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$N(\kappa)$-contact Riemann solitons with certain potential vector fields

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Abstract. In the present article we find the nature of $N(\kappa)$-contact metric manifolds admitting Riemann solitons with some restrictions on the potential vector fields. The cases, when the potential vector field is collinear with the Reeb vector field and when it is infinitesimal contact transformation are specially treated. Moreover, it is proved that if the potential vector field is a gradient vector field, then the manifold considered is isometric to a product manifold. The validity of the obtained results are ensured with two non-trivial examples.

1. Introduction

The theory of solitons is primarily associated with non-linear partial differential equations. But after the introduction of the famous theory of Hamilton's Ricci flow [19] the study of Ricci soliton [22, 23] and Yamabe soliton [1] has taken a leading role in the research area of geometric partial differential equations and Riemannian geometry. A Ricci soliton is a self similar solution, upto diffeomorphisms and scaling, of a Ricci flow which is a pseudo-parabolic heat type partial differential equation. The similar definition applies for a Yamabe soliton. Different aspects of Ricci solitons and Yamabe solitons in the context of contact and symplectic geometry have been studied by several authors [9, 10, 13, 17, 21, 24, 31–33]. Recently Falcitelli, Sarkar and Halder [16] studied conformal Ricci solitons in the perspective of α-Kenmotsu manifolds [16]. As a natural trend of Mathematics the concept of Ricci solitons was extended to Riemann solitons by Udrişte [28, 29] and subsequently it has been studied by several geometers [6–8, 14, 15, 25, 30]. The similarities and dissimilarities between Ricci solitons and Riemann solitons are pointed out by Udrişte [29]. It is known that Ricci solitons are generalizations of Einstein manifolds while Riemann solitons are natural generalizations of spaces of constant sectional curvature. In [6], Blaga analyzed Riemann solitons with some types of potential vector fields. Potential vector fields play a pivotal role to determine the nature of solitons. Sometimes the restrictions imposed on potential vector fields give finer results.

On the other hand $N(\kappa)$-contact manifolds form an important class of contact manifolds. Such manifolds bear more information than Sasakian manifolds because here κ is any real number and in particular case for $\kappa = 1$, it is Sasakian. So our study will also be applicable for Sasakian manifolds which are backbone of contact geometry. These facts motivate us to study $N(\kappa)$-contact metric manifolds admitting Riemann solitons with certain potential vector fields. Here we establish some classification results.

The paper is organized as follows: After introduction, we give some basic definitions and curvature properties of $N(\kappa)$-contact metric manifolds in the Section 2. In Section 3, we derive some characterizations

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of Riemann solitons on $N(\kappa)$-contact metric manifolds. Next section deals with gradient Riemann solitons. In the last section, we give two examples to support our results.

2. Preliminaries

A $(2m+1)$-dimensional differentiable manifold M equipped with a $(1,1)$ tensor field ϕ, a vector field ζ, a 1-form η satisfying [12]

$$\phi^2(V_1) = -V_1 + \eta(V_1)\zeta, \quad \eta(\zeta) = 1, \tag{1}$$

for any vector field $V_1 \in \chi(M)$, the set of all vector fields on M, is known as an almost contact manifold. An almost contact manifold is called an almost contact metric manifold if it admits a Riemannian metric g such that

$$g(\phi V_1, \phi V_2) = g(V_1, V_2) - \eta(V_1)\eta(V_2). \tag{2}$$

As a consequence of (1) and (2), we obtain the following:

$$\phi\zeta = 0, \quad g(V_1, \zeta) = \eta(V_1), \quad \eta(\phi V_1) = 0,$$

$$g(\phi V_1, V_2) = -g(V_1, \phi V_2),$$

$$(\nabla_{V_1}\eta)(V_2) = g(\nabla_{V_1}\zeta, V_2),$$

for any vector fields V_1, $V_2 \in \chi(M)$.

An almost contact metric manifold is called a contact metric manifold if the almost contact metric structure (ϕ, ζ, η, g) satisfy the following condition [12]

$$g(V_1, \phi V_2) = d\eta(V_1, V_2),$$

for all vector fields V_1, $V_2 \in \chi(M)$. For a contact metric manifold M, we define a symmetric $(1,1)$-tensor field h by $h = \frac{1}{2}\mathcal{L}_\zeta\phi$, where $\mathcal{L}_\zeta\phi$ denotes the Lie derivative of ϕ in the direction ζ and satisfy the following relations

$$h\zeta = 0, \quad h\phi + \phi h = 0, \quad trace(h) = trace(h\phi) = 0,$$

$$\nabla_{V_1}\zeta = -\phi V_1 - \phi h V_1. \tag{3}$$

The notion of κ-nullity distribution on a Riemannian manifold was coined by Tanno [26]. In a Riemannian manifold M, the κ-nullity distribution is defined by

$$N(\kappa) : q \longrightarrow N_q(\kappa) = \{V_3 \in T_qM : R(V_1, V_2)V_3 = \kappa[g(V_2, V_3)V_1 - g(V_1, V_3)V_2]\},$$

for any vector fields V_1, $V_2 \in T_qM$, where κ is a real number and T_qM is the Lie algebra of all vector fields at q. A $(2m+1)$-dimensional contact metric manifold is called $N(\kappa)$-contact metric manifold if the characteristic vector field ζ belongs to the κ-nullity distribution. So, for an $N(\kappa)$-contact metric manifold, we have

$$R(V_1, V_2)\zeta = \kappa\{\eta(V_2)V_1 - \eta(V_1)V_2\}. \tag{4}$$

If $\kappa = 1$, then the manifold is Sasakian manifold and for $\kappa = 0$, the manifold is locally isometric to the product of a flat $(m+1)$-dimensional manifold and a m-dimensional manifold with scalar curvature 4, provided $m>1$. In case $m = 1$ and $\kappa = 0$, the manifold is flat [2]. For proper $N(\kappa)$-contact metric manifold, $\kappa<1$. For more details see [2–4, 12].

