Filomat 37:30 (2023), 10415–10429
https://doi.org/10.2298/FIL2330415S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Further results on Berezin number inequalities and related problems

Satyajit Sahoo[a,*], **Ulaş Yamancı**[b]

[a] *School of Mathematical Sciences, National Institute of Science Education and Research Bhubaneswar (NISER) Bhubaneswar, India*
[b] *Department of Statistics, Suleyman Demirel University, Isparta, Turkey*

Abstract. In this article, we obtain certain Berezin number inequalities which extend certain earlier existing results in the literature. Also, we give some reverse Berezin number inequalities for normal operators on reproducing kernel Hilbert spaces. Moreover, we characterize the projection operators and the partial isometry operators in terms of Berezin number.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators from $\mathcal{H}$ into itself. For $T \in \mathcal{B}(\mathcal{H})$, the *numerical range* of T is defined as

$$W(T) = \{\langle Tx, x \rangle : x \in \mathcal{H}, \|x\| = 1\}.$$

The *numerical radius* of T, denoted by $w(T)$, is defined as $w(T) = \sup\{|z| : z \in W(T)\}$. It is well-known that $w(\cdot)$ defines a norm on $\mathcal{H}$, and is equivalent to the usual operator norm $\|T\| = \sup\{\|Tx\| : x \in \mathcal{H}, \|x\| = 1\}$. In fact, for every $T \in \mathcal{B}(\mathcal{H})$,

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|. \tag{1}$$

Several numerical radius inequalities that provide alternative lower and upper bounds for $w(\cdot)$ have attracted great research interest in recent years. For example, the estimation of the numerical radii of operator matrices is useful in obtaining bounds for the zeros of polynomials (see [6]). One may refer to the excellent articles [8, 11, 18, 21, 26] for the history and significance of numerical radius inequalities.

Kittaneh [16] showed that if T is an operator in $\mathcal{B}(\mathcal{H})$, then

$$w(T) \leq \frac{1}{2}\left(\|T\| + \|T^2\|^{\frac{1}{2}}\right). \tag{2}$$

2020 *Mathematics Subject Classification*. 47A30, 47A12, 15A60.
Keywords. Berezin number; Reproducing kernel; Operator matrix.
Received: 03 April 2023; Accepted: 13 June 2023
Communicated by Fuad Kittaneh
* Corresponding author: Satyajit Sahoo
ORCID iD: 0000-0002-1363-0103 (Satyajit Sahoo), 0000-0002-4709-0993 (Ulaş Yamancı)
Email addresses: ssahoo@niser.ac.in, satyajitsahoo2010@gmail.com (Satyajit Sahoo), ulasyamanci@sdu.edu.tr (Ulaş Yamancı)

Consequently, if $T^2 = 0$, then

$$w(T) = \frac{1}{2}\|T\|. \tag{3}$$

A Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ of complex valued functions on a nonempty open set $\Omega \subset \mathbb{C}$ which has the property that point evaluations are continuous, is called a *functional Hilbert space*. The point evaluations are continuous means for each $\lambda \in \Omega$, the map $f \longmapsto f(\lambda)$ is a continuous linear functional on $\mathcal{H}$. For each $\lambda \in \Omega$, there is a unique element K_λ of $\mathcal{H}$ such that $f(\lambda) = \langle f, K_\lambda \rangle$ for all $f \in \mathcal{H}$ by Riesz representation theorem. The collection $\{K_\lambda : \lambda \in \Omega\}$ is known as the *reproducing kernel* of $\mathcal{H}$. Problem 37 of [12] states that the reproducing kernel of a functional Hilbert space $\mathcal{H}$ with $\{e_n\}$ as an orthonormal basis is $K_\lambda(z) = \sum_n \overline{e_n(\lambda)} e_n(z)$. Let $\hat{k}_\lambda = K_\lambda / \|K_\lambda\|$ be the nomalized reproducing kernel of $\mathcal{H}$, where $\lambda \in \Omega$. The function $\tilde{A}$ defined on Ω by $\tilde{A}(\lambda) = \langle A\hat{k}_\lambda, \hat{k}_\lambda \rangle$ is the *Berezin symbol* of a bounded linear operator A on $\mathcal{H}$. *Berezin set* and *Berezin number* of the operator A are defined by Karaev [15]

$$\mathrm{Ber}(A) = \{\tilde{A}(\lambda) : \lambda \in \Omega\} \text{ and } \mathrm{ber}(A) = \sup\{|\tilde{A}(\lambda)| : \lambda \in \Omega\},$$

respectively. These definitions are named in honor of Felix Berezin, who introduced these notions in [7]. Clearly, the Berezin symbol $\tilde{A}$ is a bounded function on Ω whose values lie in the numerical range of the operator A, and hence

$$\mathrm{Ber}(A) \subseteq W(A) \text{ and } \mathrm{ber}(A) \le w(A).$$

Berezin number of an operator T satisfies the following properties:
(i) $\mathrm{ber}(\alpha T) = |\alpha| \mathrm{ber}(T)$ for all $\alpha \in \mathbb{C}$.
(ii) $\mathrm{ber}(T + S) \le \mathrm{ber}(T) + \mathrm{ber}(S)$.

The Berezin symbol has been studied in detail for Toeplitz and Hankel operators on Hardy and Bergman spaces. A nice property of the Berezin symbol is mentioned next. If $A, B \in \mathcal{B}(\mathcal{H})$ then $\tilde{A}(\lambda) = \tilde{B}(\lambda)$ for all $\lambda \in \Omega$, implies $A = B$. Thus the Berezin symbol uniquely determines the operator.

Let $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$ be Hilbert spaces. If $\mathcal{H} = \bigoplus_{i=1}^{n} \mathcal{H}_i$ and $S \in \mathcal{B}(\mathcal{H})$, then the operator S can be represented as an $n \times n$ operator matrix, i.e., $S = [S_{ij}]_{n\times n}$ with $S_{ij} \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i)$, the space of all bounded linear operators from $\mathcal{H}_j$ to $\mathcal{H}_i$. Operator matrices provide a useful tool for studying Hilbert space operators, which have been extensively studied in the literature. The block-norm matrix $\hat{S}$ associated with an operator matrix $S = [S_{ij}]_{n\times n}$ is defined by $\hat{S} = [\|S_{ij}\|]_{n\times n}$ which is an $n \times n$ non-negative matrix. Hou *et al.* [14] established some estimates for the numerical radii, operator norm, and spectral radii of an $n \times n$ operator matrix $S = [S_{ij}]$. In particular, they showed that if $S = (S_{ij})_{n\times n}$ is an operator matrix and $\hat{S} = (\|S_{ij}\|)_{n\times n}$ is its block-norm matrix, then

$$(i)\ w(S) \le w(\hat{S}),\ (ii)\ \|S\| \le \|\hat{S}\|,\ (iii)\ \rho(S) \le \rho(\hat{S}), \tag{4}$$

where $\rho(S)$ denotes the spectral radius of S.

