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A geometric approach to inequalities for the Hilbert–Schmidt norm

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Abstract. We show that if X and Y are two non-zero Hilbert–Schmidt operators, then for any $\lambda \geq 0$,

$$\begin{aligned}&\cos^2 \Theta_{X,Y}\\ &\leq \frac{1}{1+\lambda}\sqrt{\cos \Theta_{|X^*|,|Y^*|}}\sqrt{\cos \Theta_{|X|,|Y|}}\,\frac{|\langle X, Y\rangle|}{\|X\|_2\|Y\|_2} + \frac{\lambda}{1+\lambda}\cos \Theta_{|X^*|,|Y^*|}\cos \Theta_{|X|,|Y|}\\ &\leq \cos \Theta_{|X^*|,|Y^*|}\cos \Theta_{|X|,|Y|}.\end{aligned}$$

Here $\Theta_{A,B}$ denotes the angle between non-zero Hilbert–Schmidt operators A and B. This enables us to present some inequalities for the Hilbert–Schmidt norm. In particular, we prove that

$$\big\|X+Y\big\|_2 \leq \sqrt{\frac{\sqrt{2}+1}{2}}\,\big\||X|+|Y|\big\|_2.$$

1. Introduction and preliminaries

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators acting on a Hilbert space $\big(\mathcal{H}, [\cdot,\cdot]\big)$. For every $X \in \mathcal{B}(\mathcal{H})$, let $|X|$ denote the square root of X^*X, that is, $|X| = (X^*X)^{1/2}$. Let $X = U|X|$ be the polar decomposition of X, where U is some partial isometry. The polar decomposition satisfies

$$U^*X = |X|,\ U^*U|X| = |X|,\ U^*UX = X,\ X^* = |X|U^*,\ |X^*| = U|X|U^*. \tag{1}$$

Let $C_1(\mathcal{H})$ and $C_2(\mathcal{H})$ denote the trace-class and the Hilbert–Schmidt class in $\mathcal{B}(\mathcal{H})$, respectively. It is well known that $C_1(\mathcal{H})$ and $C_2(\mathcal{H})$ each form a two-sided $*$-ideal in $\mathcal{B}(\mathcal{H})$ and $C_2(\mathcal{H})$ is itself a Hilbert space with the inner product

$$\langle X, Y\rangle = \sum_{i=1}^{\infty}[Xe_i, Ye_i] = \mathrm{Tr}(Y^*X) \tag{2}$$

where $\{e_i\}_{i=1}^{\infty}$ is any orthonormal basis of $\mathcal{H}$ and $\mathrm{Tr}(\cdot)$ is the natural trace on $C_1(\mathcal{H})$. Three principal properties of the trace are that it is a linear functional and, for every X and Y, we have $\mathrm{Tr}(X^*) = \overline{\mathrm{Tr}(X)}$ and

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$\mathrm{Tr}(XY) = \mathrm{Tr}(YX)$. The Hilbert–Schmidt norm of $X \in \mathcal{C}_2(\mathcal{H})$ is given by $\|X\|_2 = \sqrt{\langle X, X\rangle}$. One more fact that will be needed the sequel is that if $X \in \mathcal{C}_2(\mathcal{H})$, then

$$\|X\|_2 = \|X^*\|_2 = \big\| |X| \big\|_2 = \big\| |X^*| \big\|_2. \tag{3}$$

The reader is referred to [8, 10] for further properties of the Hilbert–Schmidt class.

In Section 2, by using the angle $\Theta_{X,Y}$ between non-zero Hilbert–Schmidt operators X, Y, we present an extension of the Fuglede–Putnam theorem modulo the Hilbert–Schmidt class. We also prove that $|\langle X, Y\rangle|^2 \leq \langle |X^*|, |Y^*|\rangle\langle |X|, |Y|\rangle$ and then we apply it to obtain for any $\lambda \geq 0$,

$$\cos^2\Theta_{X,Y} \leq \frac{1}{1+\lambda}\sqrt{\cos\Theta_{|X^*|,|Y^*|}}\sqrt{\cos\Theta_{|X|,|Y|}}\,\frac{|\langle X.Y\rangle|}{\|X\|_2\|Y\|_2} + \frac{\lambda}{1+\lambda}\cos\Theta_{|X^*|,|Y^*|}\cos\Theta_{|X|,|Y|}.$$

In Section 3, by using the results in Section 2, we provide alternative proofs of some well-known inequalities for the Hilbert–Schmidt norm. For example we present a considerably briefer proof of an extension of the Araki–Yamagami inequality [1] obtained by Kittaneh [3]. In particular, we prove Lee's conjecture [6, p. 584] on the sum of the square roots of operators. Some related inequalities and numerical examples are also presented.

2. Angle between two Hilbert–Schmidt operators

Let $X, Y \in \mathcal{C}_2(\mathcal{H})$. Since $\mathcal{C}_2(\mathcal{H})$ is a Hilbert space with the inner product (2), by the Cauchy–Schwarz inequality, we have

$$-\|X\|_2\|Y\|_2 \leq -|\langle X, Y\rangle| \leq \mathrm{Re}\langle X, Y\rangle \leq |\langle X, Y\rangle| \leq \|X\|_2\|Y\|_2. \tag{4}$$

Therefore, when X and Y are non-zero operators, the inequality (4) implies

$$-1 \leq \frac{\mathrm{Re}\langle X, Y\rangle}{\|X\|_2\|Y\|_2} \leq 1.$$

This motivates defining the angle between X and Y as follows (see [12]).

