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On the generalized *ABS* index of graphs

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Abstract. The atom-bond sum-connectivity (*ABS*) index is a recently introduced variant of three earlier much-studied graph-based molecular descriptors: Randić, atom-bond connectivity, and sum-connectivity indices. The general atom-bond sum-connectivity index is defined as $ABS_\alpha(G) = \sum_{uv\in E(G)} \left(\frac{d_u+d_v-2}{d_u+d_v}\right)^\alpha$, where α is a real number. In this paper, we present some upper and lower bounds on the general atom-bond sum-connectivity index in terms of graph parameters and other graph indices.

1. Introduction

Let $G = (V, E)$ be a simple connected graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$. Let $\deg(v) = d_v$ and $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n > 0$, be the degree sequence of G. A vertex v is called isolated if $d_v = 0$. We refer the readers to consult books [5, 6] for graph-theoretical notation and terminology which is used without being defined.

A number, representing a molecular structure in graph-theoretical terms via the molecular graph, is called a topological index. In other words, a topological index is a function that associates each molecular graph with a real value. Topological indices are mainly used to unveil and model the dependence of physicochemical properties on the molecular structure since many of them correlate well with some molecular properties.

The studies on degree-based graph invariants started in the early 1970s, when Gutman and Trinajstić introduced the first and second Zagreb indices in [9]. These invariants are entirely dependent on the vertex degree as follows.

$$M_1(G) = \sum_{uv\in E(G)} (d_u + d_v)$$

and

$$M_2(G) = \sum_{uv\in E(G)} (d_u d_v)\,.$$

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The Randić index of a graph G is defined in [14] as

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}}.$$

So far, many modifications of the Randić index have been proposed. Here, we mention two topological indices which have been introduced by taking into consideration the definition of the Randić index, namely the "sum-connectivity (χ) index" [16] and the "atom-bond connectivity (ABC) index" [7]. These indices have the following definitions for a graph G:

$$\chi(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u + d_v}}$$

and

$$ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}}.$$

By utilizing the definitions of the ABC and χ indices, a novel topological index the atom-bond sum-connectivity (ABS) index has recently been proposed in [2]. For a graph G, this index is defined as

$$ABS(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u + d_v}}.$$

Another remarkable topological descriptor is the harmonic index, defined in [8] as

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d_u + d_v}.$$

The general Platt index [1] and the general sum-connectivity index [17] for the graph G are defined by

$$Pl_\alpha(G) = \sum_{uv \in E(G)} (d_u + d_v - 2)^\alpha$$

and

$$\chi_\alpha(G) = \sum_{uv \in E(G)} (d_u + d_v)^\alpha.$$

Similarly, the general atom-bond sum-connectivity (ABS_α) is defined in [3] as

$$ABS_\alpha(G) = \sum_{uv \in E(G)} \left(\frac{d_u + d_v - 2}{d_u + d_v}\right)^\alpha.$$

In paper [17] the authors have obtained some basic properties for the general sum-connectivity index, while in [2] graphs having extreme values of the $ABS_{\frac{1}{2}}$ index among (molecular) trees and general graphs with a fixed order have been characterized. Alraqad et al. [4] characterized the graphs having the maximal ABS_α value for $\alpha = \frac{1}{2}$ among trees with a fixed order and/or the number of pendent vertices. Recently, in [12], the minimum $ABS_{\frac{1}{2}}$ index of trees with a given number of pendent vertices has been presented. In this paper, we obtain new inequalities for ABS_α.

2. Auxiliary results

In this section, we recall some known results that will be used in the sequel.

Lemma 2.1. *[13] For two positive real number sequences $x_1, \ldots, x_{n-1}$ and $y_1, \ldots, y_{n-1}$ and $\alpha \in \mathbb{R} - (-1, 0)$, we have*

$$\sum_{i=1}^{n-1} \frac{x_i^{\alpha+1}}{y_i^{\alpha}} \geq \frac{\left(\sum_{i=1}^{n-1} x_i\right)^{\alpha+1}}{\left(\sum_{i=1}^{n-1} y_i\right)^{\alpha}}. \tag{1}$$

When $-1 \leq \alpha \leq 0$ the opposite inequality in (1) holds. Equality holds if and only if $\alpha = 1$, $\alpha = 0$, or $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \cdots = \frac{x_{n-1}}{y_{n-1}}$.