Several numerical radius inequalities improving and refining inequality (1) have been obtained by many authors see for examples [1, 2, 6, 14]. Among others, important facts concerning the numerical radius inequalities of $n \times n$ operator matrices are obtained by different authors which are grouped together, as follows: Let $S = [S_{ij}]$ be an $n \times n$ operator matrix with $S_{ij} \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i)$. Then

$$w(S) \le \begin{cases} w\left(\left[s_{ij}^{(1)}\right]\right), & \text{BaniDomi \& Kittaneh in [6]} \\ w\left(\left[s_{ij}^{(2)}\right]\right), & \text{AbuOmar \& Kittaneh in [1]} \end{cases}; \tag{5}$$

where

$$s_{ij}^{(1)} = \begin{cases} \frac{1}{2}\left(\|S_{ii}\| + \left\|S_{ii}^2\right\|^{1/2}\right), & i = j \\ \left\|S_{ij}\right\|, & i \neq j \end{cases},$$

and

$$s_{ij}^{(2)} = \begin{cases} w\left(S_{ii}\right), & i = j \\ \left\|S_{ij}\right\|, & i \neq j \end{cases}.$$

In the other direction, Bakherad [3] and Sahoo *et al.* [24] established certain inequalities involving the Berezin number of operators. Using the ideas given in [1], Bakherad [3] and Sahoo *et al.* [24] proved that if $S = [S_{ij}]$ is an $n \times n$ operator matrix with $S_{ij} \in \mathcal{B}(\mathcal{H}(\Omega_j), \mathcal{H}(\Omega_i))$, $1 \leq i, j \leq n$. Then

$$ber\left(S\right) \leq \begin{cases} w\left(\left[s_{ij}^{(1)}\right]\right), & \text{Bakherad in [3]} \\ \\ w\left(\left[s_{ij}^{(2)}\right]\right), & \text{Sahoo et al. in [24]} \end{cases}; \tag{6}$$

where

$$s_{ij}^{(1)} = \begin{cases} \text{ber}(S_{ij}) & \text{for } i = j \\ \|S_{ij}\| & \text{for } i \neq j. \end{cases},$$

and

$$s_{ij}^{(2)} = \begin{cases} \frac{1}{2}\text{ber}\left(\alpha|S_{ii}|^2 + (1-\alpha)|S_{ii}^*|^2\right), & for\ i = j \\ \|S_{ij}\|, \quad i \neq j. \end{cases}$$

Recently, Bakherad *et al.* [5] established Berezin number inequality via Aluthge transform of operator is presented below.

$$\text{ber}(T) \leq \frac{1}{4}\||T|^{2t} + |T|^{2(1-t)}\| + \frac{1}{2}\text{ber}(\tilde{T}_t), \tag{7}$$

where $T \in \mathcal{B}(\mathcal{H})$, $t \in [0, 1]$. For other related upper and lower bounds for Berezin number one may refer to [3, 4, 22, 23].

The main objective of this paper is to extend the inequalities (6), (7) and to obtain Berezin number inequalities for operators on reproducing kernel Hilbert space. To this end, the paper is organized as follows. Section 2 begins with the description of some useful preliminary results. In Section 3, we have established Berezin number inequalities via generalized Aluthge transform of an operator. Certain Berezin number inequalities for $n \times n$ operator matrices are also obtained. Some reverse Berezin number inequalities for normal operators on reproducing kernel Hilbert space are proved in Section 4. In Section 5, we have characterized the projection operators and partial isometry operators in terms of Berezin number.

2. Preliminaries

Here, we collect some definitions and earlier results which will be used to prove the main results in the next section. We begin with a formula for the numerical radius of a matrix with non-negative entries (see page No. 44, Problem 23(n)[13]).

Lemma 2.1. ([13], Page No. 44, Problem 23(n)) *Let $T = [t_{ij}] \in M_n(\mathbb{C})$ be such that $t_{ij} \geq 0$ for all $i, j = 1, 2, \ldots, n$. Then*

$$w(T) = \frac{1}{2}\rho([t_{ij} + t_{ji}]).$$

Here $\rho(\cdot)$ denotes the spectral radius.

A generalization of the mixed Cauchy-Schwarz inequality which is useful in proving our main results is presented below.

Lemma 2.2. ([17], Theorem 1) *Let A be an operator in $\mathcal{B}(\mathcal{H})$. If f and g are non-negative continuous functions on $[0,\infty)$ satisfying the relation $f(t)g(t) = t$ for all $t \in [0,\infty)$. Then*

$$|\langle Ax, y\rangle| \le \|f(|A|x)\|\|g(|A^*|y)\|$$

for all x, y in $\mathcal{H}$.

The following lemma is an operator version of the classical Jensen inequality.

Lemma 2.3. ([20], Theorem 1.2) *Let A be a self adjoint operator in $\mathcal{B}(\mathcal{H})$ with $\mathrm{sp}(A) \subset [m, M]$ for some scalars $m \le M$, and let $x \in \mathcal{H}$ be a unit vector. If $f(t)$ is a convex function on $[m, M]$, then*

$$f(\langle Ax, x\rangle) \le \langle f(A)x, x\rangle.$$

The next lemma follows from the spectral theorem for non-negative operators and Jensen inequality (see [17]).

Lemma 2.4. *[McCarthy inequality] Let $S \in \mathcal{B}(\mathcal{H})$, $S \ge 0$ and $x \in \mathcal{H}$ be a unit vector. Then*
(i) $\langle Sx, x\rangle^r \le \langle S^r x, x\rangle$ for $r \ge 1$;
(ii) $\langle S^r x, x\rangle \le \langle Sx, x\rangle^r$ for $0 < r \le 1$.

Lemma 2.5. [3, Corollary 2.2] *Let $S \in \mathcal{B}(\mathcal{H}(\Omega_1))$, $X \in \mathcal{B}(\mathcal{H}(\Omega_2), \mathcal{H}(\Omega_1))$, $Y \in \mathcal{B}(\mathcal{H}(\Omega_1), \mathcal{H}(\Omega_2))$ and $R \in \mathcal{B}(\mathcal{H}(\Omega_2))$. Then the following statements hold:*
(a) $ber\left(\begin{bmatrix} S & 0 \\ 0 & R \end{bmatrix}\right) \le \max\{ber(S), ber(R)\}$.
(b) $ber\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \le \frac{1}{2}(\|X\| + \|Y\|)$.
In particular,

$$ber\left(\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}\right) \le \|X\|. \tag{8}$$

We conclude this section with the following lemma from [4].

Lemma 2.6. ([4], Lemma 2.1) *Let $A \in \mathcal{B}(\mathcal{H})$, then*

$$\mathrm{ber}(A) = \sup_{\theta\in\mathbb{R}} \mathrm{ber}\left(\mathfrak{R}(e^{i\theta}A)\right) = \sup_{\theta\in\mathbb{R}} \mathrm{ber}\left(\mathfrak{I}(e^{i\theta}A)\right).$$

3. Some extended Berezin number inequalities

Let $S = U|S|$ be the polar decomposition of S. Here U is a partial isometry and $|S| = (S^*S)^{\frac{1}{2}}$. The *Aluthge transform* of the operator S, denoted by $\widetilde{S}$ is defined as $\widetilde{S} = |S|^{\frac{1}{2}}U|S|^{\frac{1}{2}}$. Okubo [19] introduced a more general notion called *t-Aluthge transform*. It is denoted by $\widetilde{S}_t$, and is defined as $\widetilde{S}_t = |S|^t U|S|^{1-t}$ for $0 \le t \le 1$. It coincides with the usual Aluthge transform for $t = \frac{1}{2}$. When $t = 1$, the operator $\widetilde{S}_1 = |S|U$ is called the *Duggal transform* of $S \in \mathcal{B}(\mathcal{H})$. Shebrawi and Bakherad [25] introduced *generalized Aluthge transform* of the operator S, denoted by $\widetilde{S}_{f,g}$. It is defined by $\widetilde{S}_{f,g} = f(|S|)Ug(|S|)$, where f, g are non-negative continuous functions such that $f(x)g(x) = x$ $(x \ge 0)$ and $S \in \mathcal{B}(\mathcal{H})$.
The following theorem is an extension of (7).