Definition 2.1. *For non-zero operators $X, Y \in \mathcal{C}_2(\mathcal{H})$, the angle $\Theta_{X,Y}$ between X and Y is defined by*

$$\cos\Theta_{X,Y} = \frac{\mathrm{Re}\langle X, Y\rangle}{\|X\|_2\|Y\|_2}; \quad 0 \leq \Theta_{X,Y} \leq \pi.$$

Example 2.2. *Let us recall that by [8, p. 66] we have*

$$\mathrm{Tr}(a\otimes b) = [a, b] \quad \textit{and} \quad \|a\otimes b\|_2 = \|a\|\|b\|,$$

for all $a, b \in \mathcal{H}$. Here, $a \otimes b$ denotes the rank one operator in $\mathcal{B}(\mathcal{H})$ defined by $(a\otimes b)c := [c, b]a$ for all $c \in \mathcal{H}$. Now, let $x, y, z \in \mathcal{H}\setminus\{0\}$. Put $X = x\otimes z$ and $Y = y\otimes z$. A simple calculation shows that $\langle X, Y\rangle = \|z\|^2[x, y]$. Thus,

$$\cos\Theta_{X,Y} = \frac{\mathrm{Re}[x, y]}{\|x\|\|y\|}.$$

Remark 2.3. *Let $X, Y \in \mathcal{C}_2(\mathcal{H})\setminus\{0\}$.*

(i) One can see that $\cos\Theta_{\gamma X,\gamma Y} = \cos\Theta_{X,Y}$ for all $\gamma \in \mathbb{C}\setminus\{0\}$.

(ii) For every $\alpha, \beta \in \mathbb{R}\setminus\{0\}$, it is easy to check that

$$\cos\Theta_{\alpha X,\beta Y} = \begin{cases} \cos\Theta_{X,Y} & \alpha\beta > 0 \\ -\cos\Theta_{X,Y} & \alpha\beta < 0. \end{cases}$$

(iii) By (2) *and Definition 2.1, we have*

$$\|X \pm Y\|_2^2 = \|X\|_2^2 + \|Y\|_2^2 \pm 2\|X\|_2\|Y\|_2 \cos\Theta_{X,Y}.$$

(iv) If X and Y are positive, then

$$\langle X, Y\rangle = \mathrm{Tr}\left(Y^{1/2}Y^{1/2}X\right) = \mathrm{Tr}\left(Y^{1/2}XY^{1/2}\right) = \mathrm{Tr}\left(\left(X^{1/2}Y^{1/2}\right)^*\left(X^{1/2}Y^{1/2}\right)\right).$$

Therefore, $\mathrm{Re}\langle X, Y\rangle = \langle X, Y\rangle \geq 0$ *and hence* $0 \leq \Theta_{X,Y} \leq \frac{\pi}{2}$.

Let us recall the Fuglede–Putnam theorem modulo the Hilbert–Schmidt class due to Weiss [11, Theorem 1]: if X and Y are normal operators and Z is an operator on $\mathcal{H}$, then $\|XZ - ZY\|_2 = \|X^*Z - ZY^*\|_2$. In the following theorem, we present an extension of the Fuglede–Putnam theorem modulo the Hilbert–Schmidt class which provided by Kittaneh in [4]. Our proof is very different from that in [4, Theorem 5].

Theorem 2.4. *If* X, Y *and* Z *are operators on* $\mathcal{H}$*, then*

$$\|XZ - ZY\|_2^2 + \|X^*Z\|_2^2 + \|ZY^*\|_2^2 = \|XZ\|_2^2 + \|ZY\|_2^2 + \|X^*Z - ZY^*\|_2^2.$$

Proof. With no loss of generality we may assume that the operators XZ, ZY, X^*Z and ZY^* are Hilbert–Schmidt; otherwise, both sides of the last equation in the theorem are infinite. We may also assume that $XZ, ZY, X^*Z, ZY^* \neq 0$ otherwise the desired equality trivially holds. By (2) and Definition 2.1 we have

$$\begin{aligned}
\|XZ\|_2\|ZY\|_2 \cos\Theta_{XZ,ZY} &= \mathrm{Re}\langle XZ, ZY\rangle = \mathrm{Re}\mathrm{Tr}\Big(Y^*Z^*XZ\Big) \\
&= \mathrm{Re}\overline{\mathrm{Tr}\Big(Z^*X^*ZY\Big)} = \mathrm{Re}\mathrm{Tr}\Big(Z^*X^*ZY\Big) \\
&= \mathrm{Re}\mathrm{Tr}\Big(YZ^*X^*Z\Big) = \mathrm{Re}\langle X^*Z, ZY^*\rangle \\
&= \|X^*Z\|_2\|ZY^*\|_2 \cos\Theta_{X^*Z,ZY^*},
\end{aligned}$$

and hence

$$\|XZ\|_2\|ZY\|_2 \cos\Theta_{XZ,ZY} = \|X^*Z\|_2\|ZY^*\|_2 \cos\Theta_{X^*Z,ZY^*}. \tag{5}$$