For a $N(\kappa)$-contact metric manifold of dimension $(2m+1), m \geq 1$, we have [12]

$$h^2 = (\kappa - 1)\phi^2,$$

$$(\nabla_{V_1}\phi)(V_2) = g(V_1 + hV_1, V_2)\zeta - \eta(V_2)(V_1 + hV_1),$$

$$R(V_1, V_2)\zeta = \kappa\{\eta(V_2)V_1 - \eta(V_1)V_2\}, \tag{5}$$

$$R(\zeta, V_1)V_2 = \kappa\{g(V_1, V_2)\zeta - \eta(V_2)V_1\},$$

$$\begin{aligned} S(V_1, V_2) =&2(m-1)\{g(V_1, V_2) + g(hV_1, V_2)\} \\ &+\{2m\kappa - 2(m-1)\}\eta(V_1)\eta(V_2), \end{aligned} \tag{6}$$

$$S(V_1, \zeta) = 2m\kappa\eta(V_1), \tag{7}$$

$$(\nabla_{V_1}\eta)(V_2) = g(V_1 + hV_1, \phi V_2), \tag{8}$$

$$\begin{aligned} (\nabla_{V_1}h)(V_2) =&\{(1-\kappa)g(V_1, \phi V_2) + g(V_1, h\phi V_2)\}\zeta \\ &+\eta(V_2)\{h(\phi V_1 + \phi hV_1)\}, \end{aligned} \tag{9}$$

$$r = 2m(2m - 2 + \kappa), \tag{10}$$

for any vector fields V_1, V_2, $\in \chi(M)$, where R, S and r are the Riemannian curvature, Ricci tensor and scalar curvature, respectively.

Remembering Blair ([5], p-72) and Tanno [27], we give the following definition

Definition 2.1. *A vector field Z on a $N(\kappa)$-contact metric manifold M is called an infinitesimal contact transformation if it fulfills*

$$\mathcal{L}_Z\eta = f\eta,$$

for some smooth function f on M. If $f = 0$, then the vector field Z is called a strict infinitesimal contact transformation.

The notion of Riemann flow was coined by Udrişte [28, 29]. The Riemann flow on a Riemannian manifold M of dimension (2m+1) is defined by

$$\frac{\partial}{\partial t}G(t) = -2R(g(t)),$$

$G = \frac{1}{2}g \otimes g$, where $\otimes$ denotes the Kulkarni-Nomizu product defined by

$$\begin{aligned} (P_1 \otimes P_2)(V_1, V_2, V_3, V_4) =&P_1(V_1, V_4)P_2(V_2, V_3) + P_1(V_2, V_3)P_2(V_1, V_4) \\ &-P_1(V_1, V_3)P_2(V_2, V_4) - P_1(V_2, V_4)P_2(V_1, V_3), \end{aligned} \tag{11}$$

for any vector fields V_1, V_2, V_3, V_4; and R is the Riemannian curvature tensor of type (0,4) corresponding to the Riemannian metric g.

In 2016, Hirică and Udrişte [18] introduced the notion of Riemann solitons. In a Riemannian manifold of dimension (2m+1), a Riemannian metric g is called a Riemann soliton if the following equation holds

$$2R + \lambda g \otimes g + g \otimes \mathcal{L}_Z g = 0, \tag{12}$$

$\mathcal{L}_Z$ being the Lie derivative along the smooth vector field Z and λ is a constant. The vector field Z is known as a potential vector field. A Riemann soliton is denoted by (g, Z, λ). The soliton is said to be expanding or steady or shrinking according as $\lambda>0$ or $\lambda = 0$ or $\lambda<0$.

If the vector field Z is the gradient of a smooth function f, then the soliton is said to be a gradient Riemann soliton. Thus for the gradient Riemann soliton the equation (12) reduces to

$$R + \frac{\lambda}{2}g \otimes g + g \otimes \nabla^2 f = 0,$$

$\nabla^2 f$ being the Hessian of the function f.

3. Riemann solitons on $N(\kappa)$-contact metric manifolds with certain potential vector fields

In this section we study Riemann solitons on $N(\kappa)$-contact metric manifolds. Before entering the main topic we prove the following Lemma.