Theorem 3.1. *Let $S \in \mathcal{B}(\mathcal{H}(\Omega))$ and f, g are two non-negative continuous functions defined on $[0,\infty)$ such that $f(t)g(t) = t$ for $t \geq 0$. Then*

$$ber(S) \leq \frac{1}{4}\|f^2(|S|) + g^2(|S|)\| + \frac{1}{2}ber(\tilde{S}_{f,g}). \tag{9}$$

Proof. Let $\hat{k}_\lambda \in \mathcal{H}$. Then we have

$$\begin{aligned}
\Re\langle e^{i\theta}S\hat{k}_\lambda, \hat{k}_\lambda\rangle &= \Re\langle e^{i\theta}U|S|\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&= \Re\langle e^{i\theta}Ug(|S|)f(|S|)\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&= \Re\langle e^{i\theta}f(|S|)\hat{k}_\lambda, g(|S|)U^*\hat{k}_\lambda\rangle \\
&= \frac{1}{4}\|(e^{i\theta}f(|S|) + g(|S|)U^*)\hat{k}_\lambda\|^2 - \frac{1}{4}\|(e^{i\theta}f(|S|) - g(|S|)U^*)\hat{k}_\lambda\|^2 \\
&\qquad \text{(by the polarization identity)} \\
&\leq \frac{1}{4}\|(e^{i\theta}f(|S|) + g(|S|)U^*)\hat{k}_\lambda\|^2 \\
&= \frac{1}{4}\langle(e^{i\theta}f(|S|) + g(|S|)U^*)\hat{k}_\lambda, (e^{i\theta}f(|S|) + g(|S|)U^*)\hat{k}_\lambda\rangle \\
&= \frac{1}{4}\langle(e^{i\theta}f(|S|) + g(|S|)U^*)(e^{-i\theta}f(|S|) + Ug(|S|))\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&= \frac{1}{4}\langle f^2(|S|) + g^2(|S|) + e^{i\theta}\tilde{S}_{f,g} + e^{-i\theta}(\tilde{S}_{f,g})^*\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&= \frac{1}{4}\langle(f^2(|S|) + g^2(|S|))\hat{k}_\lambda, \hat{k}_\lambda\rangle + \frac{1}{4}\langle(e^{i\theta}\tilde{S}_{f,g} + e^{-i\theta}(\tilde{S}_{f,g})^*)\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&= \frac{1}{4}\langle(f^2(|S|) + g^2(|S|))\hat{k}_\lambda, \hat{k}_\lambda\rangle + \frac{1}{2}\langle\Re(e^{i\theta}\tilde{S}_{f,g})\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&\leq \frac{1}{4}\|f^2(|S|) + g^2(|S|)\| + \frac{1}{2}\mathrm{ber}(\Re(e^{i\theta}\tilde{S}_{f,g}) \\
&\leq \frac{1}{4}\|f^2(|S|) + g^2(|S|)\| + \frac{1}{2}\mathrm{ber}(\tilde{S}_{f,g}).
\end{aligned}$$

By taking the supremum over $\lambda \in \Omega$, and using Lemma 2.6 we get the desired result. □

As a special case of our result we have the following result [5, Theorem 3.2].

Remark 3.2. 1. *Put $f(t) = t^\alpha, g(t) = t^{1-\alpha}, \alpha \in [0,1]$, we have*

$$ber(S) \leq \frac{1}{4}\||S|^{2\alpha} + |S|^{2(1-\alpha)}\| + \frac{1}{2}ber(\tilde{S}_t), \tag{10}$$

2. *By putting $\alpha = \frac{1}{2}$ in Theorem 3.1, we get*

$$ber(S) \leq \frac{1}{2}\|S\| + \frac{1}{2}ber(\tilde{S}), \tag{11}$$

where $S \in \mathcal{B}(\mathcal{H}(\Omega))$.

The following Theorem is the extension of [5, Theorem 3.1].

Theorem 3.3. *Let $S = \begin{bmatrix} 0 & S_1 \\ S_2 & 0 \end{bmatrix} \in \mathcal{B}(\mathcal{H}(\Omega_1) \bigoplus \mathcal{H}(\Omega_2)$ and f, g are two non-negative continuous functions defined on $[0,\infty)$ such that $f(t)g(t) = t$ for $t \geq 0$. Then*

$$ber(\tilde{S}_{f,g}) \leq \frac{1}{2}(\|f(|S_2|)g(|S_1^*|)\| + \|f(|S_1|)g(|S_2^*|)\|). \tag{12}$$

Proof. Let $S_1 = U|S_1|$ and $S_2 = V|S_2|$ be the polar decompositions of the operators S_1 and S_2 respectively. Then

$$\begin{bmatrix} 0 & S_1 \\ S_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix}\begin{bmatrix} |S_2| & 0 \\ 0 & |S_1| \end{bmatrix}$$

is the polar decomposition of S. By generalized Aluthge transform of S, we have

$$\begin{aligned}\tilde{S}_{f,g} = f(|S|)\begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix} g(|S|) &= \begin{bmatrix} f(|S_2|) & 0 \\ 0 & f(|S_1|) \end{bmatrix}\begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix}\begin{bmatrix} g(|S_2|) & 0 \\ 0 & g(|S_1|) \end{bmatrix} \\ &= \begin{bmatrix} 0 & f(|S_2|)Ug(|S_1|) \\ f(|S_1|)Vg(|S_2|) & 0 \end{bmatrix}.\end{aligned}$$

So

$$\begin{aligned}\text{ber}(\tilde{S}_{f,g}) &= \text{ber}\left(\begin{bmatrix} 0 & f(|S_2|)Ug(|S_1|) \\ f(|S_1|)Vg(|S_2|) & 0 \end{bmatrix}\right) \\ &\leq \frac{1}{2}(\|f(|S_2|)Ug(|S_1|)\| + \|f(|S_1|)Vg(|S_2|)\|). \qquad \text{(by Lemma 2.5)}\end{aligned}$$

Since, $|S_1^*|^2 = S_1S_1^* = U|S_1|^2U^*$, and $|S_2^*|^2 = S_2S_2^* = V|S_2|^2V^*$, so we have $g(|S_1|) = U^*g(|S_1^*|)U$ and $g(|S_2|) = V^*g(|S_2^*|)V$ for every non-negative continuous function g on $[0,\infty)$. Therefore,

$$\text{ber}(\tilde{S}_{f,g}) \leq \frac{1}{2}(\|f(|S_2|)g(|S_1^*|)\| + \|f(|S_1|)g(|S_2^*|)\|),$$

which proves the theorem. □

As a special case of our result we have the following remark (see [5, Theorem 3.1]).

Remark 3.4. *Put $f(t) = t^\alpha, g(t) = t^{1-\alpha}, \alpha \in [0,1]$ in Theorem 3.3, we have*

$$ber(\tilde{S}_t) \leq \frac{1}{2}(\||S_2|^\alpha|S_1^*|^{1-\alpha}\| + \||S_1|^\alpha|S_2^*|^{1-\alpha}\|). \tag{13}$$

The following theorem is an extension of the first part of the inequality (6).

