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Elementary properties of [∞,C]-symmetric operators

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Abstract. Inspired by recent works on $[m]$-complex symmetric operator, we introduce the class of [∞,C]-symmetric operators and study various properties of this class. We study the quasi-nilpotent perturbations of [∞,C]-symmetric operator. Also, we prove that the class of [∞,C]-symmetric operators is norm closed. Finally, we characterize when product of [∞,C]-symmetric operators is also [∞,C]-symmetric operator.

1. Introduction

Let $\mathcal{H}$ be a separable complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators acting on $\mathcal{H}$, and let $\mathbb{N}$, $\mathbb{C}$ be the set of natural numbers and complex numbers, respectively. An operator C on $\mathcal{H}$ is said to be conjugation if C is antilinear operator and satisfies $C^2 = I$ and $(Cx, Cy) = (y, x)$ for all $x, y \in \mathcal{H}$.

In [11], $[m]$-complex symmetric operator with conjugation C is introduced as follow: if there exists some conjugation C satisfying

$$\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}CT^iCT^{m-i} = 0,$$

T is called an $[m]$-complex symmetric operator. For an operator $T \in \mathcal{B}(\mathcal{H})$ and a conjugation C, define $w_m(T)$ as follows:

$$w_m(T) = \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}CT^iCT^{m-i}.$$

It's clear that T is $[m]$-complex symmetric if and only if $w_m(T) = 0$. Moreover,

$$CTC.w_m(T) - w_m(T).T = w_{m+1}(T)$$

holds. Hence every $[m]$-complex symmetric is $[n]$-complex symmetric for each $n \geq m$. But the converse isn't true in general, see [11]. We now introduce the class of [∞,C]-symmetric operators.

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Definition 1.1. *Let $T \in \mathcal{B}(\mathcal{H})$. If T satisfies*

$$\limsup_{m\to\infty} \|w_m(T)\|^{\frac{1}{m}} = 0,$$

then T is said to be an $[\infty,C]$-symmetric operator.

Let $T \in \mathcal{B}(\mathcal{H})$. If T is an $[m]$-complex symmetric operator for some $m \geq 1$, then T is called a finite $[m]$-complex symmetric operator with conjugation C. The class of $[\infty,C]$-symmetric operators is larger than finite $[m]$-complex symmetric operators with conjugation C.

The motivation of studying $[\infty,C]$-symmetric operator comes from recent interests in $[m]$-complex symmetric operator and m-complex symmetric operator [2–11], and $[\infty,C]$-symmetric operator enjoys many properties of $[m]$-complex symmetric operator.

2. $[\infty,C]$-symmetric operator

We next show that the following result about eigenvectors for (∞,C)-isometric operators does not extend to $[\infty,C]$-symmetric operators, see part (a) of Theorem 2.2 in [1].

Theorem 2.1. [1] *Let $T \in \mathcal{B}(\mathcal{H})$. If T is an (∞,C)-isometric operator and satisfies $(T-\alpha)x = 0$ and $(T-\beta)y = 0$ with $\alpha\beta \neq 1$, then $(Cx, y) = 0$.*

Example 2.2. *Let $\mathcal{H} = \mathbb{C}^2$ and let C be a conjugation on $\mathcal{H}$ satisfying $C\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix}$. If $T = \begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix}$ on $\mathbb{C}^2$, simple calculations show that $(T-6)\begin{pmatrix} 2 \\ 5 \end{pmatrix} = 0$, $(T+1)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$, and T is a $[2]$-complex symmetric operator, hence T is an $[\infty,C]$-symmetric operator, while $(C\begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}) = -3 \neq 0$.*

But we have the following result.

