left|< T_f,\phi >\right| = \left|\int_1^\infty f(t)\phi(t)dt\right| \leq \sup_{t\in\mathrm{supp}(f)}\left|\phi(t)\right|\int_{\mathrm{supp}(f)}\left|f(t)\right|dt$$

$$= \gamma_{0,\mathrm{supp}(f)}(\phi)\int_{\mathrm{supp}(f)}\left|f(t)\right|dt,$$

where supp$(f)$ represents the support of the function $f$, it follows that $T_f$ has order $r=0$.

Consequently, we have

$$F(\tau) = \left\langle T_f(t), {}_2F_1\left(\mu+\frac{1}{2}+i\tau,\mu+\frac{1}{2}-i\tau;\mu+1;-t\right)t^\alpha\right\rangle$$

$$= \int_0^\infty f(t)\,{}_2F_1\left(\mu+\frac{1}{2}+i\tau,\mu+\frac{1}{2}-i\tau;\mu+1;-t\right)t^\alpha dt,\ \ \tau>0, \tag{2.7}$$

for $\Re(\mu) > -1/2$.

From this fact one concludes that the index ${}_2F_1$-transform of the regular distribution generated by the function $f$ is the classical index ${}_2F_1$-transform of the function $f$.

Furthermore, by using Theorem 2.3 for the index ${}_2F_1$-transform of these regular members of $\mathcal{E}'((0,\infty))$, one obtains the following

**Corollary 2.4.** *Set* $\Re(\mu) > -1/2$ *and* $\alpha\in\mathbb{C}$. *Let* $f$ *be a locally integrable function in* $(0,\infty)$ *and such that* $f$ *has compact support on* $(0,\infty)$. *Then the function* $F$ *given by (2.7), satisfies the following:*

*(i) for any* $\gamma>0$ *one has*

$$\lim_{\tau\to0^+}\{\tau^\gamma F(\tau)\} = 0,$$

*(ii) for any* $\gamma>0$ *one has*

$$\lim_{\tau\to+\infty}\{\tau^{-\gamma} F(\tau)\} = 0.$$

## 3. Abelian theorems for the index ${}_2F_1$-transform of generalized functions

In [8], Hayek and González studied the index ${}_2F_1$-transform over certain spaces of generalized functions. In that paper it was considered the linear space $U_{a,\mu,\alpha}$ of all smooth complex-valued functions $\phi$ defined on $(0,\infty)$, such that

$$\gamma_{k,a,\mu,\alpha}(\phi) = \sup_{0<t<\infty}\left|(2t+1)^a t^{\frac{\mu}{2}-\alpha}(t+1)^{\frac{\mu}{2}}A_k^t\phi(t)\right| < \infty, \quad k\in\mathbb{N}\cup\{0\}, \tag{3.1}$$

where $A_t$ is the differential operator given by (1.2).

The space $U_{a,\mu,\alpha}$ equipped with the topology arising from the family of seminorms $\{\gamma_{k,a,\mu}\}$ is a Fréchet space.

As usual, by $U'_{a,\mu,\alpha}$ is denoted the dual space of $U_{a,\mu,\alpha}$.

By using (1.5), (1.6) and (1.7) it follows that

$${}_2F_1\left(\mu+\frac{1}{2}+i\tau,\mu+\frac{1}{2}-i\tau;\mu+1;-t\right)t^\alpha \in U_{a,\mu,\alpha},$$

for $\Re(\mu) \geq 0$, $a < 1/2$ and $\alpha \in \mathbb{C}$, and thus, as it is usual, the generalized index ${}_2F_1$-transform is defined for $f \in U'_{a,\mu,\alpha}$ by

$$F(\tau) = \left\langle f(t), {}_2F_1\left(\mu + \frac{1}{2} + i\tau, \mu + \frac{1}{2} - i\tau; \mu + 1; -t\right) t^{\alpha} \right\rangle \; \tau > 0. \tag{3.2}$$

From [12, Proposition 2, p. 97], one has that for all $f \in U'_{a,\mu,\alpha}$, there exist a $C > 0$ and a nonnegative integer $p$, all depending on $f$, such that

$$\left|\left\langle f, \phi \right\rangle\right| \leq C \max_{0\leq k\leq p} \gamma_{k,a,\mu}(\phi) = C \max_{0\leq k\leq p} \sup_{t\in(1,\infty)} \left|(2t+1)^{a} t^{\frac{\mu}{2}-\alpha}(t+1)^{\frac{\mu}{2}} A_t^k \phi(t)\right|, \tag{3.3}$$

for all $\phi \in U_{a,\mu,\alpha}$.