Theorem 3.5. *Let $S = [S_{ij}]$ be an $n \times n$ operator matrix with $S_{ij} \in \mathcal{B}(\mathcal{H}(\Omega_j), \mathcal{H}(\Omega_i))$, $1 \leq i, j \leq n$ and f, g are two non-negative continuous functions defined on $[0,\infty)$ such that $f(t)g(t) = t$ for $t \geq 0$. Then*

$$\text{ber}(S) \leq w([s_{ij}]),$$

where

$$s_{ij} = \begin{cases} \text{ber}(S_{ij}), & for\ i = j, \\ \|f^2(|S_{ij}|)\|^{\frac{1}{2}}\|g^2(|S_{ij}^*|)\|^{\frac{1}{2}} & for\ \ i \neq j. \end{cases}$$

Proof. Let $\mathcal{H} = \bigoplus_{i=1}^{n} \mathcal{H}(\Omega_i)$. For every $(\lambda_1, \ldots, \lambda_n) \in \Omega_1 \times \cdots \times \Omega_n$, let $\hat{k}_{(\lambda_1,\ldots,\lambda_n)} = \begin{bmatrix} k_{\lambda_1} \\ \vdots \\ k_{\lambda_n} \end{bmatrix}$ be the normalized reproducing kernel of $\mathcal{H}$. Using Lemma 2.2, we then have

$$\begin{aligned}
|\tilde{S}(\lambda_1, \ldots, \lambda_n)| &= |\langle S\hat{k}_{(\lambda_1,\ldots,\lambda_n)}, \hat{k}_{(\lambda_1,\ldots,\lambda_n)}\rangle| \\
&= |\sum_{i,j=1}^{n} \langle S_{ij}k_{\lambda_j}, k_{\lambda_i}\rangle| \\
&\leq \sum_{i,j=1}^{n} |\langle S_{ij}k_{\lambda_j}, k_{\lambda_i}\rangle| \\
&= \sum_{i=1}^{n} |\langle S_{ii}k_{\lambda_i}, k_{\lambda_i}\rangle| + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} |\langle S_{ij}k_{\lambda_j}, k_{\lambda_i}\rangle| \\
&\leq \sum_{i=1}^{n} ber(S_{ii})\|k_{\lambda_i}\|^2 + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \langle f^2(|S_{ij}|)k_{\lambda_j}, k_{\lambda_j}\rangle^{1/2} \langle g^2(|S_{ij}^*|)k_{\lambda_i}, k_{\lambda_i}\rangle^{1/2} \\
&\leq \sum_{i=1}^{n} ber(S_{ii})\|k_{\lambda_i}\|^2 + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \|f^2(|S_{ij}|)\|^{1/2} \|g^2(|S_{ij}^*|)\|^{1/2} \|k_{\lambda_i}\|\|k_{\lambda_j}\| \\
&= \langle [s_{ij}]x, x\rangle,
\end{aligned}$$

where $x = \begin{bmatrix} \|k_{\lambda_1}\| \\ \vdots \\ \|k_{\lambda_n}\| \end{bmatrix}$. Since $\|x\| = 1$, so $|\tilde{S}(\lambda_1, \ldots, \lambda_n)| \leq w([s_{ij}])$.

Hence

$$\text{ber}(S) = \sup_{(\lambda_1,\ldots,\lambda_n) \in \Omega_1\times\cdots\times\Omega_n} |\tilde{S}(\lambda_1, \ldots, \lambda_n)| \leq w([s_{ij}]),$$

which proves the theorem. □

One can notice that Remark 3.6 is in [3, Theorem 2.1].

Remark 3.6. *If we take $f(t) = g(t) = t^{1/2}$ in Theorem 3.5, we get*

$$\text{ber}(S) \leq w([s_{ij}]),$$

where

$$s_{ij} = \begin{cases} \text{ber}(S_{ij}), & for\ i = j, \\ \|S_{ij}\| & for\ \ i \neq j. \end{cases}$$

Corollary 3.7. *If $S = \left(\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}\right) \in \mathcal{L}(\mathcal{H}(\Omega_1) \bigoplus \mathcal{H}(\Omega_2)$, then*

$$\text{ber}(S) \leq \frac{1}{2}\left[\text{ber}(S_{11}) + \text{ber}(S_{22}) + \sqrt{(\text{ber}(S_{11}) - \text{ber}(S_{22}))^2 + (M+N)^2}\right],$$

where $M = \|f^2(|S_{12}|)\|^{\frac{1}{2}} \|g^2(|S_{12}^*|)\|^{\frac{1}{2}}$ and $N = \|f^2(|S_{21}|)\|^{\frac{1}{2}} \|g^2(|S_{21}^*|)\|^{\frac{1}{2}}$.

Proof. Using Theorem 3.5, we obtain

$$\begin{aligned} \operatorname{ber}\left(\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}\right) &\le w\left(\begin{bmatrix} \operatorname{ber}(S_{11}) & \|f^2(|S_{12}|)\|^{\frac{1}{2}}\|g^2(|S_{12}^*|)\|^{\frac{1}{2}} \\ \|f^2(|S_{21}|)\|^{\frac{1}{2}}\|g^2(|S_{21}^*|)\|^{\frac{1}{2}} & \operatorname{ber}(S_{22}) \end{bmatrix}\right) \\ &= \rho\left(\begin{bmatrix} \operatorname{ber}(S_{11}) & \frac{M+N}{2} \\ \frac{M+N}{2} & \operatorname{ber}(S_{22}) \end{bmatrix}\right) \\ &= \frac{1}{2}\left[\operatorname{ber}(S_{11}) + \operatorname{ber}(S_{22}) + \sqrt{(\operatorname{ber}(S_{11}) - \operatorname{ber}(S_{22}))^2 + (M+N)^2}\right], \end{aligned}$$

which proves the corollary. □

Remark 3.8. *If we take $f(t) = g(t) = t^{1/2}$ in Corollary 3.7, we get [3, Corollary 2.2].*

$$\operatorname{ber}\left(\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}\right) \le \frac{1}{2}\left[\operatorname{ber}(S_{11}) + \operatorname{ber}(S_{22}) + \sqrt{(\operatorname{ber}(S_{11}) - \operatorname{ber}(S_{22}))^2 + (\|(S_{12}\| + \|(S_{21}\|)^2}\right].$$

The following theorem is an extension of the second part of the inequality (6).

Theorem 3.9. *Let $S = [S_{ij}]$ be an $n \times n$ operator matrix with $S_{ij} \in \mathcal{B}(\mathcal{H}(\Omega_j), \mathcal{H}(\Omega_i))$, $1 \le i, j \le n$ and f, g are two non-negative continuous functions defined on $[0, \infty)$ such that $f(t)g(t) = t$ for $t \ge 0$. Then*

$$\operatorname{ber}(S) \le w([s_{ij}]),$$

where

$$s_{ij} = \begin{cases} \frac{1}{2}\operatorname{ber}(f^2(|S_{ii}|) + g^2(|S_{ii}^*|)), \\ \|f^2(|S_{ij}|)\|^{\frac{1}{2}}\|g^2(|S_{ij}^*|)\|^{\frac{1}{2}} \quad for \;\; i \ne j. \end{cases}$$