Theorem 2.3. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ is an $[\infty,C]$-symmetric operator.*
(i) *If there exist nonzero vectors x, y such that $(T-\alpha)x = 0$ and $(T^*-\beta)y = 0$ with $\alpha \neq \beta$, then $(Cx, y) = 0$.*
(ii) *If there exists nonzero vector x such that $(T-\alpha)x = 0$ and $(T^*-\beta)Cx = 0$, then $\alpha = \beta$.*
(iii) *If there exist sequences of unit vectors $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty}(T-\alpha)x_n = 0$ and $\lim_{n\to\infty}(T^*-\beta)y_n = 0$ with $\alpha \neq \beta$, then $\{(Cx_n, y_n)\}$ has a subsequence $\{(Cx_{nl}, y_{nl})\}$ which converges to 0.*
(iv) *If there exists a sequence of unit vectors $\{x_n\}$ such that $\lim_{n\to\infty}(T-\alpha)x_n = 0$ and $\lim_{n\to\infty}(T^*-\beta)Cx_n = 0$, then $\alpha = \beta$.*

Proof. (i) Let x, y be nonzero vectors such that $(T-\alpha)x = 0$ and $(T^*-\beta)y = 0$. Then

$$\begin{aligned}
(Cw_m(T)x, y) &= (C\sum_{i=0}^{m}(-1)^{m-i}\tbinom{m}{i}CT^iCT^{m-i}x, y) \\
&= (\sum_{i=0}^{m}(-1)^{m-i}\tbinom{m}{i}T^i\overline{\alpha}^{m-i}Cx, y) \\
&= \sum_{i=0}^{m}(-1)^{m-i}\tbinom{m}{i}(\overline{\alpha}^{m-i}Cx, \beta^i y) \\
&= \sum_{i=0}^{m}(-1)^{m-i}\tbinom{m}{i}\overline{\alpha}^{m-i}\overline{\beta}^i(Cx, y) \\
&= (\overline{\beta}-\overline{\alpha})^m(Cx, y),
\end{aligned}$$

and hence

$$|\overline{\beta}-\overline{\alpha}||(Cx,y)|^{\frac{1}{m}} = |(Cw_m(T)x,y)|^{\frac{1}{m}} \leq \|w_m(T)\|^{\frac{1}{m}}\|x\|^{\frac{1}{m}}\|y\|^{\frac{1}{m}}.$$

Since T is an $[\infty,C]$-symmetric operator, we have

$$\lim_{m\to\infty}|\overline{\beta}-\overline{\alpha}||(Cx,y)|^{\frac{1}{m}} \leq \lim_{m\to\infty}\|w_m(T)\|^{\frac{1}{m}}\|x\|^{\frac{1}{m}}\|y\|^{\frac{1}{m}} = 0. \tag{2.1}$$

Since $\alpha \neq \beta$, it follows from (2.1) that

$$\lim_{m\to\infty}|(Cx,y)|^{\frac{1}{m}} = 0.$$

This implies $(Cx,y) = 0$.

(ii) Assume that $\alpha \neq \beta$. Set $y = Cx$. Then y is a nonzero vector. By (i), $\|x\|^2 = (Cx,Cx) = 0$, which contradicts with the fact that x is a nonzero vector. Hence $\alpha = \beta$.

(iii) Let $\{x_n\}$ and $\{y_n\}$ be sequences of unit vectors such that

$$\lim_{n\to\infty}(T-\alpha)x_n = 0 \text{ and } \lim_{n\to\infty}(T^*-\beta)y_n = 0.$$

Since $\{(Cx_n,y_n)\}_{n=1}^{\infty}$ is bounded, there exists a convergent subsequence $\{(Cx_{nl},y_{nl})\}$. Set $\lim_{l\to\infty}(Cx_{nl},y_{nl}) = \mu$. For $\forall m \geq 1$,