So, by (5) and Remark 2.3(iii), we have

$$\begin{aligned}
&\|XZ - ZY\|_2^2 + \|X^*Z\|_2^2 + \|ZY^*\|_2^2 \\
&\quad = \|XZ\|_2^2 + \|ZY\|_2^2 - 2\|XZ\|_2\|ZY\|_2 \cos\Theta_{XZ,ZY} + \|X^*Z\|_2^2 + \|ZY^*\|_2^2 \\
&\quad = \|XZ\|_2^2 + \|ZY\|_2^2 - 2\|X^*Z\|_2\|ZY^*\|_2 \cos\Theta_{X^*Z,ZY^*} + \|X^*Z\|_2^2 + \|ZY^*\|_2^2 \\
&\quad = \|XZ\|_2^2 + \|ZY\|_2^2 + \|X^*Z - ZY^*\|_2^2.
\end{aligned}$$

□

Remark 2.5. *Let* X *and* Y *be normal operators and let* Z *be an operator on* $\mathcal{H}$*. By* (2) *we have*

$$\begin{aligned}
\|X^*Z\|_2^2 + \|ZY^*\|_2^2 &= \langle X^*Z, X^*Z\rangle + \langle ZY^*, ZY^*\rangle \\
&= \mathrm{Tr}\Big(Z^*XX^*Z\Big) + \mathrm{Tr}\Big(YZ^*ZY^*\Big) \\
&= \mathrm{Tr}\Big(Z^*XX^*Z\Big) + \mathrm{Tr}\Big(Z^*ZY^*Y\Big) \\
&= \mathrm{Tr}\Big(Z^*X^*XZ\Big) + \mathrm{Tr}\Big(Z^*ZYY^*\Big) \\
&= \mathrm{Tr}\Big(Z^*X^*XZ\Big) + \mathrm{Tr}\Big(Y^*Z^*ZY\Big) \\
&= \langle XZ, XZ\rangle + \langle ZY, ZY\rangle = \|XZ\|_2^2 + \|ZY\|_2^2.
\end{aligned}$$

Therefore if in Theorem 2.4, X and Y are assumed to be normal operators, then we retain the Fuglede-Putnam theorem modulo the Hilbert-Schmidt class.

We now state a interesting inequality for Hilbert–Schmidt operators.

Theorem 2.6. *If X and Y are Hilbert–Schmidt operators on $\mathcal{H}$, then*

$$|\langle X, Y\rangle|^2 \leq \langle |X^*|, |Y^*|\rangle\langle |X|, |Y|\rangle. \tag{6}$$

Moreover, the inequality in (6) becomes an equality if and only if ζY^*X is positive for some scalar ζ.

Proof. Let $X = U|X|$ and $Y = V|Y|$ be the polar decompositions of X and Y, respectively. By (1), (2) and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
|\langle X, Y\rangle|^2 &= \Big|\mathrm{Tr}\Big(|Y|V^*U|X|\Big)\Big|^2 \\
&= \Big|\mathrm{Tr}\Big(|X|^{1/2}|Y|V^*U|X|^{1/2}\Big)\Big|^2 \\
&= \Big|\Big\langle |Y|^{1/2}V^*U|X|^{1/2}, |Y|^{1/2}|X|^{1/2}\Big\rangle\Big|^2 \\
&\leq \Big\||Y|^{1/2}V^*U|X|^{1/2}\Big\|_2^2\Big\||Y|^{1/2}|X|^{1/2}\Big\|_2^2 \\
&= \mathrm{Tr}\Big(|X|^{1/2}U^*V|Y|V^*U|X|^{1/2}\Big)\mathrm{Tr}\Big(|X|^{1/2}|Y||X|^{1/2}\Big) \\
&= \mathrm{Tr}\Big(V|Y|V^*U|X|U^*\Big)\mathrm{Tr}\Big(|Y||X|\Big) \\
&= \mathrm{Tr}\Big(|Y^*||X^*|\Big)\mathrm{Tr}\Big(|Y||X|\Big) \\
&= \mathrm{Tr}\Big(|Y^*|^*|X^*|\Big)\mathrm{Tr}\Big(|Y|^*|X|\Big) = \langle |X^*|, |Y^*|\rangle\langle |X|, |Y|\rangle.
\end{aligned}$$

The inequality in (6) becomes an equality if and only if

$$|Y|^{1/2}|X|^{1/2} = \zeta|Y|^{1/2}V^*U|X|^{1/2}$$

for some $\zeta \in \mathbb{C}$, and hence $|Y|\,|X| = \zeta|Y|V^*U|X|$. So, by (1), we obtain $|Y|\,|X| = \zeta Y^*X$. Therefore, the inequality in (6) becomes an equality if and only if ζY^*X is positive for some $\zeta \in \mathbb{C}$. □

As a consequence of Theorem 2.6, we have the following result for trace-class operators.

Corollary 2.7. *If X is a trace-class operator on $\mathcal{H}$, then*

$$|\mathrm{Tr}X| \leq \mathrm{Tr}|X|. \tag{7}$$

Moreover, the inequality in (7) becomes an equality if and only if ζX is positive for some scalar $|\zeta| = 1$.