Lemma 3.1. *If $N(\kappa)$-contact metric manifolds admit Riemann solitons, then the divergence of the potential vector field is constant.*

Proof. Let M be a $N(\kappa)$-contact metric manifold of dimension $(2m+1)$ admitting Riemann soliton. Then, from (11) and (12), we obtain

$$\begin{aligned} 2R(V_1, V_2, V_3, V_4) + 2\lambda\{g(V_1, V_4)g(V_2, V_3) - g(V_1, V_3)g(V_2, V_4)\} \\ +g(V_1, V_4)(\mathcal{L}_Z g)(V_2, V_3) + g(V_2, V_3)(\mathcal{L}_Z g)(V_1, V_4) \\ -g(V_1, V_3)(\mathcal{L}_Z g)(V_2, V_4) - g(V_2, V_4)(\mathcal{L}_Z g)(V_1, V_3) = 0. \end{aligned} \tag{13}$$

Contracting V_1 and V_4 in the above equation, we obtain

$$2S(V_2, V_3) + 2(2m\lambda + divZ)g(V_2, V_3) + (2m-1)(\mathcal{L}_Z g)(V_2, V_3) = 0, \tag{14}$$

where '*div*' is the divergence operator. Again contracting the above equation, we get

$$divZ = -\frac{r + 2m(2m+1)\lambda}{4m}. \tag{15}$$

Using (10) in (15), we get

$$divZ = -\frac{(2m-2+\kappa) + (2m+1)\lambda}{2}, \tag{16}$$

which is a constant. □

Using (16) in (14), we obtain

$$\begin{aligned} 2S(V_2, V_3)+\{(2m-1)\lambda - (2m-2+\kappa)\}g(V_2, V_3) \\ +(2m-1)(\mathcal{L}_Z g)(V_2, V_3) = 0. \end{aligned} \tag{17}$$

Theorem 3.2. *If a (2m+1)-dimensional $N(\kappa)$-contact metric manifold admits Riemann solitons and the potential vector field is pointwise collinear with the Reeb vector field ζ, then the potential vector field is a constant multiple of the Reeb vector field ζ.*

Proof. Let the potential vector field Z be pointwise collinear with the Reeb vector field ζ, i.e., $Z = \rho\zeta$, where ρ is a function on the manifold. Then from (17), we get

$$\begin{aligned} 2S(V_2, V_3)+\{(2m-1)\lambda - (2m-2+\kappa)\}g(V_2, V_3) + (2m-1)\{(V_2\rho)\eta(V_3) \\ +(V_3\rho)\eta(V_2) + \rho g(\nabla_{V_2}\zeta, V_3) + \rho g(\nabla_{V_3}\zeta, V_2)\} = 0. \end{aligned} \tag{18}$$

Using (3) in (18), we obtain

$$\begin{aligned} 2S(V_2, V_3)+\{(2m-1)\lambda - (2m-2+\kappa)\}g(V_2, V_3) + (2m-1)\{(V_2\rho)\eta(V_3) \\ +(V_3\rho)\eta(V_2) - 2\rho g(\phi h V_2, V_3)\} = 0. \end{aligned} \tag{19}$$

Putting $V_3 = \zeta$ in (19) and using (7), we get

$$\begin{aligned} 4m\kappa\eta(V_2)+\{(2m-1)\lambda - (2m-2+\kappa)\}\eta(V_2) \\ +(2m-1)\{(V_2\rho) + (\zeta\rho)\eta(V_2)\} = 0. \end{aligned} \tag{20}$$

Again putting $V_2 = \zeta$ in (20), we obtain

$$(\zeta\rho) = \frac{2(m-1) - (4m-1)\kappa - (2m-1)\lambda}{2(2m-1)}. \tag{21}$$

Using (21) in (20), we infer

$$(V_2\rho) = \frac{2(m-1) - (4m-1)\kappa - (2m-1)\lambda}{2(2m-1)}\eta(V_2), \tag{22}$$

which implies

$$d\rho = \frac{2(m-1) - (4m-1)\kappa - (2m-1)\lambda}{2(2m-1)}\eta. \tag{23}$$

Taking exterior derivative of (23) and then taking wedge product with η, we get

$$\frac{2(m-1) - (4m-1)\kappa - (2m-1)\lambda}{2(2m-1)}\eta \wedge d\eta = 0. \tag{24}$$

Since $\eta \wedge (d\eta)^m$ is the volume element, we have $\eta \wedge d\eta \neq 0$. Thus from (24), we get

$$\lambda = \frac{2(m-1) - (4m-1)\kappa}{2m-1}. \tag{25}$$

Using (25) in (22), we have

$$(V_2\rho) = 0,$$

from which we conclude that ρ is a constant. □

Corollary 3.3. *Let a $N(\kappa)$-contact metric manifold of dimension (2m+1) admit Riemann solitons. If the potential vector field is the Reeb vector field ζ, then the soliton is steady, provided $m = 1$. For $m>1$, the soliton is expanding.*

Proof. If the potential vector field is the Reeb vector field ζ, then from (19), we get

$$\begin{aligned} 2S(V_2, V_3) + &\{(2m-1)\lambda - (2m-2+\kappa)\}g(V_2, V_3) \\ &-2(2m-1)g(\phi h V_2, V_3) = 0. \end{aligned} \tag{26}$$

Putting $V_2 = V_3 = \delta_i$ in (26), where $\{\delta_i\}, i = 1, 2, \cdots, (2m+1)$ is the orthonormal basis of the tangent space at each point of the manifold. Then summing over i, we get

$$2r + \{(2m-1)\lambda - (2m-2+\kappa)\}(2m+1) = 0. \tag{27}$$

Using (10) in (27), we get

$$\lambda = -\frac{2m-2+\kappa}{2m+1}. \tag{28}$$

Again, substituting $V_2 = V_3 = \zeta$ in (26), one obtains

$$\kappa = \frac{(2m-1)\lambda - 2(m-1)}{4m-1}. \tag{29}$$

Applying (29) in (28), we infer

$$\lambda = \frac{(m-1)(2m-1)}{2m^2}. \tag{30}$$

From the foregoing equation, we see that for $m = 1$, $\lambda = 0$ and when $m>1$, we find $\lambda>0$.
The above equation ensures the validity of the corollary. □