$$\begin{aligned}
|(\overline{\alpha}-\overline{\beta})^m\mu| &= |(\overline{\alpha}-\overline{\beta})^m|\lim_{l\to\infty}|(Cx_{nl},y_{nl})| \\
&= \lim_{l\to\infty}|\sum_{i=0}^{m}(-1)^{m-i}\tbinom{m}{i}\overline{\alpha}^{m-i}\overline{\beta}^{i}(Cx_{nl},y_{nl})| \\
&= \lim_{l\to\infty}|\sum_{i=0}^{m}(-1)^{m-i}\tbinom{m}{i}(CT^{m-i}x_{nl},T^{*i}y_{nl})| \\
&= \lim_{l\to\infty}|(C\sum_{i=0}^{m}(-1)^{m-i}\tbinom{m}{i}CT^{i}CT^{m-i}x_{nl},y_{nl})| \\
&= \lim_{l\to\infty}|(Cw_m(T)x_{nl},y_{nl})| \\
&\leq \|w_m(T)\|.
\end{aligned}$$

Since T is an $[\infty,C]$-symmetric operator, we have

$$|\overline{\alpha}-\overline{\beta}|\lim_{m\to\infty}|\mu|^{\frac{1}{m}} \leq \limsup_{m\to\infty}\|w_m(T)\|^{\frac{1}{m}} = 0.$$

Since $\alpha \neq \beta$, it follows that $\mu = 0$, i.e., $\lim_{l\to\infty}(Cx_{nl},y_{nl}) = 0$.

(iv) Assume that $\alpha \neq \beta$. Set $y_n = Cx_n$. It follows from (iii) that $\{(Cx_n,Cx_n)\}$ has a subsequence $\{(Cx_{nl},Cx_{nl})\}$ which converges to 0. While $(Cx_{nl},Cx_{nl}) = 1$, which is a contradiction. Hence $\alpha = \beta$. □

Theorem 2.4. *Suppose that $T \in \mathcal{B}(\mathcal{H})$. If $TCTC = CTCT$, then*

$$\limsup_{m\to\infty}\|w_m(T)\|^{\frac{1}{m}} = r(T-CTC),$$

where $r(A)$ denotes the spectral radius of A. In particular, if $r(T-CTC)=0$, then T is an $[\infty,C]$-symmetric operator.

Proof. Since $TCTC = CTCT$, we have

$$w_m(T) = \sum_{i=0}^{m}(-1)^{m-i}\tbinom{m}{i}CT^{m}CT^{m-i} = (CTC-T)^m,$$

and hence

$$\limsup_{m\to\infty} \|w_m(T)\|^{\frac{1}{m}} = \limsup_{m\to\infty} \|(T - CTC)^m\|^{\frac{1}{m}} = r(T - CTC).$$

In particular, if $r(T - CTC) = 0$, then T is an $[\infty, C]$-symmetric operator. □

Lemma 2.5. *Suppose that $T, Q \in \mathcal{B}(\mathcal{H})$ satisfy $TQ = QT$ and $TCQC = CQCT$. Then, for $m \geq 2$,*

$$\|w_m(T + Q)\| \leq M^m(\max_{j\leq n\leq m} \|w_n(T)\| + \max_{j\leq n\leq m} \|Q^n\|),$$

where $M = 2(2\|T\| + 2\|Q\| + 1)$ and $j = [\frac{m}{3}]$ is the integer part of $\frac{m}{3}$.

Proof. Since

$$\begin{aligned}[(a + b) - (c + d)]^m =&[(a - c) + b - d]^m \\ =& \sum_{m_1+m_2+m_3=m} (-1)^{m_2} \binom{m}{m_1, m_2, m_3}(a - c)^{m_1} d^{m_2} b^{m_3},\end{aligned}$$

we have

$$w_m(T + Q) = \sum_{m_1+m_2+m_3=m} (-1)^{m_2} \binom{m}{m_1, m_2, m_3} w_{m_1}(T)(CQC)^{m_2} Q^{m_3}.$$

Suppose that $j = [\frac{m}{3}]$ is the integer part of $\frac{m}{3}$. Put

$$M_i = \sum_{m_1+m_2+m_3=m, m_i\geq j} \binom{m}{m_1, m_2, m_3}\|w_{m_1}(T)(CQC)^{m_2} Q^{m_3}\|, i = 1, 2, 3.$$