Proof. Let $\mathcal{H} = \bigoplus_{i=1}^{n} \mathcal{H}(\Omega_i)$. For every $(\lambda_1, \ldots, \lambda_n) \in \Omega_1 \times \cdots \times \Omega_n$, let $\hat{k}_{(\lambda_1,\ldots,\lambda_n)} = \begin{bmatrix} k_{\lambda_1} \\ \vdots \\ k_{\lambda_n} \end{bmatrix}$ be the normalized reproducing kernel of $\mathcal{H}$. Using Lemma 2.2, we then have

$$
\begin{aligned}
&|\tilde{S}(\lambda_1,\ldots,\lambda_n)| \\
&= |\langle S\hat{k}_{(\lambda_1,\ldots,\lambda_n)}, \hat{k}_{(\lambda_1,\ldots,\lambda_n)}\rangle| \\
&= |\sum_{i,j=1}^{n}\langle S_{ij}k_{\lambda_j}, k_{\lambda_i}\rangle| \\
&\leq \sum_{i,j=1}^{n}|\langle S_{ij}k_{\lambda_j}, k_{\lambda_i}\rangle| \\
&= \sum_{i=1}^{n}|\langle S_{ii}k_{\lambda_i}, k_{\lambda_i}\rangle| + \sum_{\substack{i,j=1\\ i\neq j}}^{n}|\langle S_{ij}k_{\lambda_j}, k_{\lambda_i}\rangle| \\
&\leq \sum_{i=1}^{n}\langle f^2(|S_{ii}|)k_{\lambda_i}, k_{\lambda_i}\rangle^{1/2}\langle g^2(|S_{ii}^*|)k_{\lambda_i}, k_{\lambda_i}\rangle^{1/2} + \sum_{\substack{i,j=1\\ i\neq j}}^{n}\langle f^2(|S_{ij}|)k_{\lambda_j}, k_{\lambda_j}\rangle^{1/2}\langle g^2(|S_{ij}^*|)k_{\lambda_i}, k_{\lambda_i}\rangle^{1/2} \\
&\leq \frac{1}{2}\sum_{i=1}^{n}\Big[\langle f^2(|S_{ii}|)k_{\lambda_i}, k_{\lambda_i}\rangle + \langle g^2(|S_{ii}^*|)k_{\lambda_i}, k_{\lambda_i}\rangle\Big] + \sum_{\substack{i,j=1\\ i\neq j}}^{n}\langle f^2(|S_{ij}|)k_{\lambda_j}, k_{\lambda_j}\rangle^{1/2}\langle g^2(|S_{ij}^*|)k_{\lambda_i}, k_{\lambda_i}\rangle^{1/2} \\
&\leq \frac{1}{2}\sum_{i=1}^{n}\langle\Big(f^2(|S_{ii}|) + g^2(|S_{ii}^*|)\Big)k_{\lambda_i}, k_{\lambda_i}\rangle + \sum_{\substack{i,j=1\\ i\neq j}}^{n}\|f^2(|S_{ij}|)\|^{\frac{1}{2}}\|g^2(|S_{ij}^*|)\|^{\frac{1}{2}}\|k_{\lambda_j}\|\|k_{\lambda_i}\| \\
&\leq \frac{1}{2}\sum_{i=1}^{n}\mathrm{ber}\Big(f^2(|S_{ii}|) + g^2(|S_{ii}^*|)\Big)\|k_{\lambda_i}\|^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{n}\|f^2(|S_{ij}|)\|^{\frac{1}{2}}\|g^2(|S_{ij}^*|)\|^{\frac{1}{2}}\|k_{\lambda_j}\|\|k_{\lambda_i}\| \\
&= \langle [s_{ij}]x, x\rangle,
\end{aligned}
$$

where $x = \begin{bmatrix}\|k_{\lambda_1}\|\\ \vdots \\ \|k_{\lambda_n}\|\end{bmatrix}$. Since $\|x\| = 1$, so $|\tilde{S}(\lambda_1,\ldots,\lambda_n)| \leq w([s_{ij}])$.

Hence

$$\mathrm{ber}(S) = \sup_{(\lambda_1,\ldots,\lambda_n)\in\Omega_1\times\cdots\times\Omega_n}|\tilde{S}(\lambda_1,\ldots,\lambda_n)| \leq w([s_{ij}]),$$

which proves the theorem. □

For $\alpha \in (0,1)$, putting $f(t) = t^{\alpha}$, $g(t) = t^{1-\alpha}$ in Theorem 3.9, we obtain the following inequality see [24, Corollary 3.6].

Corollary 3.10. *Let $S = [S_{ij}]$ be an $n\times n$ operator matrix, where $S_{ij} \in \mathcal{L}(\mathcal{H}(\Omega_j), \mathcal{H}(\Omega_i))$, $1 \leq i, j \leq n$. Then*

$$\mathrm{ber}(S) \leq w([s_{ij}]),$$

where

$$s_{ij} = \begin{cases}\frac{1}{2}\mathrm{ber}\Big(\alpha|S_{ii}|^2 + (1-\alpha)|S_{ii}^*|^2\Big), & for\ i = j\\ \|S_{ij}\|, & i \neq j.\end{cases}$$

Corollary 3.11. *Let* $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \in \mathcal{L}(\mathcal{H}_1(\Omega) \bigoplus \mathcal{H}_2(\Omega))$, *then*

$$\mathrm{ber}(S) \leq \frac{1}{2}\left(\tilde{S}_{11} + \tilde{S}_{22} + \sqrt{(\tilde{S}_{11} - \tilde{S}_{22})^2 + (M+N)^2}\right),$$

where $\tilde{S}_{ii} = \frac{1}{2}\mathrm{ber}\Big(f^2(|S_{ii}|) + g^2(|S_{ii}^*|)\Big)$, $i = 1, 2$, $M = \|f^2(|S_{12}|)\|^{\frac{1}{2}}\|g^2(|S_{12}^*|)\|^{\frac{1}{2}}$ *and* $N = \|f^2(|S_{21}|)\|^{\frac{1}{2}}\|g^2(|S_{21}^*|)\|^{\frac{1}{2}}$.

Proof. Proof is very easy by using spectral radius formula. □

Remark 3.12. *If we take* $f(t) = g(t) = t^{1/2}$ *in Corollary 3.11, we get*

$$\mathrm{ber}\left(\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}\right) \leq \frac{1}{2}\left(\tilde{S}_{11} + \tilde{S}_{22} + \sqrt{(\tilde{S}_{11} - \tilde{S}_{22})^2 + (\|S_{12}\| + \|S_{21}\|)^2}\right),$$

where $\tilde{S}_{ii} = \frac{1}{2}\mathrm{ber}\Big(|S_{ii}| + |S_{ii}^*|\Big)$, $i = 1, 2$.

4. Some reverse Berezin number inequalities for normal operators

In this section, we give some reverse Berezin number inequalities for normal operators on reproducing kernel Hilbert spaces $\mathcal{H} = \mathcal{H}(\Omega)$.

The first theorem is as follows:

Theorem 4.1. *Let* $\mathcal{H} = \mathcal{H}(\Omega)$ *be a reproducing kernel Hilbert space and let* $A : \mathcal{H} \to \mathcal{H}$ *be a normal operator. If* $\mu \in \mathbb{C}\setminus\{0\}$ *and* $k > 0$ *are such that*

$$\left\|A - \mu A^*\right\| \leq k, \tag{4.1}$$

then

$$\frac{1 + |\mu|^2}{2|\mu|}\left\|A\widehat{k}_\lambda\right\|^2 \leq ber\left(A^2\right) + \frac{k^2}{2|\mu|} \tag{4.2}$$

for all $\lambda \in \Omega$.

Proof. The inequality (4.1) is clearly equivalent to

$$\left\|A\widehat{k}_\lambda\right\|^2 + |\mu|^2\left\|A^*\widehat{k}_\lambda\right\|^2 \leq 2\,\mathrm{Re}\left[\overline{\mu}\left\langle A\widehat{k}_\lambda, A^*\widehat{k}_\lambda\right\rangle\right] + k^2 \tag{4.3}$$

for all $\lambda \in \Omega$. Since A is a normal operator, then $\left\|A\widehat{k}_\lambda\right\| = \left\|A^*\widehat{k}_\lambda\right\|$ for any $\lambda \in \Omega$ and by (4.3) we have

$$\left(1 + |\mu|^2\right)\left\|A\widehat{k}_\lambda\right\|^2 \leq 2\,\mathrm{Re}\left[\overline{\mu}\left\langle A^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right] + k^2$$

for any $\lambda \in \Omega$.