Theorem 3.4. *If the metric (g, Z, λ) of a (2m+1)-dimensional $N(\kappa)$-contact metric manifold is a Riemann soliton, then*

$$\kappa = \frac{2(m-1)}{4m-1} - \frac{2m-1}{4m-1}\lambda.$$

Proof. From (17), we have

$$(\mathcal{L}_Z g)(V_2, V_3) = -\frac{2}{2m-1}S(V_2, V_3) - \frac{(2m-1)\lambda - (2m-2+\kappa)}{2m-1}g(V_2, V_3). \tag{31}$$

Taking covariant derivative of (31) along the arbitrary vector field V_1, we have

$$(\nabla_{V_1}\mathcal{L}_Z g)(V_2, V_3) = -\frac{2}{2m-1}(\nabla_{V_1}S)(V_2, V_3). \tag{32}$$

From Yano [30], we have

$$\begin{aligned}(\mathcal{L}_Z\nabla_{V_1}g - \nabla_{V_1}\mathcal{L}_Z g - \nabla_{[Z,V_1]}g)(V_2, V_3) = &- g((\mathcal{L}_Z\nabla)(V_1, V_2), V_3)\\ &- g((\mathcal{L}_Z\nabla)(V_1, V_3), V_2).\end{aligned}$$

Due to symmetry property of $\mathcal{L}_Z\nabla$, we have from the above formula

$$\begin{aligned}2g((\mathcal{L}_Z\nabla)(V_1, V_2), V_3) =&(\nabla_{V_1}\mathcal{L}_Z g)(V_2, V_3) + (\nabla_{V_2}\mathcal{L}_Z g)(V_1, V_3)\\ &-(\nabla_{V_3}\mathcal{L}_Z g)(V_1, V_2).\end{aligned} \tag{33}$$

Taking covariant derivative of (6) along the vector field V_1, we obtain

$$\begin{aligned}(\nabla_{V_1}S)(V_2, V_3) =&2(m-1)g((\nabla_{V_1}h)(V_2), V_3)\\ &+\{2m\kappa - 2(m-1)\}\{(\nabla_{V_1}\eta)(V_2)\eta(V_3)\\ &+\eta(V_2)(\nabla_{V_1}\eta)(V_3)\}.\end{aligned} \tag{34}$$

Using (8) and (9) in (34), we obtain

$$\begin{aligned}(\nabla_{V_1}S)(V_2, V_3) =&2\kappa\{g(V_1, \phi V_2)\eta(V_3) - g(V_1, \phi V_3)\eta(V_2)\}\\ &+2m\kappa\{g(V_1, h\phi V_2)\eta(V_3) + g(V_1, h\phi V_3)\eta(V_2)\}.\end{aligned} \tag{35}$$

Using (32) and (35) in (33), we obtain

$$g((\mathcal{L}_Z\nabla)(V_1, V_2), V_3) = -\frac{4\kappa}{2m-1}\{g(V_1, \phi V_2) + mg(V_1, \phi V_2)\}\eta(V_3).$$

From the above equation, we have

$$(\mathcal{L}_Z\nabla)(V_1, V_2) = -\frac{4\kappa}{2m-1}(g(V_1, \phi V_2) + mg(V_1, h\phi V_2))\zeta. \tag{36}$$

Putting $V = \zeta$ in the above equation, we infer

$$(\mathcal{L}_Z\nabla)(V_1, \zeta) = 0. \tag{37}$$

Therefore,

$$\begin{aligned}\nabla_{V_2}(\mathcal{L}_Z\nabla)(V_1, \zeta) =&(\nabla_{V_2}\mathcal{L}_Z\nabla)(V_1, \zeta) + (\mathcal{L}_Z\nabla)(\nabla_{V_2}V_1, \zeta)\\ &+(\mathcal{L}_Z\nabla)(V_1, \nabla_{V_2}\zeta) = 0.\end{aligned} \tag{38}$$

Using (3) and (37) in (38), we get

$$(\nabla_{V_2}\mathcal{L}_Z\nabla)(V_1,\zeta) = (\mathcal{L}_Z\nabla)(V_1,\phi V_2) + (\mathcal{L}_Z\nabla)(V_1,\phi h V_2).$$

Using (36) in the above equation, we obtain

$$(\nabla_{V_2}\mathcal{L}_Z\nabla)(V_1,\zeta) = -\frac{4\kappa}{2m-1}[((\kappa-1)m-1)(g(V_1,V_2) \\ -\eta(V_1)\eta(V_2))\zeta - (m+1)g(V_1,hV_2)\zeta]. \tag{39}$$

It is well known that [30]

$$(\mathcal{L}_Z R)(V_1,V_2)V_3 = (\nabla_{V_1}\mathcal{L}_Z\nabla)(V_2,V_3) - (\nabla_{V_2}\mathcal{L}_Z\nabla)(V_1,V_3). \tag{40}$$