We will use the following particular case of Jensen's inequality.

Lemma 2.2. *If f is a convex function in an interval I and $a_1, a_2, \ldots, a_m \in I$, then*

$$f\left(\frac{a_1 + a_2 + \cdots + a_m}{m}\right) \leq \frac{1}{m}\left(f(a_1) + \cdots + f(a_m)\right).$$

The following result appears in [15].

Theorem 2.3. *If $\alpha \geq 1$ is an integer and $0 \leq x_1, x_2, \ldots, x_k \leq k - 1$, then*

$$(k-1)^{1-\alpha} \sum_{i=1}^{k} x_i^{\alpha} \leq \left(\sum_{i=1}^{k} x_i^{\frac{1}{\alpha}}\right)^{\alpha}.$$

In [11], the following lemma is proved.

Lemma 2.4. *Let $0 < a \leq x_i \leq A$ and $0 < b \leq y_i \leq B$ for $1 \leq i \leq n$, then*

$$\left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} y_i^2\right)^{\frac{1}{2}} \leq \frac{1}{2}\left(\sqrt{\frac{AB}{ab}} + \sqrt{\frac{ab}{AB}}\right)\left(\sum_{i=1}^{n} x_i y_i\right). \tag{2}$$

3. Main results

In this section, we obtain several new bounds for the general atom-bond sum-connectivity index and characterize graphs for which these bounds are best possible.

First, we present a relationship between the general atom-bond sum-connectivity index, the general sum-connectivity index, and the general Platt index of graphs.

Theorem 3.1. *Let G be a graph with t isolated edges and $0 < \alpha < 1$.*

$$ABS_{\alpha}(G) \leq Pl_{\alpha}(G)\left(\chi_{\frac{-\alpha}{1-\alpha}}(G) - t\right)^{1-\alpha}.$$

The equality in this bound is attained for the union of any regular or biregular graph and t isolated edges; if G is the union of a connected graph and t isolated edges, then the equality in this bound is attained if and only if G is the union of any regular or biregular connected graph and t isolated edges.

Proof. Since $ABS_\alpha(P_2) = 0$ and $\chi_\alpha(P_2) = 2^\alpha$, it suffices to prove the theorem for the case $t = 0$, i.e., when G is a graph without isolated edges. Hence, $\Delta \geq 2$.
Hölder's inequality gives:

$$\begin{aligned}
ABS_\alpha(G) &= \sum_{uv\in E(G)} \left(\frac{d_u + d_v - 2}{d_u + d_v}\right)^\alpha \\
&\leq \left(\sum_{uv\in E(G)} \left((d_u + d_v - 2)^\alpha\right)^{\frac{1}{\alpha}}\right)^\alpha \left(\sum_{uv\in E(G)} \left(\frac{1}{(d_u + d_v)^\alpha}\right)^{\frac{1}{1-\alpha}}\right)^{1-\alpha} \\
&= \left(\sum_{uv\in E(G)} d_u + d_v - 2\right)^\alpha \left(\sum_{uv\in E(G)} (d_u + d_v)^{\frac{-\alpha}{1-\alpha}}\right)^{1-\alpha} \\
&= Pl_\alpha(G)\chi_{\frac{-\alpha}{1-\alpha}}(G)^{1-\alpha}.
\end{aligned}$$

This implies the result stated in the theorem.
If G is a regular or biregular graph with m edges, then:

$$Pl_\alpha(G)\chi_{\frac{-\alpha}{1-\alpha}}(G)^{1-\alpha} = ((\Delta + \delta - 2)m)^\alpha \left((\Delta + \delta)^{\frac{-\alpha}{1-\alpha}} m\right)^{1-\alpha} = \frac{(\Delta + \delta - 2)^\alpha}{(\Delta + \delta)^\alpha} m = ABS_\alpha(G).$$

Assume that G is connected and that the equality in the first inequality is attained. $\square$

The following results provide inequalities relating to the general atom-bond sum-connectivity index and the general sum-connectivity index.