Also we have that

$$\begin{aligned}\mathrm{Re}\left[\overline{\mu}\left\langle A^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right] &\leq |\mu|\left|\widetilde{A^2}(\lambda)\right| \\ &\leq |\mu|\sup_{\lambda\in\Omega}\left|\widetilde{A^2}(\lambda)\right| \\ &= |\mu|\, ber\left(A^2\right).\end{aligned}$$

Hence, we deduce (4.2) together with inequality (4.3) and above inequality. □

For a normal operator A, we see that

$$\left|\widetilde{A^2}(\lambda)\right| = \left|\left\langle A\widehat{k}_\lambda, A^*\widehat{k}_\lambda\right\rangle\right| \le \left\|A\widehat{k}_\lambda\right\| \left\|A^*\widehat{k}_\lambda\right\| = \left\|A\widehat{k}_\lambda\right\|^2$$

for any $\lambda \in \Omega$. Therefore,

$$\left\|A\widehat{k}_\lambda\right\| - \left|\left\langle A\widehat{k}_\lambda, A^*\widehat{k}_\lambda\right\rangle\right|^{\frac{1}{2}} \ge 0$$

for any $\lambda \in \Omega$.

Denote $\delta(A) = \inf_{\lambda\in\Omega}\left[\left\|A\widehat{k}_\lambda\right\| - \left|\left\langle A\widehat{k}_\lambda, A^*\widehat{k}_\lambda\right\rangle\right|^{\frac{1}{2}}\right] \ge 0$. We can state the following result.

Theorem 4.2. *Let A be a normal operator satisfying Theorem 4.1. Then we have*

$$\left\|A\widehat{k}_\lambda\right\|^2 - ber\left(A^2\right) \le k^2 - 2\left|\mu\right|\delta(A)\,\eta(A),$$

for any $\lambda \in \Omega$, where $\eta(A) = \inf_{\lambda\in\Omega}\left|\widetilde{A^2}(\lambda)\right|^{1/2}$.

Proof. From the inequality (4.3), we get

$$\left\|A\widehat{k}_\lambda\right\|^2 - \left|\widetilde{A^2}(\lambda)\right| \le 2\,\mathrm{Re}\left[\overline{\mu}\widetilde{A^2}(\lambda)\right] - \left|\widetilde{A^2}(\lambda)\right| - \left|\mu\right|^2\left\|A\widehat{k}_\lambda\right\|^2 + k^2 \tag{4.4}$$

for any $\lambda \in \Omega$.

We can write the right hand side of above inequality as follows:

$$I = k^2 + 2\,\mathrm{Re}\left[\overline{\mu}\widetilde{A^2}(\lambda)\right] - 2\left|\mu\right|\left|\widetilde{A^2}(\lambda)\right|^{1/2}\left\|A\widehat{k}_\lambda\right\| - \left(\left|\widetilde{A^2}(\lambda)\right|^{1/2} - \left|\mu\right|\left\|A\widehat{k}_\lambda\right\|\right)^2.$$

Since, clearly,

$$\mathrm{Re}\left[\overline{\mu}\widetilde{A^2}(\lambda)\right] \le \left|\mu\right|\left|\widetilde{A^2}(\lambda)\right|$$

and

$$\left(\left|\widetilde{A^2}(\lambda)\right|^{1/2} - \left|\mu\right|\left\|A\widehat{k}_\lambda\right\|\right)^2 \ge 0,$$

then

$$\begin{aligned} I &\le k^2 - 2\left|\mu\right|\left|\widetilde{A^2}(\lambda)\right|^{1/2}\left(\left\|A\widehat{k}_\lambda\right\| - \left|\widetilde{A^2}(\lambda)\right|^{1/2}\right) \\ &\le k^2 - 2\left|\mu\right|\eta(A)\left|\widetilde{A^2}(\lambda)\right|^{1/2}. \end{aligned}$$

Using the inequality (4.4), we have

$$\begin{aligned} \left\|A\widehat{k}_\lambda\right\|^2 &\le \left|\widetilde{A^2}(\lambda)\right| - 2\left|\mu\right|\eta(A)\left|\widetilde{A^2}(\lambda)\right|^{1/2} + k^2 \\ &\sup_{\lambda\in\Omega}\left|\widetilde{A^2}(\lambda)\right| - 2\left|\mu\right|\eta(A)\inf_{\lambda\in\Omega}\left|\widetilde{A^2}(\lambda)\right|^{1/2} + k^2 \\ &\le ber\left(A^2\right) - 2\left|\mu\right|\eta(A)\,\delta(A) + k^2 \end{aligned}$$

for any $\lambda \in \Omega$, which gives the desired result. □

Notice that for a normal operator A and $\mu \in \mathbb{C}\setminus\{0\}$, $k > 0$, the following two conditions are equivalent

$$(i) \quad \left\|\widehat{Ak_\lambda} - \mu A^*\widehat{k_\lambda}\right\| \leq k \leq |\mu| \left\|\widehat{Ak_\lambda}\right\| \text{ for any } \lambda \in \Omega$$

and

$$(ii) \quad \|A - \mu A^*\| \leq k \text{ and } \Phi(A) = \inf_{\lambda\in\Omega} \left\|\widehat{Ak_\lambda}\right\| \geq \frac{k}{|\mu|}.$$

Theorem 4.3. *Let A be a normal operator on a reproducing kernel Hilbert space $\mathcal{H}$ satisfying either (i) or, equivalently, (ii) for $\mu \in \mathbb{C}\setminus\{0\}$ and $k > 0$. Then we have*
(i_1)

$$\left\|\widehat{Ak_\lambda}\right\|^4 - ber^2\left(A^2\right) \leq k^2 \left\|\widehat{Ak_\lambda}\right\|^2$$

(i_2)

$$\left\|\widehat{Ak_\lambda}\right\| \left(\Phi^2(A) - \frac{k^2}{|\mu|^2}\right)^{1/2} \leq ber\left(A^2\right)$$

for all $\lambda \in \Omega$.

Proof. We know from Dragomir result (see [9, 10]):

$$\|x\|^2 \|a\|^2 - [\mathrm{Re}\langle x, a\rangle]^2 \leq k^2 \|x\|^2$$

if $\|x - a\| \leq k \leq \|a\|$.
Putting $x = \widehat{Ak_\lambda}$ and $a = \mu A^*\widehat{k_\lambda}$, we get

$$\left\|\widehat{Ak_\lambda}\right\|^2 \left\|\mu A^*\widehat{k_\lambda}\right\|^2 - \left|\left\langle \widehat{Ak_\lambda}, \mu A^*\widehat{k_\lambda}\right\rangle\right|^2 \leq k^2 \left\|\mu A^*\widehat{k_\lambda}\right\|^2$$

and hence

$$\left\|\widehat{Ak_\lambda}\right\|^4 \leq ber^2\left(A^2\right) + k^2 \left\|A^*\widehat{k_\lambda}\right\|^2$$

which gives the desired result (i_1).
Also we know that provided $\|x - a\| \leq k \leq \|a\|$, then (see [9, 10])

$$\|x\|^2 \left(\|a\|^2 - k^2\right)^{1/2} \leq [\mathrm{Re}\langle x, a\rangle],$$