By (39) and (40), we infer

$$(\mathcal{L}_Z R)(V_1,\zeta)\zeta = 0. \tag{41}$$

Again taking Lie-derivative of $R(V_1,\zeta)\zeta = \kappa(V_1 - \eta(V_1)\zeta)$ along the vector field Z, we obtain

$$(\mathcal{L}_Z R)(V_1,\zeta)\zeta = -\kappa[(\mathcal{L}_Z\eta)(V_1)\zeta - \eta(V_1)\mathcal{L}_Z\zeta] \\ -R(V_1,\mathcal{L}_Z\zeta)\zeta - R(V_1,\zeta)\mathcal{L}_Z\zeta. \tag{42}$$

Putting $V_3 = \zeta$ in (31) and using (7), we obtain

$$(\mathcal{L}_Z\eta)(V_2) = g(V_2,\mathcal{L}_Z\zeta) - \frac{(4m-1)\kappa + (2m-1)\lambda - 2(m-1)}{2m-1}\eta(V_2). \tag{43}$$

Using (43) in (42), we have

$$(\mathcal{L}_Z R)(V_1,\zeta)\zeta = -\kappa[g(V_1,\mathcal{L}_Z\zeta)\zeta - \frac{(4m-1)\kappa + (2m-1)\lambda - 2(m-1)}{2m-1}\eta(V_1)\zeta \\ -\eta(V_1)\mathcal{L}_Z\zeta] - R(V_1,\mathcal{L}_Z\zeta)\zeta - R(V_1,\zeta)\mathcal{L}_Z\zeta. \tag{44}$$

From (41) and (44), we obtain

$$R(V_1,\mathcal{L}_Z\zeta)\zeta + R(V_1,\zeta)\mathcal{L}_Z\zeta = -\kappa[g(V_1,\mathcal{L}_Z\zeta)\zeta \\ -\frac{(4m-1)\kappa + (2m-1)\lambda - 2(m-1)}{2m-1}\eta(V_1)\zeta - \eta(V_1)\mathcal{L}_Z\zeta].$$

Contracting the above equation, we have

$$S(\mathcal{L}_Z\zeta,\zeta) = \frac{\{(4m-1)\kappa + (2m-1)\lambda - 2(m-1)\}\kappa}{2(2m-1)}.$$

By virtue of equation (7), the above equation reduces to

$$g(\mathcal{L}_Z\zeta,\zeta) = \frac{(4m-1)\kappa + (2m-1)\lambda - 2(m-1)}{4m(2m-1)}. \tag{45}$$

Putting $V_2 = V_3 = \zeta$ in (31) and using (7), we obtain

$$g(\mathcal{L}_Z\zeta,\zeta) = \frac{(4m-1)\kappa + (2m-1)\lambda - 2(m-1)}{2(2m-1)}. \tag{46}$$

Comparing (45) and (46), we obtain

$$\kappa = \frac{2(m-1)}{4m-1} - \frac{2m-1}{4m-1}\lambda. \tag{47}$$

This completes the proof of the theorem. □

Since $\kappa<1$, from (47), we can state the following corollary

Corollary 3.5. *Let a (2m+1)-dimensional $N(\kappa)$-contact metric manifold admits Riemann solitons.*
For $\frac{2(m-1)}{4m-1}<\kappa<1$, the soliton is shrinking.
For $\kappa = \frac{2(m-1)}{4m-1}$, the soliton is steady.
For $\kappa<\frac{2(m-1)}{4m-1}$, the soliton is expanding.

In three-dimensional $N(\kappa)$-contact metric manifolds, $\kappa = -\frac{1}{3}\lambda$. Thus we can state the following

Corollary 3.6. *If $\kappa>0$, then the soliton is shrinking.*
If $\kappa = 0$, then the soliton is steady.
If $\kappa<0$, then the soliton is expanding.

Theorem 3.7. *If a $(2m+1)$-dimensional $N(\kappa)$-contact metric manifold admits Riemann solitons, then the potential vector field Z is an infinesimal contact transformation.*

Proof. Substituting $V_3 = \zeta$ in (17), one obtains

$$(\mathcal{L}_Z g)(V_2, \zeta) = -\frac{(4m-1)\kappa + (2m-1)\lambda - 2(m-1)}{2m-1}\eta(V_2). \tag{48}$$

Applying $V_2 = \zeta$ in (48), we have

$$g(\mathcal{L}_Z\zeta, \zeta) = \frac{(4m-1)\kappa + (2m-1)\lambda - 2(m-1)}{2(2m-1)}. \tag{49}$$

The above equation gives

$$\mathcal{L}_Z\zeta = \frac{(4m-1)\kappa + (2m-1)\lambda - 2(m-1)}{2(2m-1)}\zeta. \tag{50}$$

Now, Lie derivative of $\eta(V_2) = g(V_2, \zeta)$ over Z confers

$$(\mathcal{L}_Z\eta)(V_2) = (\mathcal{L}_Z g)(V_2, \zeta) + g(V_2, \mathcal{L}_Z\zeta). \tag{51}$$

Equations (48) and (50) together with equation (51) gives

$$(\mathcal{L}_Z\eta)(V_2) = -\frac{(4m-1)\kappa + (2m-1)\lambda - 2(m-1)}{2(2m-1)}\eta(V_2), \tag{52}$$

which conferms that the potential vector field Z is an infinitesimal contact transformation. □

From (52), we may establish the following corollary

Corollary 3.8. *If the potential vector field Z of a Riemann soliton is strict infinitesimal contact transformation, then* $\lambda = \frac{2(m-1)}{2m-1} - \frac{(4m-1)\kappa}{2m-1}$.