Since $m_1 + m_2 + m_3 = m$, then there exists some $m_i \geq j$, $i = 1, 2, 3$, and

$$\begin{aligned}\|w_m(T + Q)\| \leq& \sum_{m_1+m_2+m_3=m} \binom{m}{m_1, m_2, m_3}\|w_{m_1}(T)(CQC)^{m_2} Q^{m_3}\| \\ \leq& M_1 + M_2 + M_3.\end{aligned}$$

On the other hand,

$$\begin{aligned}M_1 =& \sum_{m_1+m_2+m_3=m, m_1\geq j} \binom{m}{m_1, m_2, m_3}\|w_{m_1}(T)(CQC)^{m_2} Q^{m_3}\| \\ \leq& \sum_{m_1+m_2+m_3=m, m_1\geq j} \binom{m}{m_1, m_2, m_3}\|w_{m_1}(T)\|\|Q\|^{m_2}\|Q\|^{m_3} \\ \leq& \max_{j\leq n\leq m} \|w_n(T)\|(2\|Q\| + 1)^m \\ \leq& (\frac{M}{2})^m \max_{j\leq n\leq m} \|w_n(T)\|.\end{aligned}$$

Since $\|w_k(T)\| \leq (2\|T\|)^k$ for all $k \in \mathbb{N}$, by the similar way, we have

$$\begin{aligned}M_2 \leq& \max_{j\leq n\leq m} \|Q^n\|(2\|T\| + \|Q\| + 1)^m \\ \leq& (\frac{M}{2})^m \max_{j\leq n\leq m} \|Q^n\|\end{aligned}$$

and

$$\begin{aligned}M_3 \leq& \max_{j\leq n\leq m} \|Q^n\|.(2\|T\| + \|Q\| + 1)^m \\ \leq& (\frac{M}{2})^m \max_{j\leq n\leq m} \|Q^n\|.\end{aligned}$$

Hence

$$\begin{aligned}\|w_m(T+Q)\| \leq& (\frac{M}{2})^m \max_{j\leq n\leq m}\|w_n(T)\| + 2(\frac{M}{2})^m \max_{j\leq n\leq m}\|Q^n\|\\ \leq& M^m(\max_{j\leq n\leq m}\|w_n(T)\| + \max_{j\leq n\leq m}\|Q^n\|).\end{aligned}$$

□

Theorem 2.6. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ and C is a conjugation on $\mathcal{H}$. Then the following statements hold:*
(i) If T is an $[\infty,C]$-symmetric operator, $Q^n = 0$ for some $n \in \mathbb{N}$, $TQ = QT$ and $TCQC = CQCT$, then $T + Q$ is an $[\infty,C]$-symmetric operator.
(ii) If T_n is a sequence of commuting $[\infty,C]$-symmetric operators which satisfy $\lim_{n\to\infty}\|T_n - T\| = 0$, then T is an $[\infty,C]$-symmetric operator.

Proof. (i) Let T be an $[\infty,C]$-symmetric operator and $Q^n = 0$ for some $n \in \mathbb{N}$, Then for a given $0 < \varepsilon < 1$, there exists N which satisfies

$$\|w_n(T)\| \leq \varepsilon^n \text{ and } \|Q^n\| \leq \varepsilon^n$$

for all $n \geq N$. It follows from Lemma 2.5, for $m \geq 3N$ and $j = [\frac{m}{3}] \geq N$,

$$\begin{aligned}\|w_m(T+Q)\|^{\frac{1}{m}} \leq& M(\max_{j\leq n\leq m}\|w_n(T)\| + \max_{j\leq n\leq m}\|Q^n\|)^{\frac{1}{m}}\\ \leq& M(2\varepsilon^n)^{\frac{1}{m}} \leq M(2\varepsilon^j)^{\frac{1}{m}}\\ =& 2^{\frac{1}{m}}M\varepsilon^{\frac{j}{m}} = 2^{\frac{1}{m}}M\varepsilon^{\frac{1}{m}[\frac{m}{3}]}.\end{aligned}$$

Since ε is arbitrary, $\limsup_{m\to\infty}\|w_m(T+Q)\|^{\frac{1}{m}} = 0$, i.e., $T + Q$ is an $[\infty,C]$-symmetric operator.