Proof. Let $X = U|X|$ be the polar decomposition of X. Since $|X|^{1/2}U$ and $|X|^{1/2}$ are Hilbert–Schmidt operators on $\mathcal{H}$, by Theorem 2.6 and (1), the result follows. □

Remark 2.8. *Let X and Y be Hilbert–Schmidt operators on $\mathcal{H}$ and let $\lambda \in \mathbb{R}^{\geq 0}$. By Theorem 2.6, we have*

$$\begin{aligned}
|\langle X, Y\rangle|^2 &\leq \sqrt{\langle |X^*|, |Y^*|\rangle}\sqrt{\langle |X|, |Y|\rangle}\,|\langle X, Y\rangle| \\
&\leq \sqrt{\langle |X^*|, |Y^*|\rangle}\sqrt{\langle |X|, |Y|\rangle}\,|\langle X, Y\rangle| + \lambda\Big(\langle |X^*|, |Y^*|\rangle\langle |X|, |Y|\rangle - |\langle X, Y\rangle|^2\Big).
\end{aligned}$$

This implies

$$|\langle X, Y\rangle|^2 \leq \frac{1}{1+\lambda}\sqrt{\langle |X^*|, |Y^*|\rangle}\sqrt{\langle |X|, |Y|\rangle}\,|\langle X, Y\rangle| + \frac{\lambda}{1+\lambda}\langle |X^*|, |Y^*|\rangle\langle |X|, |Y|\rangle. \tag{8}$$

Note that by Theorem 2.6 we have

$$\begin{aligned}
&\frac{1}{1+\lambda}\sqrt{\langle |X^*|, |Y^*|\rangle}\sqrt{\langle |X|, |Y|\rangle}\,|\langle X, Y\rangle| + \frac{\lambda}{1+\lambda}\langle |X^*|, |Y^*|\rangle\langle |X|, |Y|\rangle \\
&\quad \le \frac{1}{1+\lambda}\langle |X^*|, |Y^*|\rangle\langle |X|, |Y|\rangle + \frac{\lambda}{1+\lambda}\langle |X^*|, |Y^*|\rangle\langle |X|, |Y|\rangle \\
&\quad = \langle |X^*|, |Y^*|\rangle\langle |X|, |Y|\rangle,
\end{aligned}$$

and hence the inequality (8) *refines the inequality* (6).

Here we present one of the main results of this paper. In fact, the following theorem enables us to provide alternative proofs of some well-known inequalities for the Hilbert–Schmidt norm.

Theorem 2.9. *For non-zero Hilbert–Schmidt operators X and Y on $\mathcal{H}$ the following properties hold.*

(i) $\cos^2\Theta_{X,Y} \le \cos\Theta_{|X^*|,|Y^*|}\cos\Theta_{|X|,|Y|}$.

(ii) $\left|\cos\Theta_{X,Y}\right| \le \min\left\{\sqrt{\cos\Theta_{|X^*|,|Y^*|}},\ \sqrt{\cos\Theta_{|X|,|Y|}}\right\}$.

(iii) $\sin^2\Theta_{|X^*|,|Y^*|} + \sin^2\Theta_{|X|,|Y|} \le 2\sin^2\Theta_{X,Y}$.

Proof. (i) By (3), (4), Definition 2.1 and Theorem 2.6 we have

$$\begin{aligned}
\cos^2\Theta_{X,Y} &= \left(\frac{\mathrm{Re}\langle X, Y\rangle}{\|X\|_2\|Y\|_2}\right)^2 \\
&\le \frac{|\langle X, Y\rangle|^2}{\|X\|_2^2\|Y\|_2^2} \\
&\le \frac{\langle |X^*|, |Y^*|\rangle}{\|X\|_2\|Y\|_2}\,\frac{\langle |X|, |Y|\rangle}{\|X\|_2\|Y\|_2} \\
&= \frac{\mathrm{Re}\langle |X^*|, |Y^*|\rangle}{\big\||X^*|\big\|_2\big\||Y^*|\big\|_2}\,\frac{\mathrm{Re}\langle |X|, |Y|\rangle}{\big\||X|\big\|_2\big\||Y|\big\|_2} \\
&= \cos\Theta_{|X^*|,|Y^*|}\cos\Theta_{|X|,|Y|}.
\end{aligned} \tag{9}$$

(ii) The proof follows immediately from (i).

(iii) By the arithmetic-geometric mean inequality and (i) we have

$$\begin{aligned}
\sin^2\Theta_{|X^*|,|Y^*|} + \sin^2\Theta_{|X|,|Y|} &= 2 - \left(\cos^2\Theta_{|X^*|,|Y^*|} + \cos^2\Theta_{|X|,|Y|}\right) \\
&\le 2 - 2\cos\Theta_{|X^*|,|Y^*|}\cos\Theta_{|X|,|Y|} \\
&\le 2 - 2\cos^2\Theta_{X,Y} = 2\sin^2\Theta_{X,Y}.
\end{aligned}$$