4. Gradient Riemann solitons on $N(\kappa)$-contact metric manifolds

In this section we consider gradient Riemann soliton as the metric of a $N(\kappa)$-contact metric manifold and prove the following results:

Theorem 4.1. *If a (2m+1)-dimensional $N(\kappa)$-contact metric manifold admits a gradient Riemann soliton, then the manifold is locally isometric to the product of a flat (m+1)-dimensional manifold and a m-dimensional manifold with scalar curvature 4, provided $m>1$. For $m = 1$, the manifold is flat.*

Proof. Let M be a (2m+1)-dimensional $N(\kappa)$-contact metric manifold which admit gradient Riemann solitons. Let the potential vector filed Z be the gradient of a non-zero smooth function $f : M \to \mathbb{R}$, that is, $Z = Df$, where D be the gradient operator. Then, from (14), we have

$$\nabla_{V_2} Df = -\frac{1}{2m-1}(Q + 2m\lambda + divDf)V_2. \tag{53}$$

Taking covariant derivative of (53) along the vector field V_1, we obtain

$$\nabla_{V_1}\nabla_{V_2} Df = -\frac{1}{2m-1}(\nabla_{V_1} QV_2 + V_1(divDf)V_2 + (2m\lambda + divDf)\nabla_{V_1} V_2). \tag{54}$$

Interchanging V_1 and V_2 in (54), we get

$$\nabla_{V_2}\nabla_{V_1} Df = -\frac{1}{2m-1}(\nabla_{V_2} QV_1 + V_2(divDf)V_1 + (2m\lambda + divDf)\nabla_{V_2} V_1).$$

Also, from (53), we have

$$\nabla_{[V_1,V_2]} Df = -\frac{1}{2m-1}(Q + 2m\lambda + divDf)[V_1, V_2].$$

Therefore,

$$\begin{aligned} R(V_1, V_2)Df =& -\frac{1}{2m-1}((\nabla_{V_1} Q)(V_2) - (\nabla_{V_2} Q)(V_1) \\ &+V_1(divDf)V_2 - V_2(divDf)V_1). \end{aligned} \tag{55}$$

From (6), we have

$$QV_2 = 2(m-1)(V_2 + hV_2) + (2m\kappa - 2(m-1)\eta(V_2)\zeta). \tag{56}$$

Taking covariant derivative of (56) along the vector field V_1, we obtain

$$\begin{aligned} (\nabla_{V_1} Q)(V_2) =& 2\kappa(g(V_1, \phi V_2)\zeta - \eta(V_2)\phi V_1) \\ &+ 2m\kappa(g(V_1, h\phi V_2)\zeta + \eta(V_2)h\phi V_1). \end{aligned} \tag{57}$$

Using (57) in (55), we get

$$\begin{aligned} R(V_1, V_2)Df =& -\frac{1}{2m-1}[2\kappa(2g(V_1, \phi V_2)\zeta - \eta(V_2)\phi V_1 + \eta(V_1)\phi V_2) \\ &+2m\kappa(\eta(V_2)h\phi V_1 - \eta(V_1)h\phi V_2) + V_1(divDf)V_2 - V_2(divDf)V_1]. \end{aligned} \tag{58}$$

Taking inner product of (58) with the vector field ζ, we obtain

$$\begin{aligned} g(R(V_1, V_2)Df, \zeta) =& -\frac{1}{2m-1}[4\kappa g(V_1, \phi V_2) + V_1(divDf)\eta(V_2) \\ &-V_2(divDf)\eta(V_1)]. \end{aligned} \tag{59}$$

Again taking inner product of (5) with Df, we get

$$g(R(V_1, V_2)\zeta, Df) = \kappa((V_1 f)\eta(V_2) - (V_2 f)\eta(V_1)). \tag{60}$$

As $g(R(V_1, V_2)V_3, V_4) = -g(R(V_1, V_2)V_4, V_3)$ for any vector fields V_1, V_2, V_3, V_4 on M, from (59) and (60), we get

$$\begin{aligned} &\frac{1}{2m-1}[4\kappa g(V_1, \phi V_2) + V_1(divDf)\eta(V_2) - V_2(divDf)\eta(V_1)] \\ &\qquad= \kappa((V_1 f)\eta(V_2) - (V_2 f)\eta(V_1)). \end{aligned} \tag{61}$$

Replacing V_1 by ϕV_1 and V_2 by ϕV_2 in the above equation, we obtain

$$\kappa g(\phi V_1, V_2) = 0,$$

which gives $\kappa = 0$. Thus, in view of [2], the statement of the theorem is proved. □

Corollary 4.2. *If a $N(\kappa)$-contact metric manifold admits gradient Riemann soliton with smooth potential function f, then divergence of Df is a constant, where D denotes gradient operator.*

Proof. Putting $V_1 = \zeta$ in (61) and using $\kappa = 0$, we get

$$V_2(divDf) = \zeta(divDf)\eta(V_2). \tag{62}$$

Contracting (58), we get

$$S(V_2, Df) = \frac{2m}{2m-1} V_2(divDf).$$

Using (6) and (62) in the above equation, we obtain

$$2(m-1)((V_2 f) + (hV_2 f) - (\zeta f)\eta(V_2)) = \frac{2m}{2m-1}\zeta(divDf)\eta(V_2). \tag{63}$$