Theorem 3.2. *If G is a graph with m edges and t isolated edges and $\alpha \in \mathbb{R}$, then:*

$$ABS_\alpha(G) \leq (m - t - 1)^\alpha \left(\chi_{-\alpha}(G) - t\right), \ \text{if } \alpha > 0,$$
$$ABS_\alpha(G) \geq (m - 1)^\alpha \chi_{-\alpha}(G), \ \text{if } \alpha < 0 \text{ and } t = 0.$$

The equality in the first bound is attained if and only if G is the union of a star graph and t isolated edges. The equality in the second bound is attained if and only if G is a star graph.

Proof. Since $ABS_\alpha(P_2) = 0$ and $\chi_\alpha(P_2) = 2^\alpha$, it suffices to prove the theorem for the case $t = 0$, i.e., when G is a graph without isolated edges. Hence, $\Delta \geq 2$.
In any graph, the inequality $d_u + d_v \leq m + 1$ holds for every $uv \in E(G)$. If $\alpha > 0$, then:

$$\frac{\left(\frac{d_u+d_v-2}{d_u+d_v}\right)^\alpha}{\left(\frac{1}{d_u+d_v}\right)^\alpha} = (d_u + d_v - 2)^\alpha \leq (m - 1)^\alpha,$$

$$\left(\frac{d_u + d_v - 2}{d_u + d_v}\right)^\alpha \leq (m - 1)^\alpha \chi_{-\alpha}(G).$$

The last inequality leads to the desired bound.
If $\alpha < 0$, then we obtain the converse inequality.
If G is a star graph, then $d_u + d_v = m + 1$ for every $uv \in E(G)$, and the equality is attained for every α.
If equality is attained in some inequality, the previous argument gives that $d_u + d_v = m + 1$ for every $uv \in E(G)$. In particular, G is a connected graph. If $m = 2$, then $\{d_u, d_v\} = \{1, 2\}$ for every $uv \in E(G)$, and so, $G = P_3 = S_3$. Assume now $m \geq 3$. Assume that $\{d_u, d_v\} \neq \{1, m\}$ for some $uv \in E(G)$. Since $d_u + d_v = m + 1$, we have $2 \leq d_u, d_v \leq m - 1$, and so, there exist two different vertices $x, y \in V(G)/\{u, v\}$ with $ux, vy \in E(G)$. Since vy is not incident on u and x, we have $d_u + d_x < m + 1$, a contradiction. Hence, $\{d_u, d_v\} = \{1, m\}$ for every $uv \in E(G)$, and so, G is a star graph. $\square$

In the next result, we give a relation between the general atom-bond sum-connectivity index, the general sum-connectivity index, and the harmonic index of a graph.

Theorem 3.3. *Let G be a simple connected graph with n vertices and m edges. Then, for any $\alpha \geq 2$,*

$$(m + 4\chi_{-2}(G) - 2H(G))\left(1 - \frac{1}{\delta}\right)^{\alpha-2} \leq ABS_{\alpha}(G) \leq (m + 4\chi_{-2}(G) - 2H(G))\left(1 - \frac{1}{\Delta}\right)^{\alpha-2}.$$

If $\alpha \geq 1$, then

$$(m - H(G))\left(1 - \frac{1}{\delta}\right)^{\alpha-1} \leq ABS_{\alpha}(G) \leq (m - H(G))\left(1 - \frac{1}{\Delta}\right)^{\alpha-1}.$$

If $\alpha \geq 0$, then

$$m\left(1 - \frac{1}{\delta}\right)^{\alpha} \leq ABS_{\alpha}(G) \leq m\left(1 - \frac{1}{\Delta}\right)^{\alpha}.$$

Equalities in the above inequalities are attained, respectively, for $\alpha = 2, \alpha = 1, \alpha = 0$, or if G is regular. When $\alpha \leq 2, \alpha \leq 1$ and $\alpha \leq 0$, respectively, the opposite inequalities are valid.