(ii) Suppose that $T_nT_k = T_kT_n$ for all $k, n \in \mathbb{N}$. Then $TT_n = T_nT$ for all $n \geq 1$. For a given $0 < \varepsilon < 1$, there exists n_0 which satisfies

$$\|T - T_{n_0}\| \leq \varepsilon \text{ and } \|w_n(T_{n_0})\| \leq \varepsilon^n$$

for all $n \geq n_0$. It follows from Lemma 2.5, for $m \geq 3n_0$ and $j = [\frac{m}{3}] \geq n_0$,

$$\begin{aligned}\|w_m(T)\|^{\frac{1}{m}} =& \|w_m(T_{n_0} + T - T_{n_0})\|^{\frac{1}{m}}\\ \leq& M(\max_{j\leq n\leq m}\|w_n(T_{n_0})\| + \max_{j\leq n\leq m}\|T - T_{n_0}\|^n)^{\frac{1}{m}}\\ \leq& 2^{\frac{1}{m}}M\varepsilon^{\frac{j}{m}} = 2^{\frac{1}{m}}M\varepsilon^{\frac{1}{m}[\frac{m}{3}]}.\end{aligned}$$

Since ε is arbitrary, $\limsup_{m\to\infty}\|w_m(T)\|^{\frac{1}{m}} = 0$, i.e., T is an $[\infty,C]$-symmetric operator. □

We use Theorem 2.6 (ii) to illustrate the following example.

Example 2.7. *Let $C_n : \mathbb{C}^n \to \mathbb{C}^n$ be the conjugation given by*

$$C_n(x_1, x_2, \cdots, x_n)^T = (\overline{x_1}, \overline{x_2}, \cdots, \overline{x_n})^T.$$

Put $T = \oplus_{n=1}^{\infty} T_n$, where T_n is an nth order matrix such that

$$\begin{aligned}T_n =& I_n + N_n\\ =& \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0\\ 0 & 1 & 0 & \cdots & 0 & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & \ddots & 1 & 0\\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0\\ \frac{1}{2n} & 0 & 0 & \cdots & 0 & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & \ddots & 0 & 0\\ 0 & 0 & 0 & \cdots & \frac{1}{2n} & 0 \end{pmatrix}.\end{aligned}$$

Since N_n is a nilpotent operator of order n, it follows from [11] that T_n is a $[2n-1]$-complex symmetric operator with conjugation C_n, we have T is an $[\infty, C]$-symmetric operator with a conjugation $C = \oplus_{n=1}^{\infty} C_n$. In fact, Set $S_n = T_1 \oplus \cdots \oplus T_n \oplus I \oplus I \oplus \cdots$. Then S_n is a $[2n-1]$-complex symmetric operator with conjugation C and $S_n S_k = S_k S_n$ for all $n, k \geq 1$. Since $S_n \to T$ in the operator norm, it follows from Theorem 2.6 (ii) that T is an $[\infty, C]$-symmetric operator.

In the following, we study the product properties of $[\infty,C]$-symmetric operators.

Lemma 2.8. *Suppose that $T, R \in \mathcal{B}(\mathcal{H})$ satisfy $TR = RT$ and $T(CRC) = (CRC)T$. Then*

$$w_m(TR) = \sum_{i=0}^{m} \binom{m}{i} CT^i C w_{m-i}(T) w_i(R) R^{m-i},$$

where $w_0(*) = I$.