which gives

$$\begin{aligned}\left\|\widehat{Ak_\lambda}\right\| \left(|\mu|^2 \left\|\widehat{Ak_\lambda}\right\|^2 - k^2\right)^{1/2} &\leq \mathrm{Re}\left\langle \widehat{Ak_\lambda}, \mu A^*\widehat{k_\lambda}\right\rangle \\ &\leq |\mu| \left|\widetilde{A^2}(\lambda)\right| \\ &\leq |\mu|\, ber\left(A^2\right),\end{aligned}$$

that is,

$$\left\|\widehat{Ak_\lambda}\right\| \left(\left\|\widehat{Ak_\lambda}\right\|^2 - \frac{k^2}{|\mu|^2}\right)^{1/2} \leq ber\left(A^2\right)$$

for any $\lambda \in \Omega$. Since, clearly,

$$\left(\left\|\widehat{Ak_\lambda}\right\|^2 - \frac{k^2}{|\mu|^2}\right)^{1/2} \geq \left(\Phi^2(A) - \frac{k^2}{|\mu|^2}\right)^{1/2},$$

then we have

$$\left\|\widehat{Ak_\lambda}\right\| \left(\Phi^2(A) - \frac{k^2}{|\mu|^2}\right)^{1/2} \leq ber\left(A^2\right)$$

which gives the desired inequality (i_2). □

Theorem 4.4. *Let A be a normal operator on a reproducing kernel Hilbert space $\mathcal{H}$ satisfying either (i) or, equivalently, (ii) for $\mu \in \mathbb{C}\setminus\{0\}$ and $k > 0$. Then we have*

$$\left\|\widehat{Ak_\lambda}\right\|^4 - ber^2\left(A^2\right) \leq 2ber\left(A^2\right)\left\|\widehat{Ak_\lambda}\right\|\left[|\mu|\left\|\widehat{Ak_\lambda}\right\| - \left(|\mu|^2\Phi^2(A) - k^2\right)^{1/2}\right]$$

for any $\lambda \in \Omega$.

Proof. The reverse of the Schwarz inequality obtained in [9] is as following:

$$0 \leq \|x\|^2\|a\|^2 - |\langle x,a\rangle|^2 \leq 2|\langle x,a\rangle|\,\|a\|\left(\|a\| - \sqrt{\|a\|^2 - k^2}\right)$$

if $\|x - a\| \leq k \leq \|a\|$.

Choosing $x = \widehat{Ak_\lambda}$ and $a = \mu A^*\widehat{k_\lambda}$, we have

$$\begin{aligned}&\left\|\widehat{Ak_\lambda}\right\|^2|\mu|^2\left\|A^*\widehat{k_\lambda}\right\| - |\mu|^2\left|\widetilde{A^2}(\lambda)\right|^2\\ &\leq 2|\mu|^2\left|\widetilde{A^2}(\lambda)\right|\left\|A^*\widehat{k_\lambda}\right\|\left[|\mu|\left\|A^*\widehat{k_\lambda}\right\| - \left(|\mu|^2\left\|A^*\widehat{k_\lambda}\right\|^2 - k^2\right)^{1/2}\right]\end{aligned}$$

and hence

$$\left\|\widehat{Ak_\lambda}\right\|^4 - \sup_{\lambda\in\Omega}\left|\widetilde{A^2}(\lambda)\right|^2 \leq 2\sup_{\lambda\in\Omega}\left|\widetilde{A^2}(\lambda)\right|\left\|\widehat{Ak_\lambda}\right\|\left[|\mu|\left\|A^*\widehat{k_\lambda}\right\| - \left(|\mu|^2\inf_{\lambda\in\Omega}\left\|A^*\widehat{k_\lambda}\right\|^2 - k^2\right)^{1/2}\right]$$

for all $\lambda \in \Omega$.

Therefore, we get

$$\left\|\widehat{Ak_\lambda}\right\|^4 - ber^2\left(A^2\right) \leq 2ber\left(A^2\right)\left\|\widehat{Ak_\lambda}\right\|\left[|\mu|\left\|A^*\widehat{k_\lambda}\right\| - \left(|\mu|^2\Phi^2(A) - k^2\right)^{1/2}\right],$$

which proves the theorem. □

5. Berezin number and projection operator

In this section, we characterize the projection operators and partial isometry operators in terms of Berezin number. Recall that A is a projection operator if $A^* = A = A^2$, and A is a partial isometry if $A = AA^*A$.

Definition 5.1. *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a reproducing kernel Hilbert space of complex-valued functions defined on some set Ω. We say that $\mathcal{H}$ has (Ber) property, if for any two operators $A_1, A_2 \in \mathcal{B}(\mathcal{H})$, $\widetilde{A_1}(\lambda) = \widetilde{A_2}(\lambda)$ for all $\lambda \in \Omega$ means that $A_1 = A_2$.*

Notice that any reproducing kernel Hilbert space of analytic functions in the unit disc $\mathbb{D}$ (including Bergman and Hardy spaces) possesses the (*Ber*) property (see Zhu [27]).

The main results is as following:

Theorem 5.2. *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a reproducing kernel Hilbert space with the (Ber) property and $A \in \mathcal{B}(\mathcal{H})$ be an idempotent operator ($A^2 = A$). Then A is projection if and only if $ber(A^*A) \leq 1$.*

Proof. From the definition of Berezin number, we have that

$$ber(A^*A) \leq 1 \text{ if and only if } \left\|A\widehat{k}_\lambda\right\| \leq 1 \ (\forall\, \lambda \in \Omega). \tag{5.1}$$

Taking into consideration assertion (5.1), we obtain for all $\lambda \in \Omega$ that

$$\begin{aligned}
\left\|(A - A^*A)\widehat{k}_\lambda\right\|^2 &= \left\langle (A - A^*A)\widehat{k}_\lambda, (A - A^*A)\widehat{k}_\lambda \right\rangle \\
&= \left\|A\widehat{k}_\lambda\right\|^2 - \left\langle A\widehat{k}_\lambda, A^*A\widehat{k}_\lambda \right\rangle - \left\langle A^*A\widehat{k}_\lambda, A\widehat{k}_\lambda \right\rangle + \left\|A^*A\widehat{k}_\lambda\right\|^2 \\
&= \left\|A\widehat{k}_\lambda\right\|^2 - \left\langle A^2\widehat{k}_\lambda, A\widehat{k}_\lambda \right\rangle - \left\langle A\widehat{k}_\lambda, A^2\widehat{k}_\lambda \right\rangle + \left\|A^*A\widehat{k}_\lambda\right\|^2 \\
&= \left\|A\widehat{k}_\lambda\right\|^2 - \left\|A\widehat{k}_\lambda\right\|^2 - \left\|A\widehat{k}_\lambda\right\|^2 + \left\|A^*A\widehat{k}_\lambda\right\|^2 \ (\text{by } A^2 = A) \\
&= \left\|A^*A\widehat{k}_\lambda\right\|^2 - \left\|A\widehat{k}_\lambda\right\|^2 \leq 0
\end{aligned}$$

and hence $(A - A^*A)k_\lambda = 0$ for all $\lambda \in \Omega$. Since $\{k_\lambda : \lambda \in \Omega\}$ is a total set, we reach that $A = A^*A$, that is, A is self-adjoint. So, A is a projection. □

Next result characterizes the partial isometry operator in terms of Berezin number.