Proof. Note that

$$ABS_2(G) = \sum_{uv \in E(G)} \left(1 - \frac{2}{d_u + d_v}\right)^2 = m + 4\chi_{-2}(G) - 2H(G).$$

Since

$$ABS_{\alpha}(G) = \sum_{uv \in E(G)} \left(1 - \frac{2}{d_u + d_v}\right)^{\alpha} = \sum_{uv \in E(G)} \left(1 - \frac{2}{d_u + d_v}\right)^{\alpha-2} \left(1 - \frac{2}{d_u + d_v}\right)^2.$$

For $\alpha \geq 2$ holds

$$(m + 4\chi_{-2}(G) - 2H(G))\left(1 - \frac{1}{\delta}\right)^{\alpha-2} \leq ABS_{\alpha}(G) \leq (m + 4\chi_{-2}(G) - 2H(G))\left(1 - \frac{1}{\Delta}\right)^{\alpha-2}.$$

By a similar procedure, the remaining inequalities can be proved. □

In the next theorem, we determine an upper bound on the general atom-bond sum-connectivity index or a tree in terms of a graph order.

Theorem 3.4. *Let G be a tree with $n \geq 3$ vertices. If $\alpha > 0$, then:*

$$ABS_{\alpha}(G) \leq (n-1)\left(1 - \frac{2}{n}\right)^{\alpha},$$

with equality if and only if $G \cong S_n$.

If $\alpha < 0$, then the above inequalities on $ABS_{\alpha}(G)$ is reversed.

Proof. Here, we only prove the case $\alpha > 0$. Let uv be any edge of G. Obviously, $d_u + d_v \leq n$. Thus:

$$ABS_{\alpha}(G) \leq \sum_{uv \in E(G)} \left(1 - \frac{2}{n}\right)^{\alpha} = (n-1)\left(1 - \frac{2}{n}\right)^{\alpha}$$

with equality if and only if $d_u + d_v = n$ for every edge uv of G if and only if G is a complete bipartite graph that is a tree, i.e., $G = S_n$.

□

Now, we present a connection between the ABS_{α} and $M_1(G)$.

Theorem 3.5. *For any connected graph G of order $n \geq 3$ and for any real $\alpha \in \mathbb{R} - (-1, 0)$, we have*

$$ABS_\alpha(G) \geq \frac{(M_1(G) - 2m)^{\alpha+1}}{(2\Delta - 2)\,(M_1(G))^\alpha}. \tag{3}$$

If $-1 \leq \alpha \leq 0$, the opposite inequality is valid. Equality holds if and only if either $\alpha = 0$, $\alpha = -1$, or G is a regular graph.

Proof. If we take $x_i = d_u + d_v - 2$, $y_i = d_u + d_v$, where summation goes over all adjacent vertices u and v of G, i.e., over all edges, the inequality (1) for $\alpha \in \mathbb{R} - (-1, 0)$ becomes

$$\sum_{uv\in E(G)} \frac{(d_u + d_v - 2)^{\alpha+1}}{(d_u + d_v)^\alpha} \geq \frac{\left(\sum\limits_{uv\in E(G)} (d_u + d_v - 2)\right)^{\alpha+1}}{\left(\sum\limits_{uv\in E(G)} (d_u + d_v)\right)^{\alpha}}. \tag{4}$$

Since, for any graph, we have $\delta \leq d_v \leq \Delta$, therefore,

$$(2\Delta - 2) \sum_{uv\in E(G)} \frac{(d_u + d_v - 2)^{\alpha}}{(d_u + d_v)^\alpha} \geq \sum_{uv\in E(G)} \frac{(d_u + d_v - 2)^{\alpha+1}}{(d_u + d_v)^\alpha}. \tag{5}$$

Note that

$$\sum_{uv\in E(G)} (d_u + d_v - 2) = M_1(G) - 2m. \tag{6}$$

Combining (5) and (6) with (4) leads to the desired inequality. Likewise, when $-1 \leq \alpha \leq 0$ the opposite inequality in (3) holds. □

The next theorem reveals a connection between the general atom-bond sum-connectivity index and harmonic index.