Proof. Suppose that $TR = RT$ and $T(CRC) = (CRC)T$. Since

$$\begin{aligned}(ab-cd)^m &= [(a-c)b + (b-d)c]^m \\ &= \sum_{i=0}^{m} \binom{m}{i} c^i (a-c)^{m-i} (b-d)^i b^{m-i},\end{aligned}$$

it follows that

$$\begin{aligned}w_m(TR) &= \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} C(TR)^i C(TR)^{m-i} \\ &= \sum_{i=0}^{m} \binom{m}{i} CT^i C w_{m-i}(T) w_i(R) R^{m-i}.\end{aligned}$$

□

Theorem 2.9. *Suppose that T and R are $[\infty,C]$-symmetric operators. If $TR = RT$ and $T(CRC) = (CRC)T$, then TR is an $[\infty,C]$-symmetric operator.*

Proof. Suppose that T and R are $[\infty,C]$-symmetric operators. Then for a given $0 < \varepsilon < 1$, there exist N_1 and N_2 such that

$$\|w_{n_1}(T)\| \leq \varepsilon^n \text{ and } \|w_{n_2}(R)\| \leq \varepsilon^n$$

for $n_1 \geq N_1$ and $n_2 \geq N_2$. Set $N = \max\{N_1, N_2\}$. Then it suffices to show that there exists a constant $M > 0$ which satisfies for $m \geq 2N$,

$$\|w_m(TR)\| \leq M^m \varepsilon^{\frac{m}{2}}.$$

Let $j = [\frac{m}{2}]$ denote the integer part of $\frac{m}{2}$. It follows from Lemma 2.8 that

$$\begin{aligned}w_m(TR) &= \sum_{i=0}^{j} \binom{m}{i} CT^i C w_{m-i}(T) w_i(R) R^{m-i} \\ &\quad + \sum_{i=j+1}^{m} \binom{m}{i} CT^i C w_{m-i}(T) w_i(R) R^{m-i}.\end{aligned}$$

If $i \le j = [\frac{m}{2}]$, then $m - i \ge [\frac{m}{2}] = j \ge N$, and so $\|w_{m-i}(T)\| \le \varepsilon^{m-i} \le \varepsilon^j$. Since $\|C\| = 1$, $\|w_i(R)\| \le (2\|R\|)^i$ for all $i \ge 1$. Thus we have

$$\begin{aligned}
&\|\sum_{i=0}^{j} \binom{m}{i} CT^i C w_{m-i}(T) w_i(R) R^{m-i}\| \\
&\le \sum_{i=0}^{j} \binom{m}{i} \|w_{m-i}(T)\| \|CT^i C\| \|R^{m-i}\| \|w_i(R)\| \\
&\le \sum_{i=0}^{j} \binom{m}{i} \varepsilon^j \|T\|^i \|R\|^{m-i} (2\|R\|)^i \\
&\le \varepsilon^j (2\|T\|\|R\| + \|R\|)^m.
\end{aligned}$$

Similarly, if $i \ge j + 1 \ge N$, then $\|w_i(R)\| \le \varepsilon^j$, and hence we have

$$\|\sum_{i=j+1}^{m} \binom{m}{i} CT^i C w_{m-i}(T) w_i(R) R^{m-i}\| \le \varepsilon^j (\|T\| + 2\|T\|\|R\|)^m.$$

Then for $m \ge 2N$

$$\|w_m(TR)\| \le \varepsilon^{[\frac{m}{2}]}((2\|T\|\|R\| + \|R\|)^m + (\|T\| + 2\|T\|\|R\|)^m).$$

Hence $\limsup_{m\to\infty} \|w_m(TR)\|^{\frac{1}{m}} = 0$, i.e., TR is an $[\infty,C]$-symmetric operator. □

We use Theorem 2.9 to illustrate the following example.