□

Remark 2.10. *Let X and Y be non-zero Hilbert–Schmidt operators on $\mathcal{H}$. For any $\lambda \ge 0$, by* (3) *and* (8), *we have*

$$\begin{aligned}
\left(\frac{\mathrm{Re}\langle X, Y\rangle}{\|X\|_2\|Y\|_2}\right)^2 &\le \frac{|\langle X, Y\rangle|^2}{\|X\|_2^2\|Y\|_2^2} \\
&\le \frac{\frac{1}{1+\lambda}\sqrt{\langle |X^*|, |Y^*|\rangle}\sqrt{\langle |X|, |Y|\rangle}\,|\langle X, Y\rangle| + \frac{\lambda}{1+\lambda}\langle |X^*|, |Y^*|\rangle\langle |X|, |Y|\rangle}{\|X\|_2^2\|Y\|_2^2} \\
&= \frac{1}{1+\lambda}\sqrt{\frac{\mathrm{Re}\langle |X^*|, |Y^*|\rangle}{\big\||X^*|\big\|_2\big\||Y^*|\big\|_2}}\sqrt{\frac{\mathrm{Re}\langle |X|, |Y|\rangle}{\big\||X|\big\|_2\big\||Y|\big\|_2}}\,\frac{|\langle X, Y\rangle|}{\|X\|_2\|Y\|_2} \\
&\qquad + \frac{\lambda}{1+\lambda}\,\frac{\mathrm{Re}\langle |X^*|, |Y^*|\rangle}{\big\||X^*|\big\|_2\big\||Y^*|\big\|_2}\,\frac{\mathrm{Re}\langle |X|, |Y|\rangle}{\big\||X|\big\|_2\big\||Y|\big\|_2}.
\end{aligned}$$

So, by Definition 2.1 it follows that

$$\cos^2\Theta_{X,Y} \leq \frac{1}{1+\lambda}\sqrt{\cos\Theta_{|X^*|,|Y^*|}}\sqrt{\cos\Theta_{|X|,|Y|}}\frac{|\langle X,Y\rangle|}{\|X\|_2\|Y\|_2} + \frac{\lambda}{1+\lambda}\cos\Theta_{|X^*|,|Y^*|}\cos\Theta_{|X|,|Y|}. \tag{10}$$

Note that by (9) *we have*

$$\begin{aligned}
&\frac{1}{1+\lambda}\sqrt{\cos\Theta_{|X^*|,|Y^*|}}\sqrt{\cos\Theta_{|X|,|Y|}}\frac{|\langle X,Y\rangle|}{\|X\|_2\|Y\|_2} + \frac{\lambda}{1+\lambda}\cos\Theta_{|X^*|,|Y^*|}\cos\Theta_{|X|,|Y|}\\
&\quad\leq \frac{1}{1+\lambda}\cos\Theta_{|X^*|,|Y^*|}\cos\Theta_{|X|,|Y|} + \frac{\lambda}{1+\lambda}\cos\Theta_{|X^*|,|Y^*|}\cos\Theta_{|X|,|Y|}\\
&\quad= \cos\Theta_{|X^*|,|Y^*|}\cos\Theta_{|X|,|Y|}.
\end{aligned}$$

Therefore, the inequality (10) *is a refinement of the inequality in Theorem 2.9(i).*

3. Inequalities for the Hilbert–Schmidt norm

In this section, by using Theorem 2.9, we provide alternative proof of some well-known inequalities for the Hilbert–Schmidt norm. First we present a considerably briefer proof of an extension of the Araki–Yamagami inequality [1] obtained by Kittaneh [3, Theorem 2].

Theorem 3.1. *If X and Y are Hilbert–Schmidt operators on $\mathcal{H}$, then*

$$\big\|\,|X^*|-|Y^*|\,\big\|_2^2 + \big\|\,|X|-|Y|\,\big\|_2^2 \leq 2\|X-Y\|_2^2.$$

Proof. Since the desired inequality trivially holds when $X=0$ or $Y=0$, we may assume $X,Y\neq 0$. By (3), Remark 2.3(iii), Theorem 2.9(i) and the arithmetic-geometric mean inequality we have

$$\begin{aligned}
\big\|\,|X^*|-|Y^*|\,\big\|_2^2 + \big\|\,|X|-|Y|\,\big\|_2^2 &= \big\|\,|X^*|\,\big\|_2^2 + \big\|\,|Y^*|\,\big\|_2^2 - 2\big\|\,|X^*|\,\big\|_2\big\|\,|Y^*|\,\big\|_2\cos\Theta_{|X^*|,|Y^*|}\\
&\quad + \big\|\,|X|\,\big\|_2^2 + \big\|\,|Y|\,\big\|_2^2 - 2\big\|\,|X|\,\big\|_2\big\|\,|Y|\,\big\|_2\cos\Theta_{|X|,|Y|}\\
&= 2\|X\|_2^2 + 2\|Y\|_2^2 - 2\|X\|_2\|Y\|_2\Big(\cos\Theta_{|X^*|,|Y^*|} + \cos\Theta_{|X|,|Y|}\Big)\\
&\leq 2\|X\|_2^2 + 2\|Y\|_2^2 - 4\|X\|_2\|Y\|_2\sqrt{\cos\Theta_{|X^*|,|Y^*|}\cos\Theta_{|X|,|Y|}}\\
&\leq 2\|X\|_2^2 + 2\|Y\|_2^2 - 4\|X\|_2\|Y\|_2\big|\cos\Theta_{X,Y}\big|\\
&\leq 2\|X\|_2^2 + 2\|Y\|_2^2 - 4\|X\|_2\|Y\|_2\cos\Theta_{X,Y} = 2\|X-Y\|_2^2.
\end{aligned}$$

□

As an immediate consequence of Theorem 3.1, we get the Araki–Yamagami inequality [1, Theorem 1].