Replacing V_2 by hV_2 in the above equation, we have

$$(hV_2 f) = (\zeta f)\eta(V_2) - (V_2 f). \tag{64}$$

Using (64) in (63), we infer

$$\zeta(divDf) = 0.$$

Therefore, with the help of (62), we conclude that $divDf$ is a constant. □

5. Examples

Example 5.1. Let us consider the manifold $M = \{x, y, z \in \mathbb{R}^3 : z \neq 0\}$ of dimension 3, where (x, y, z) are standard coordinates in $\mathbb{R}^3$. We choose the vector fields δ_1, δ_2 and δ_3 which satisfies

$$[\delta_1, \delta_2] = 3\delta_3, \quad [\delta_1, \delta_3] = \delta_2, \quad [\delta_2, \delta_3] = 2\delta_1.$$

Let the metric tensor g be defined by

$$g(\delta_1, \delta_1) = g(\delta_2, \delta_2) = g(\delta_3, \delta_3) = 1$$

and

$$g(\delta_1, \delta_2) = g(\delta_1, \delta_3) = g(\delta_2, \delta_3) = 0.$$

The 1-form η is defined by

$$\eta(V_1) = g(V_1, \delta_1),$$

for all V_1 on M. Let ϕ be the $(1, 1)$-tensor field defined by

$$\phi(\delta_1) = 0, \quad \phi(\delta_2) = \delta_3, \quad \phi(\delta_3) = -\delta_2.$$

Then we find that

$$\eta(\delta_1) = 1, \quad \phi^2 V_1 = -V_1 + \eta(V_1)\delta_1,$$

$$g(\phi V_1, \phi V_2) = g(V_1, V_2) - \eta(V_1)\eta(V_2), \quad d\eta(V_1, V_2) = g(V_1, \phi V_2),$$

for any vector fields V_1, V_2 on M. Thus $(\phi, \delta_1, \eta, g)$ defines a contact structure.

Let ∇ be the Levi-Civita connection on M, then by Koszul's formula, we obtain

$$\nabla_{\delta_1}\delta_1 = 0, \quad \nabla_{\delta_1}\delta_2 = 0, \quad \nabla_{\delta_1}\delta_3 = 0,$$

$$\nabla_{\delta_2}\delta_2 = 0, \quad \nabla_{\delta_2}\delta_1 = -3\delta_3, \quad \nabla_{\delta_2}\delta_3 = 3\delta_1,$$

$$\nabla_{\delta_3}\delta_3 = 0, \quad \nabla_{\delta_3}\delta_1 = -\delta_2, \quad \nabla_{\delta_3}\delta_2 = \delta_1.$$

From the above expressions of ∇, we obtain

$$h\delta_1 = 0, \quad h\delta_2 = 2\delta_2, \quad h\delta_3 = -2\delta_3.$$

We also have

$$R(\delta_1, \delta_2)\delta_2 = -3\delta_1, \quad R(\delta_2, \delta_1)\delta_1 = -3\delta_2, \quad R(\delta_2, \delta_3)\delta_3 = 3\delta_2,$$

$$R(\delta_3, \delta_2)\delta_2 = 3\delta_3, \quad R(\delta_1, \delta_3)\delta_3 = -3\delta_1, \quad R(\delta_3, \delta_1)\delta_1 = -3\delta_3,$$

$$R(\delta_1, \delta_2)\delta_3 = 0, \quad R(\delta_2, \delta_3)\delta_1 = 0, \quad R(\delta_1, \delta_3)\delta_2 = 0.$$

Thus the manifold is a $N(\kappa)$-contact metric manifold with $\kappa = -3$. From the above expressions of $R(\delta_i, \delta_j)\delta_k$, the curvature tensor R is given by

$$\begin{aligned} R(V_1, V_2)V_3 =& 3(g(V_2, V_3)V_1 - g(V_1, V_3)V_2) + 4(g(V_2, V_3)\eta(V_1)\zeta \\ & - g(V_1, V_3)\eta(V_2)\zeta + \eta(V_2)\eta(V_3)V_1 - \eta(V_1)\eta(V_3)V_2), \end{aligned} \tag{65}$$

for any vector fields V_1, V_2, V_3.

From the expressions of curvature tensor, we get

$$S(\delta_1, \delta_1) = -6, \quad S(\delta_2, \delta_2) = 0, \quad S(\delta_3, \delta_3) = 0.$$

Thus

$$S(V_1, V_2) = -6\eta(V_1)\eta(V_2),$$

for any vector fields V_1, V_2 on the manifold.

The scalar curvature r of the manifold is given by

$$r = S(\delta_1, \delta_1) + S(\delta_2, \delta_2) + S(\delta_3, \delta_3) = -6.$$

Let us take the potential vector $Z = a\delta_1 + b\delta_2 + c\delta_3$ where a, b, c are real constants. Then

$$(\mathcal{L}_Z g)(\delta_1, \delta_1) = 0, \quad (\mathcal{L}_Z g)(\delta_2, \delta_2) = 0, \quad (\mathcal{L}_Z g)(\delta_3, \delta_3) = 0.$$

From the expressions of $\mathcal{L}_Z$ along with equation (65), the equation (13) is satisfied for $\lambda = 3$. Thus g is a Riemann soliton. Since $\lambda > 0$, the soliton is expanding. Also,

$$(\mathcal{L}_Z \eta)(\delta_1) = 0, \quad (\mathcal{L}_Z \eta)(\delta_2) = 2c, \quad (\mathcal{L}_Z \eta)(\delta_3) = -2b.$$

Hence Z is a strict infinitesimal contact transformation with $b = c = 0$.