Theorem 5.3. *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a reproducing kernel Hilbert space with the (Ber) property and $A = A^k$ for some positive integer $k \geq 2$. Then $A \in \mathcal{B}(\mathcal{H})$ is partial isometry if and only if $ber(A^*A) \leq 1$.*

Proof. Since $A^{2(k-1)} = A^{k-2}A^k = A^{k-2}A = A^{k-1}$, A^{k-1} is an idempotent operator, and A^{k-1} is a projection by Theorem 5.2. Also, we know from Theorem 5.2 that $ber(A^*A) \leq 1$ if and only if $\left\|A\widehat{k}_\lambda\right\| \leq 1$ ($\forall\, \lambda \in \Omega$). Then,

$$\begin{aligned}
\left\|(A - AA^*A)\widehat{k}_\lambda\right\|^2 &= \left\langle (A - AA^*A)\widehat{k}_\lambda, (A - AA^*A)\widehat{k}_\lambda \right\rangle \\
&= \left\|A\widehat{k}_\lambda\right\|^2 - \left\langle A\widehat{k}_\lambda, AA^*A\widehat{k}_\lambda \right\rangle - \left\langle AA^*A\widehat{k}_\lambda, A\widehat{k}_\lambda \right\rangle + \left\|AA^*A\widehat{k}_\lambda\right\|^2 \\
&= \left\|A\widehat{k}_\lambda\right\|^2 - 2\left\|A^*A\widehat{k}_\lambda\right\|^2 + \left\|AA^*A\widehat{k}_\lambda\right\|^2 \\
&\leq \left\|A\widehat{k}_\lambda\right\|^2 - 2\left\|A^*A\widehat{k}_\lambda\right\|^2 + \left\|A^*A\widehat{k}_\lambda\right\|^2 \\
&= \left\|A\widehat{k}_\lambda\right\|^2 - \left\|A^*A\widehat{k}_\lambda\right\|^2 \\
&= \left\|A^k\widehat{k}_\lambda\right\|^2 - \left\|A^*A\widehat{k}_\lambda\right\|^2 \\
&= \left\|A^{k-1}A\widehat{k}_\lambda\right\|^2 - \left\|A^*A\widehat{k}_\lambda\right\|^2 \\
&= \left\|A^{*k-2}A^*A\widehat{k}_\lambda\right\|^2 - \left\|A^*A\widehat{k}_\lambda\right\|^2 \\
&\leq \|A^*\|^{k-2}\left\|A^*A\widehat{k}_\lambda\right\|^2 - \left\|A^*A\widehat{k}_\lambda\right\|^2 \leq 0.
\end{aligned}$$

Therefore, $(A - AA^*A)k_\lambda = 0$ for all $\lambda \in \Omega$. Since $\{k_\lambda : \lambda \in \Omega\}$ is a total set, we obtain that $A = AA^*A$, that is, A is partial isometric. □

Declarations

Authors declare that data sharing is not applicable to this article as no datasets were generated or analysed during the current study. Authors also declare that there is no financial or non-financial interests that are directly or indirectly related to the work submitted for publication. The authors declare that there is no conflict of interest.

Acknowledgements

The authors thank the referee for the valuable suggestions and comments on an earlier version. Incorporating appropriate responses to these in the article has led to a better presentation of the results.

References

[1] Abu-Omar, A., Kittaneh, F., Numerical radius inequalities for $n \times n$ operator matrices, *Linear Algebra Appl*. **468** (2015), 18–26.
[2] Abu-Omar, A., Kittaneh, F., Upper and lower bounds for the numerical radius with an application to involution operators, *Rocky Mountain J. Math*, **45** (4) (2015), 1055–1065.
[3] Bakherad, M., Some Berezin number inequalities for operator matrices, *Czechoslovak Math. J.*, **68** (2018), 997–1009.
[4] Bakherad, M., Garayev, M. T., Berezin number inequalities for operators, *Concrete Operators*, **6** (2019), 33–43.
[5] Bakherad, M., Hajmohamadi, M., Rahmatollah, L., Sahoo, S., Some extensions of Berezin number inequalities on operators, *Rocky Mountain J. Math*, **51** (6) (2021), 1941–1951
[6] Bani-Dom, W., Kittaneh, F., Numerical radius inequalities for operator matrices, *Linear Multilinear Algebra*, **57** (2009), 421–427.
[7] Berezin, F. A., Covariant and contravariant symbols of operators, *Math. USSR, Izv.* **6** 1972, (1973) 1117–1151. (In English. Russian original.); translation from Russian Izv. Akad. Nauk SSSR, Ser. Mat. **36** (1972), 1134–1167.
[8] Bhatia, R.: Matrix analysis, Graduate Texts in Mathematics, **169**, Springer-Verlag, New York, 1997.
[9] Dragomir, S. S., Reverses of the Schwarz inequality in inner product spaces generalising a Klamkin-McLenaghan result, *Bull. Austral. Math. Soc.* **73** (1) (2006), 69-78.
[10] Dragomir, S. S., Reverses of the Schwarz , triangle and Bessel inequalities in inner product spaces, *J. Ineq. Pure Appl. Math.* **5** (3) (2004), Art. 76.
[11] El-Haddad, M., Kittaneh, F., Numerical radius inequalities for Hilbert space operators. II, *Studia Math.* **182** (2007), 133–140.
[12] Halmos, P. R., A Hilbert space problem book. Second edition. Graduate Texts in Mathematics, 19. Encyclopedia of Mathematics and its Applications, **17**, Springer-Verlag, New York-Berlin, 1982
[13] Horn, R. A., Johnson, C. R., Topics in Matrix Analysis. Cambridge University Press, Cambridge, 1991.
[14] Hou, J. C., Du, H. K., Norm inequalities of positive operator matrices, *Integral Equ. Oper. Theory*, **22** (1995), 281–294.
[15] Karaev, M.T., Berezin symbol and invertibility of operators on the functional Hilbert spaces, *J. Funct. Anal.* **238** (2006), 181-192.
[16] Kittaneh, F., A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix, *Studia Math.* **158** (2003), 11–17.
[17] Kittaneh, F., Notes on some inequalities for Hilbert space operators, *Publ. Res. Inst. Math. Sci.* **24** (1988), 283–293.
[18] Kittaneh, F., Numerical radius inequalities for Hilbert space operators, *Studia Math.* **168** (2005) 73–80.
[19] Okubo, K., On weakly unitarily invariant norm and the Aluthge transformation, *Linear Algebra Appl*. **371** (2003), 369–375.
[20] Pečarić, J., Furuta, T., Hot, J. M., Seo, Y., Mond-Pečarić method in operator inequalities, inequalities for bounded selfadjoint operators on a Hilbert space. Monographs in Inequalities, Element, Zagreb, 2005
[21] Sahoo, S., Das, N., Mishra, D., Numerical radius inequalities for operator matrices, *Adv. Oper. Theory,* **4** (2019), 197–214
[22] Sahoo, S., Das, N., Mishra, D., Berezin number and numerical radius inequalities for operators on Hilbert spaces, *Adv. Oper. Theory*, (2020), DOI:10.1007/s43036-019-00035-8
[23] S. Sahoo, M. Bakherad, Some extended Berezin number inequalities, *Filomat*, **35** (6) (2021), 2043–2053.
[24] S. Sahoo, N. Das, N. C. Rout, On Berezin number inequalities for operator matrices, *Acta Math. Sin. Engl. Ser.* **37** (2021), 873–892.
[25] Shebrawi, K., Bakherad, M., Generalizations of the Aluthge transform of operators, *Filomat*,**32** (18) (2018), 6465–6474.
[26] Yamazaki, T., On lower bounds for the numerical radius and an equality condition. *Studia Math.* **178** (2007), 83–89.
[27] Zhu K., Operator theory in function spaces. Second ed., Mathematical Surveys and Monographs, Vol. 138, American Mathematical Society, Providence, R.I. 2007.