Corollary 3.2. *If X and Y are Hilbert–Schmidt operators on $\mathcal{H}$, then*

$$\big\|\,|X|-|Y|\,\big\|_2 \leq \sqrt{2}\|X-Y\|_2.$$

Remark 3.3. *In [1], Araki and Yamagami remarked that $\sqrt{2}$ is the best possible coefficient for a general X and Y. Now let X and Y be normal operators. Since X and Y are normal operators, the spectral theorem (see [8]) implies that $|X^*|=|X|$, $|Y^*|=|Y|$ and hence, by Theorem 3.1 we obtain*

$$\|\,|X|-|Y|\,\|_2 \leq \|X-Y\|_2.$$

Therefore, if X and Y are restricted to be normal, then the best coefficient in the Araki–Yamagami inequality is 1 *instead of $\sqrt{2}$.*

The following result is a special case of [6, Theorem 2.1].

Theorem 3.4. *If X and Y are Hilbert–Schmidt operators on $\mathcal{H}$, then*

$$\|X+Y\|_2^2 \le \big\| |X^*|+|Y^*| \big\|_2 \big\| |X|+|Y| \big\|_2.$$

Proof. We may assume that $X, Y \neq 0$ otherwise the desired inequality trivially holds. By (3), Remark 2.3(iii), Theorem 2.9(i) and the arithmetic-geometric mean inequality we have

$$\begin{aligned}
\big\| |X^*|+|Y^*| \big\|_2^2 \big\| |X|+|Y| \big\|_2^2 &= \Big(\big\| |X^*| \big\|_2^2 + \big\| |Y^*| \big\|_2^2 + 2\big\| |X^*| \big\|_2 \big\| |Y^*| \big\|_2 \cos\Theta_{|X^*|,|Y^*|}\Big)\\
&\quad\times\Big(\big\| |X| \big\|_2^2 + \big\| |Y| \big\|_2^2 + 2\big\| |X| \big\|_2 \big\| |Y| \big\|_2 \cos\Theta_{|X|,|Y|}\Big)\\
&= \Big(\|X\|_2^2 + \|Y\|_2^2 + 2\|X\|_2\|X\|_2 \cos\Theta_{|X^*|,|Y^*|}\Big)\\
&\quad\times\Big(\|X\|_2^2 + \|Y\|_2^2 + 2\|X\|_2\|Y\|_2 \cos\Theta_{|X|,|Y|}\Big)\\
&= \Big(\|X\|_2^2 + \|Y\|_2^2\Big)^2 + 4\|X\|_2^2\|Y\|_2^2 \cos\Theta_{|X^*|,|Y^*|}\cos\Theta_{|X|,|Y|}\\
&\quad + 2\|X\|_2\|Y\|_2\Big(\|X\|_2^2 + \|Y\|_2^2\Big)\Big(\cos\Theta_{|X^*|,|Y^*|} + \cos\Theta_{|X|,|Y|}\Big)\\
&\ge \Big(\|X\|_2^2 + \|Y\|_2^2\Big)^2 + 4\|X\|_2^2\|Y\|_2^2 \cos\Theta_{|X^*|,|Y^*|}\cos\Theta_{|X|,|Y|}\\
&\quad + 4\|X\|_2\|Y\|_2\Big(\|X\|_2^2 + \|Y\|_2^2\Big)\sqrt{\cos\Theta_{|X^*|,|Y^*|}\cos\Theta_{|X|,|Y|}}\\
&\ge \Big(\|X\|_2^2 + \|Y\|_2^2\Big)^2 + 4\|X\|_2^2\|Y\|_2^2 \cos^2\Theta_{X,Y}\\
&\quad + 4\|X\|_2\|Y\|_2\Big(\|X\|_2^2 + \|Y\|_2^2\Big)\Big|\cos\Theta_{X,Y}\Big|\\
&\ge \Big(\|X\|_2^2 + \|Y\|_2^2\Big)^2 + 4\|X\|_2^2\|Y\|_2^2 \cos^2\Theta_{X,Y}\\
&\quad + 4\|X\|_2\|Y\|_2\Big(\|X\|_2^2 + \|Y\|_2^2\Big)\cos\Theta_{X,Y}\\
&= \Big(\|X\|_2^2 + \|Y\|_2^2 + 2\|X\|_2\|Y\|_2\cos\Theta_{X,Y}\Big)^2 = \|X+Y\|_2^4.
\end{aligned}$$

□

Next, we provide alternative proof of an inequality for the Hilbert–Schmidt norm due to Kittaneh [5, Theorem 2.1].

Theorem 3.5. *If X and Y are Hilbert–Schmidt operators on $\mathcal{H}$, then*

$$\big\| |X|-|Y| \big\|_2^2 \le \|X+Y\|_2\|X-Y\|_2.$$

Proof. Since the desired inequality trivially holds when $X = 0$ or $Y = 0$, we may assume $X, Y \neq 0$. By the arithmetic-geometric mean inequality we have

$$\|X\|_2\|Y\|_2\Big(1+\cos\Theta_{|X|,|Y|}\Big) \le 2\|X\|_2\|Y\|_2 \le \|X\|_2^2 + \|Y\|_2^2.$$