In the next example, we verify the existance of Riemann soliton on Sasakian manifold (i.e., for $\kappa = 1$).

Example 5.2. Let us consider the manifold $M = \{x, y, z \in \mathbb{R}^3 : z \neq 0\}$ of dimension 3, where (x, y, z) are standard coordinates in $\mathbb{R}^3$. We choose the vector fields

$$\delta_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \quad \delta_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \quad \delta_3 = \frac{\partial}{\partial z},$$

which are linearly independent at each point of M. We get the following by direct computations

$$[\delta_1, \delta_2] = -2\delta_3, \quad [\delta_1, \delta_3] = 0, \quad [\delta_2, \delta_3] = 0.$$

Let the metric tensor g be defined by

$$g(\delta_1, \delta_1) = g(\delta_2, \delta_2) = g(\delta_3, \delta_3) = 1$$

and

$$g(\delta_1, \delta_2) = g(\delta_1, \delta_3) = g(\delta_2, \delta_3) = 0.$$

The 1-form η is defined by $\eta(V_1) = g(V_1, \delta_3)$, for all V_1 on M. Let ϕ be the $(1,1)$-tensor field defined by

$$\phi(\delta_1) = -\delta_2, \quad \phi(\delta_2) = \delta_1, \quad \phi(\delta_3) = 0.$$

Then we find that

$$\eta(\delta_3) = 1, \qquad \phi^2 V_1 = -V_1 + \eta(V_1)\delta_3,$$
$$g(\phi V_1, \phi V_2) = g(V_1, V_2) - \eta(V_1)\eta(V_2), \quad d\eta(V_1, V_2) = g(V_1, \phi V_2),$$

for any vector fields V_1, V_2 on M. Thus $(\phi, \delta_3, \eta, g)$ defines a contact structure.

Let ∇ be the Levi-Civita connection on M, then we have

$$\nabla_{\delta_1}\delta_1 = 0, \quad \nabla_{\delta_1}\delta_2 = -\delta_3, \quad \nabla_{\delta_1}\delta_3 = \delta_2,$$
$$\nabla_{\delta_2}\delta_1 = \delta_3, \quad \nabla_{\delta_2}\delta_2 = 0, \quad \nabla_{\delta_2}\delta_3 = -\delta_1,$$
$$\nabla_{\delta_3}\delta_1 = \delta_2, \quad \nabla_{\delta_3}\delta_2 = -\delta_1, \quad \nabla_{\delta_3}\delta_3 = 0.$$

From the above expressions of ∇, we obtain

$$h\delta_1 = 0, \quad h\delta_2 = 0, \quad h\delta_3 = 0.$$

Thus the components of curvature tensor are given by

$$R(\delta_1, \delta_2)\delta_2 = -3\delta_1, \quad R(\delta_2, \delta_1)\delta_1 = -3\delta_2, \quad R(\delta_2, \delta_3)\delta_3 = \delta_2,$$
$$R(\delta_3, \delta_2)\delta_2 = \delta_3, \quad R(\delta_1, \delta_3)\delta_3 = \delta_1, \quad R(\delta_3, \delta_1)\delta_1 = \delta_3,$$
$$R(\delta_1, \delta_2)\delta_3 = 0, \quad R(\delta_2, \delta_3)\delta_1 = 0, \quad R(\delta_1, \delta_3)\delta_2 = 0.$$

Hence the manifold is a $N(\kappa)$-contact metric manifold with $\kappa = 1$, i.e., a Sasakian manifold. From the above expressions of $R(\delta_i, \delta_j)\delta_k$, the curvature tensor is given by

$$\begin{aligned} R(V_1, V_2)V_3 = &- 3(g(V_2, V_3)V_1 - g(V_1, V_3)V_2) + 4(g(V_2, V_3)\eta(V_1)\zeta \\ &-g(V_1, V_3)\eta(V_2)\zeta + \eta(V_2)\eta(V_3)V_1 - \eta(V_1)\eta(V_3)V_2), \end{aligned} \tag{66}$$

for any vector fields V_1, V_2, V_3.

The components of Ricci tensor are given by

$$S(\delta_1, \delta_1) = -2, \quad S(\delta_2, \delta_2) = -2, \quad S(\delta_3, \delta_3) = 2.$$

Thus

$$S(V_1, V_2) = -2g(V_1, V_2) + 4\eta(V_1)\eta(V_2),$$

for any vector fields V_1, V_2.
The scalar curvature r is given by

$$r = S(\delta_1, \delta_1) + S(\delta_2, \delta_2) + S(\delta_3, \delta_3) = -2.$$

Let us take $Z = x\delta_1 + y\delta_2 - 3z\delta_3$, then

$$(\mathcal{L}_Z g)(V_1, V_2) = 2g(V_1, V_2) - 8\eta(V_1)\eta(V_2), \tag{67}$$

for any vector fields V_1, V_2 on M. With the help of equations (66), (67) and $\lambda = 1$, the equation (13) is verified. Thus g is an expanding Riemann soliton.

Suggested future works: The present work deals with certain restrictions on potential vector fields. It will be a good problem if one can remove the restrictions and analyze the cases in future. Moreover, (κ, μ)-manifolds is a promising field of study in this regard.

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