Theorem 3.6. *For any connected graph G of order $n \geq 2$ and for any real $\alpha \in \mathbb{R} - (-1, 0)$, we have*

$$ABS_{\alpha+1}(G) \geq \frac{(2m - H(G))^{\alpha+1}}{m^\alpha}. \tag{7}$$

If $-1 \leq \alpha \leq 0$, the opposite inequality is valid. Equality holds if and only if either $\alpha = 0$, $\alpha = -1$, or G is a regular graph.

Proof. For $x_i = \frac{d_u+d_v-2}{d_u+d_v}$, $y_i = 1$, where summation goes over all adjacent vertices u and v of G, i.e., over all edges, the inequality (1) for $\alpha \in \mathbb{R} - (-1, 0)$ becomes

$$\sum_{uv\in E(G)} \left(\frac{d_u + d_v - 2}{d_u + d_v}\right)^{\alpha+1} \geq \frac{\left(\sum\limits_{uv\in E(G)} \left(\frac{d_u + d_v - 2}{d_u + d_v}\right)\right)^{\alpha+1}}{m^\alpha}. \tag{8}$$

Note that

$$\sum_{uv\in E(G)} \left(\frac{d_u + d_v - 2}{d_u + d_v}\right) = \sum_{uv\in E(G)} \left(1 - \frac{2}{d_u + d_v}\right) = 2m - H(G). \tag{9}$$

Combining inequality (8) with (9) leads to the desired inequality. Likewise, when $-1 \leq \alpha \leq 0$ the opposite inequality in (7) holds. □

In the next theorem, we determine an upper bound and a lower bound on the general atom-bond sum-connectivity index of a graph in terms of $ABS_1(G)$ and $ABS_2(G)$.

Theorem 3.7. *Let G be a connected graph of order n. Then*

$$ABS_\alpha(G) \geq \frac{(ABS_1(G))^{2-\alpha}}{(ABS_2(G))^{1-\alpha}} \quad \text{if } \alpha < 0, 0 < \alpha < 1,$$
$$ABS_\alpha(G) \leq \frac{(ABS_1(G))^{2-\alpha}}{(ABS_2(G))^{1-\alpha}} \quad \text{if } \alpha > 2, 1 < \alpha < 2.$$

Proof. Let $x_1, x_2, \ldots, x_t$ be positive real numbers and let s be a real number with $s \neq 0, 1, \frac{1}{2}$. If $s < 0$ or $s > 1$, it is clear that $\frac{2s-1}{s} > 0$. By Hölder's inequality, we have

$$\sum_{i=1}^{t} x_i^s = \sum_{i=1}^{t} x_i^{\frac{s}{2s-1}} x_i^{\frac{2s^2-2s}{2s-1}} \leq \left(\sum_{i=1}^{t} \left(x_i^{\frac{s}{2s-1}}\right)^{\frac{2s-1}{s}}\right)^{\frac{s}{2s-1}} \left(\sum_{i=1}^{t} \left(x_i^{\frac{2s(s-1)}{2s-1}}\right)^{\frac{2s-1}{s-1}}\right)^{\frac{s-1}{2s-1}}$$
$$= \left(\sum_{i=1}^{t} x_i\right)^{\frac{s}{2s-1}} \left(\sum_{i=1}^{t} x_i^{2s}\right)^{\frac{s-1}{2s-1}}$$

that is

$$\sum_{i=1}^{t} x_i \geq \frac{\left(\sum_{i=1}^{t} x_i^s\right)^{\frac{2s-1}{s}}}{\left(\sum_{i=1}^{t} x_i^{2s}\right)^{\frac{s-1}{s}}}. \tag{10}$$