Then, since $\cos\Theta_{|X|,|Y|} \ge 0$, we obtain

$$\|X\|_2\|Y\|_2\Big(\cos\Theta_{|X|,|Y|} + \cos^2\Theta_{|X|,|Y|}\Big) \le \cos\Theta_{|X|,|Y|}\Big(\|X\|_2^2 + \|Y\|_2^2\Big),$$

and so

$$\|X\|_2\|Y\|_2\cos\Theta_{|X|,|Y|} \le \cos\Theta_{|X|,|Y|}\Big(\|X\|_2^2 + \|Y\|_2^2\Big) - \|X\|_2\|Y\|_2\cos^2\Theta_{|X|,|Y|}. \tag{11}$$

Therefore by (3), Remark 2.3(iii), Theorem 2.9(ii) and (11) we have

$$\begin{aligned}\|X+Y\|_2^2\|X-Y\|_2^2 &= \left(\|X\|_2^2+\|Y\|_2^2\right)^2 - 4\|X\|_2^2\|Y\|_2^2\cos^2\Theta_{X,Y}\\ &\geq \left(\|X\|_2^2+\|Y\|_2^2\right)^2 - 4\|X\|_2^2\|Y\|_2^2\cos\Theta_{|X|,|Y|}\\ &\geq \left(\|X\|_2^2+\|Y\|_2^2\right)^2 - 4\|X\|_2\|Y\|_2\left(\|X\|_2^2+\|Y\|_2^2\right)\cos\Theta_{|X|,|Y|} + 4\|X\|_2^2\|Y\|_2^2\cos^2\Theta_{|X|,|Y|}\\ &= \left(\|X\|_2^2+\|Y\|_2^2 - 2\|X\|_2\|Y\|_2\cos\Theta_{|X|,|Y|}\right)^2\\ &= \left(\big\||X|\big\|_2^2+\big\||Y|\big\|_2^2 - 2\big\||X|\big\|_2\big\||Y|\big\|_2\cos\Theta_{|X|,|Y|}\right)^2 = \big\||X|-|Y|\big\|_2^4.\end{aligned}$$

□

The following result may be stated as well.

Theorem 3.6. *If X and Y are Hilbert–Schmidt operators on $\mathcal{H}$, then*

$$\big\||X^*|+|Y^*|\big\|_2 \leq \sqrt{2}\big\||X|+|Y|\big\|_2.$$

Proof. We may assume that $X, Y \neq 0$ otherwise the desired inequality trivially holds. Since $-2 \leq \cos\Theta_{|X^*|,|Y^*|} - 2\cos\Theta_{|X|,|Y|} \leq 1$, by the arithmetic-geometric mean inequality we get

$$2\|X\|_2\|Y\|_2\left(\cos\Theta_{|X^*|,|Y^*|} - 2\cos\Theta_{|X|,|Y|}\right) \leq 2\|X\|_2\|Y\|_2 \leq \|X\|_2^2+\|Y\|_2^2.$$

Hence

$$2\|X\|_2\|Y\|_2\cos\Theta_{|X^*|,|Y^*|} \leq \|X\|_2^2+\|Y\|_2^2+4\|X\|_2\|Y\|_2\cos\Theta_{|X|,|Y|}. \tag{12}$$

So, by (3), Remark 2.3(iii) and (12) we have

$$\begin{aligned}\big\||X^*|+|Y^*|\big\|_2^2 &= \big\||X^*|\big\|_2^2+\big\||Y^*|\big\|_2^2+2\big\||X^*|\big\|_2\big\||Y^*|\big\|_2\cos\Theta_{|X^*|,|Y^*|}\\ &= \|X\|_2^2+\|Y\|_2^2+2\|X\|_2\|Y\|_2\cos\Theta_{|X^*|,|Y^*|}\\ &\leq 2\|X\|_2^2+2\|Y\|_2^2+4\|X\|_2\|Y\|_2\cos\Theta_{|X|,|Y|}\\ &= 2\left(\big\||X|\big\|_2^2+\big\||Y|\big\|_2^2+2\big\||X|\big\|_2\big\||Y|\big\|_2\cos\Theta_{|X|,|Y|}\right) = 2\big\||X|+|Y|\big\|_2^2.\end{aligned}$$

□

In [6, p. 584], Lee conjectured that for arbitrary n-by-n matrices A and B, the inequality

$$\big\|A+B\big\|_2 \leq \sqrt{\frac{\sqrt{2}+1}{2}}\big\||A|+|B|\big\|_2$$

holds. Very recently, a proof of Lee's conjecture has been presented in [7] as our work was in progress. We end this section by a proof of Lee's conjecture for operators.

Theorem 3.7. *If X and Y are Hilbert–Schmidt operators on $\mathcal{H}$, then*

$$\big\|X+Y\big\|_2 \leq \sqrt{\frac{\sqrt{2}+1}{2}}\big\||X|+|Y|\big\|_2.$$

Proof. Since the desired inequality trivially holds when $X = 0$ or $Y = 0$, we may assume $X, Y \neq 0$. By (3), Remark 2.3(iii), Theorem 2.9(ii) we have