Now we prove Abelian theorems for the transform (3.2). First we prove a previous result

**Lemma 3.1.** *Set $\Re(\mu) \geq 0$, $a < 1/2$ and $\alpha \in \mathbb{C}$. Let $f$ be in $U'_{a,\mu,\alpha}$, and let $F$ be defined by (3.2). Then there exist $M > 0$ and a nonnegative integer $p$, all depending on $f$, such that*

$$|F(\tau)| \leq M \max_{0\leq k\leq p} \left[\left(|\mu| + \frac{1}{2}\right)^2 + \tau^2\right]^k, \forall \tau > 0. \tag{3.4}$$

*Proof.* From (1.5) and (3.3) one has

$$|F(\tau)| = \left|\left\langle f(t), {}_2F_1\left(\mu + \frac{1}{2} + i\tau, \mu + \frac{1}{2} - i\tau; \mu + 1; -t\right) t^{\alpha}\right\rangle\right|$$

$$\leq C \max_{0\leq k\leq p} \sup_{t\in(0,\infty)} \left|(2t+1)^{a} t^{\frac{\mu}{2}-\alpha}(t+1)^{\frac{\mu}{2}} A_t^k {}_2F_1\left(\mu + \frac{1}{2} + i\tau, \mu + \frac{1}{2} - i\tau; \mu + 1; -t\right) t^{\alpha}\right|$$

$$\leq C \max_{0\leq k\leq p} \sup_{t\in(0,\infty)} \left|\left[\left(\mu + \frac{1}{2}\right)^2 + \tau^2\right]^k \frac{\left|\Gamma(\mu+1)\right| \Gamma(\Re(\mu) + \frac{1}{2})}{\sqrt{\pi} \left|\Gamma\left(\mu + \frac{1}{2}\right)\right| \Gamma(\Re(\mu)+1)} {}_2F_1\left(\Re(\mu) + \frac{1}{2}, \Re(\mu) + \frac{1}{2}; \Re(\mu) + 1; -t\right) t^{\Re(\alpha)}\right|.$$

Now, from (1.6) and (1.7), and taking into account the fact that $\Re(\mu) \geq 0$ and $a < 1/2$, it follows that

$$|F(\tau)| \leq M \max_{0\leq k\leq p} \left[\left(|\mu| + \frac{1}{2}\right)^2 + \tau^2\right]^k, \quad \forall \tau > 0,$$

for certain $M > 0$. □

As it is usual, the smallest integer $p$ which verifies the inequality (3.3) is called the order of the generalized function $f$.

The next statement gives an Abelian theorem for the index ${}_2F_1$-transform of generalized functions in $U'_{a,\mu,\alpha}$.

**Theorem 3.2.** *(Abelian theorem) Set $\Re(\mu) \geq 0$, $a < 1/2$ and $\alpha \in \mathbb{C}$. If $f$ is a generalized function on $U'_{a,\mu,\alpha}$, of order $r \in \mathbb{N} \cup \{0\}$, and $F$ is given by (3.2), then*