Example 2.10. *Let C be the conjugation on $\mathcal{H}$ given by*

$$C(x_1, x_2, \cdots, x_n, \cdots)^T = (\overline{x_1}, \overline{x_2}, \cdots, \overline{x_n}, \cdots)^T.$$

Suppose that $T, S \in \mathcal{B}(\mathcal{H})$ satisfy $Te_n = \alpha e_n$ and $Se_n = \beta_n e_{n+1}$ with $\beta_n = \frac{1}{n}$ for all $n \ge 1$. Then T and $S + I$ are $[\infty,C]$-symmetric operators, and it is easy to compute

$$TCSCe_n = TCSe_n = TC(\beta_n e_{n+1}) = T\overline{\beta_n} e_{n+1} = \alpha\overline{\beta_n} e_{n+1}$$

and

$$CSCTe_n = CSC(\alpha e_n) = CS(\overline{\alpha} e_n) = C(\overline{\alpha}\beta_n e_{n+1}) = \alpha\overline{\beta_n} e_{n+1}.$$

Moreover, $TSe_n = T\beta_n e_{n+1} = \beta_n \alpha e_{n+1}$ and $STe_n = S\alpha e_n = \alpha\beta_n e_{n+1}$. Hence $TCSC = CSCT$ and $TS = ST$, it follows from Theorem 2.9 that $T(I + S)$ is an $[\infty,C]$-symmetric operator.

Corollary 2.11. *Suppose that T is an $[\infty,C]$-symmetric operator. If $T(CTC) = (CTC)T$, then T^n is an $[\infty,C]$-symmetric operator for any $n \in \mathbb{N}$.*

Proof. We shall prove T^n is an $[\infty,C]$-symmetric operator by induction. It's easy to show that T^2 is an $[\infty,C]$-symmetric operator by Theorem 2.9. Assume that T^{n-1} is an $[\infty,C]$-symmetric operator. Since $T^{n-1}CTC = CTCT^{n-1}$, it follows from Theorem 2.9 that T^n is an $[\infty,C]$-symmetric operator. □

Theorem 2.12. *Suppose that $T \in \mathcal{B}(\mathcal{H})$. Then the following statements hold:*
(i) *T is an $[\infty,C]$-symmetric operator if and only if T^* is an $[\infty,C]$-symmetric operator.*
(ii) *If T is an invertible $[\infty,C]$-symmetric operator, then T^{-1} is an $[\infty,C]$-symmetric operator.*

Proof. (i) Let T be an $[\infty,C]$-symmetric operator. Since

$$w_m(T^*) = \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}CT^{*i}CT^{*m-i},$$

then

$$\begin{aligned} w_m(T^*) &= \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}CT^{*i}CT^{*m-i} \\ &= C\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}T^{*i}CT^{*m-i}CC \\ &= \begin{cases} C(w_m(T))^*C, & if\ m\ is\ even, \\ -C(w_m(T))^*C, & if\ m\ is\ odd. \end{cases} \end{aligned}$$

Therefore,

$$\begin{aligned} \limsup_{m\to\infty} \|w_m(T^*)\|^{\frac{1}{m}} &= \limsup_{m\to\infty} \|C(w_m(T))^*C\|^{\frac{1}{m}} \\ &\leq \limsup_{m\to\infty} \|(w_m(T))^*\|^{\frac{1}{m}} \\ &= \limsup_{m\to\infty} \|w_m(T)\|^{\frac{1}{m}} \\ &= 0, \end{aligned}$$

i.e., T^* is an $[\infty,C]$-symmetric operator. The converse implication holds by a similar way.

(ii) Note for any $b, c \in \mathbb{C}$,

$$b^m(c^{-1} - b^{-1})^m c^m = (b-c)^m = \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}b^i c^{m-i}.$$

Take $c = T$ and $b = CTC$. Then we have

$$w_m(T) = (-1)^m(CTC)^m w_m(T^{-1})T^m.$$

Therefore,

$$(-1)^m(CTC)^{-m}w_m(T)T^{-m} = w_m(T^{-1}).$$

Hence

$$\limsup_{m\to\infty} \|w_m(T^{-1})\|^{\frac{1}{m}} \leq \limsup_{m\to\infty} \|T^{-1}\|\|w_m(T)\|^{\frac{1}{m}}\|T^{-1}\| = 0,$$

i.e., T^{-1} is an $[\infty,C]$-symmetric operator. □

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