$$\begin{aligned}
\frac{\sqrt{2}+1}{2}\big\||X|+|Y|\big\|_2^2 &= \frac{\sqrt{2}+1}{2}\Big(\big\||X|\big\|_2^2+\big\||Y|\big\|_2^2+2\big\||X|\big\|_2\big\||Y|\big\|_2\cos\Theta_{|X|,|Y|}\Big)\\
&= \frac{\sqrt{2}+1}{2}\Big(\|X\|_2^2+\|Y\|_2^2+2\|X\|_2\|Y\|_2\cos\Theta_{|X|,|Y|}\Big)\\
&\geq \frac{\sqrt{2}+1}{2}\Big(\|X\|_2^2+\|Y\|_2^2+2\|X\|_2\|Y\|_2\cos^2\Theta_{X,Y}\Big)\\
&= \|X\|_2^2+\|Y\|_2^2+2\|X\|_2\|Y\|_2\cos\Theta_{X,Y}+\frac{\sqrt{2}-1}{2}\Big(\|X\|_2^2+\|Y\|_2^2\Big)\\
&\qquad +\|X\|_2\|Y\|_2\Big((\sqrt{2}+1)\cos^2\Theta_{X,Y}-2\cos\Theta_{X,Y}\Big)\\
&= \big\|X+Y\big\|_2^2+\frac{\sqrt{2}-1}{2}\Big(\|X\|_2-\|Y\|_2\Big)^2\\
&\qquad +\|X\|_2\|Y\|_2\Big((\sqrt{2}-1)+(\sqrt{2}+1)\cos^2\Theta_{X,Y}-2\cos\Theta_{X,Y}\Big)\\
&= \big\|X+Y\big\|_2^2+\frac{\sqrt{2}-1}{2}\Big(\|X\|_2-\|Y\|_2\Big)^2\\
&\qquad +\|X\|_2\|Y\|_2\Big(\sqrt{\sqrt{2}-1}-\sqrt{\sqrt{2}+1}\cos\Theta_{X,Y}\Big)^2\\
&\geq \big\|X+Y\big\|_2^2.
\end{aligned}$$

□

Remark 3.8. *Suppose $\mathcal{M}_2(\mathbb{C})$ is the algebra of all complex 2×2 matrices. Let $\mathrm{Det}(A)$ denote the determinant of $A \in \mathcal{M}_2(\mathbb{C})$. Recall (see e.g. [2, p. 460]) that for $A \in \mathcal{M}_2(\mathbb{C})$ we have*

$$|A| = \frac{1}{\sqrt{\mathrm{Tr}(A^*A)+2\sqrt{\mathrm{Det}(A^*A)}}}\Big(\sqrt{\mathrm{Det}(A^*A)}\,I + A^*A\Big).$$

Now, let $B = \begin{bmatrix} 0 & 0\\ -1 & 0\end{bmatrix}$, $C = \begin{bmatrix} 0 & 0\\ 0 & 1\end{bmatrix}$ *and* $D = \begin{bmatrix} 0 & 0\\ 1-\sqrt{2} & \sqrt{\sqrt{8}-2}\end{bmatrix}$. *Then simple computations show that* $|B| = \begin{bmatrix} 1 & 0\\ 0 & 0\end{bmatrix}$, $|B^*| = \begin{bmatrix} 0 & 0\\ 0 & 1\end{bmatrix}$, $|C| = |C^*| = \begin{bmatrix} 0 & 0\\ 0 & 1\end{bmatrix}$ *and* $|D| = \begin{bmatrix} 3-\sqrt{8} & -\sqrt{\sqrt{200}-14}\\ -\sqrt{\sqrt{200}-14} & \sqrt{8}-2\end{bmatrix}$. *Therefore,*

$$\big\||B^*|+|C^*|\big\|_2 = \left\|\begin{bmatrix} 2 & 0\\ 0 & 0\end{bmatrix}\right\|_2 = \sqrt{\mathrm{Tr}\left(\begin{bmatrix} 4 & 0\\ 0 & 0\end{bmatrix}\right)} = 2$$

and

$$\sqrt{2}\big\||B|+|C|\big\|_2 = \sqrt{2}\left\|\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}\right\|_2 = \sqrt{2}\sqrt{\mathrm{Tr}\left(\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}\right)} = 2.$$

Hence the inequality in Theorem 3.6 is sharp. In addition, we have

$$\big\|B+D\big\|_2 = \left\|\begin{bmatrix} 0 & 0\\ -\sqrt{2} & \sqrt{\sqrt{8}-2}\end{bmatrix}\right\|_2 = \sqrt{\mathrm{Tr}\left(\begin{bmatrix} 2 & -\sqrt{\sqrt{32}-4}\\ -\sqrt{\sqrt{32}-4} & \sqrt{8}-2\end{bmatrix}\right)} = \sqrt[4]{8}$$

and

$$\sqrt{\frac{\sqrt{2}+1}{2}}\,\big\|\,|B|+|D|\,\big\|_2 = \sqrt{\frac{\sqrt{2}+1}{2}}\left\|\begin{bmatrix} 4-\sqrt{8} & -\sqrt{\sqrt{200}-14} \\ -\sqrt{\sqrt{200}-14} & \sqrt{8}-2 \end{bmatrix}\right\|_2$$

$$= \sqrt{\frac{\sqrt{2}+1}{2}}\sqrt{\mathrm{Tr}\left(\begin{bmatrix} 10-\sqrt{72} & -\sqrt{\sqrt{3200}-56} \\ -\sqrt{\sqrt{3200}-56} & \sqrt{8}-2 \end{bmatrix}\right)}$$

$$= \sqrt{\frac{\sqrt{2}+1}{2}}\sqrt{2\sqrt{8}-4} = \sqrt[4]{8}.$$

So, the inequality in Theorem 3.7 is also sharp.

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