*(i) for any $\gamma > 0$ one has*

$$\lim_{\tau\to 0^+} \{\tau^{\gamma} F(\tau)\} = 0,$$

*(ii) for any $\gamma > 0$ one has*

$$\lim_{\tau\to +\infty} \{\tau^{-2r-\gamma} F(\tau)\} = 0.$$

*Proof.* From Lemma 3.1 one has

$$|F(\tau)| \leq M \max_{0\leq k\leq r}\left[\left(|\mu| + \frac{1}{2}\right)^2 + \tau^2\right]^k, \quad \forall\tau > 0,$$

for some $M > 0$, and hence the conclusion follows. □

Otherwise, from Proposition 2.1 (v) in [8], a function $f$ defined on $(0,\infty)$ such that $(2t+1)^{-a}t^{\alpha-\frac{\mu}{2}}(t+1)^{-\frac{\mu}{2}}f(t)$, $\Re(\mu) \geq 0$, $a < 1/2$, is Lebesgue integrable on $(0,\infty)$, gives rise to a regular generalized function $T_f$ on $U'_{a,\mu,\alpha}$ or order $r = 0$ through

$$< T_f, \phi >= \int_0^\infty f(t)\phi(t)dt, \quad \forall\phi \in U_{a,\mu,\alpha}.$$

In fact, taking into account that

$$\left|\left\langle T_f, \phi\right\rangle\right| = \left|\int_0^\infty f(t)\phi(t)dt\right|$$

$$= \left|\int_0^\infty (2t+1)^{-a}t^{\alpha-\frac{\mu}{2}}(t+1)^{-\frac{\mu}{2}}f(t)(2t+1)^a t^{\frac{\mu}{2}-\alpha}(t+1)^{\frac{\mu}{2}}\phi(t)dt\right|$$

$$\leq \sup_{t\in(0,\infty)}\left|(2t+1)^a t^{\frac{\mu}{2}-\alpha}(t+1)^{\frac{\mu}{2}}\phi(t)\right|\int_0^\infty \left|(2t+1)^{-a}t^{\alpha-\frac{\mu}{2}}(t+1)^{-\frac{\mu}{2}}f(t)\right|dt$$

$$= \gamma_{0,a,\mu,\alpha}(\phi)\cdot\int_0^\infty (2t+1)^{-a}t^{\Re(\alpha)-\frac{\Re(\mu)}{2}}(t+1)^{-\frac{\Re(\mu)}{2}}\left|f(t)\right|dt,$$

it follows that $T_f$ is a distribution of order $r = 0$.

In this case,

$$F(\tau) = \left\langle T_f(t), {}_2F_1\left(\mu+\frac{1}{2}+i\tau, \mu+\frac{1}{2}-i\tau; \mu+1; -t\right)t^\alpha\right\rangle$$

$$= \int_0^\infty f(t)\,{}_2F_1\left(\mu+\frac{1}{2}+i\tau, \mu+\frac{1}{2}-i\tau; \mu+1; -t\right)t^\alpha dt, \ \tau > 0, \tag{3.5}$$

for $\Re(\mu) \geq 0$.

Again, as in the case of the regular distributions of compact support, it follows that the index ${}_2F_1$-transform of the regular generalized function generated by the function $f$ is the classical index ${}_2F_1$-transform of the function $f$.

Consequently, by Theorem 3.2, one obtains the following

**Corollary 3.3.** *Set $\Re(\mu) \geq 0$, $a < 1/2$ and $\alpha \in \mathbb{C}$. Let $f$ be a function defined on $(0,\infty)$ such that $(2t+1)^{-a}t^{\alpha-\frac{\mu}{2}}(t+1)^{-\frac{\mu}{2}}f(t)$ is Lebesgue integrable on $(0,\infty)$, and $F$ is given by (3.5). Then*

*(i) for any $\gamma > 0$ one has*

$$\lim_{\tau\to 0^+}\{\tau^\gamma F(\tau)\} = 0,$$

*(ii) for any $\gamma > 0$ one has*

$$\lim_{\tau\to+\infty}\{\tau^{-\gamma}F(\tau)\} = 0.$$

## 4. Conclusions

The behaviour of the Gauss hypergeometric function, used as the kernel of the index ${}_2F_1$-transform, allows us to establish Abelian theorems for this transform over distributions of compact support on $(0,\infty)$ and over the space of generalized functions $U'_{a,\mu,\alpha}$ introduced in [8] under the conditions $\Re(\mu) \geq 0$, $a < 1/2$ and $\alpha \in \mathbb{C}$.

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