For $x_{uv} = \left(\frac{d_u+d_v-2}{d_u+d_v}\right)^{\alpha}$ and $s = \frac{1}{\alpha}$, where summation goes over all adjacent vertices of G, i.e., over all edges, the inequality (10) becomes

$$\sum_{uv\in E(G)} \left(\frac{d_u + d_v - 2}{d_u + d_v}\right)^{\alpha} \geq \frac{\left(\sum_{uv\in E(G)} \left(\frac{d_u + d_v - 2}{d_u + d_v}\right)\right)^{2-\alpha}}{\left(\sum_{uv\in E(G)} \left(\frac{d_u + d_v - 2}{d_u + d_v}\right)^{2}\right)^{1-\alpha}}$$

for all $\alpha < 0$ or $0 < \alpha < 1$. If $1 < \alpha < 2$ or $\alpha > 2$, then $\frac{1}{2} < s < 1$ or $0 < s < \frac{1}{2}$. Let $p = \frac{2s-1}{s}$ and $q = \frac{2s-1}{s-1}$. If $\frac{1}{2} < s < 1$, then $p > 0; q < 0$ and if $0 < s < \frac{1}{2}$, then $p < 0; q > 0$. Therefore, in each of these cases Hölder's inequality gets reversed and so the result follows. □

In the next result, we give a relation between the general atom-bond sum-connectivity index, the general sum-connectivity index, and the general Platt index of graphs.

Theorem 3.8. *Let G be a connected graph of order n and any real number α. Then*

$$ABS_\alpha(G) \leq \sqrt{Pl_{2\alpha}(G)\chi_{-2\alpha}(G)}.$$

Proof. For $1 \leq i \leq n$ let a_i and b_i be real numbers. In this proof, we use Cauchy-Schwarz inequality (see [10]):

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right). \tag{11}$$

For $a_i = (d_u + d_v - 2)^\alpha$ and $b_i = \frac{1}{(d_u+d_v)^\alpha}$, where summation goes over all adjacent vertices of G, i.e. over all edges, the inequality (11) becomes

$$\left(\sum_{uv\in E(G)}\left(\frac{d_u+d_v-2}{d_u+d_v}\right)^\alpha\right)^2 \leq \left(\sum_{uv\in E(G)}(d_u+d_v-2)^{2\alpha}\right)\left(\sum_{uv\in E(G)}\left(\frac{1}{(d_u+d_v)^{2\alpha}}\right)\right),$$

the above inequality leads to the desired bound. □

We now present a relation between the general atom-bond sum-connectivity index and the general Platt index of graphs.

Theorem 3.9. *Let G be a nontrivial connected graph with maximum degree Δ and minimum degree δ, and $\alpha \in \mathbb{R}$. Then*

$$Pl_\alpha\left(\frac{1}{2\delta}\right)^\alpha \leq ABS_\alpha(G) \leq Pl_\alpha\left(\frac{1}{2\Delta}\right)^\alpha \quad \text{if } \alpha < 1$$
$$Pl_\alpha\left(\frac{1}{2\Delta}\right)^\alpha \leq ABS_\alpha(G) \leq Pl_\alpha\left(\frac{1}{2\delta}\right)^\alpha \quad \text{if } \alpha \geq 1$$

and the equality holds in each inequality if and only if G is regular.

Proof. If $\alpha \geq 1$, then

$$ABS_\alpha(G) = \sum_{uv\in E(G)}(d_u+d_v-2)^\alpha\left(\frac{1}{d_u+d_v}\right)^\alpha \leq Pl_\alpha\left(\frac{1}{2\delta}\right)^\alpha$$

and

$$ABS_\alpha(G) = \sum_{uv\in E(G)}(d_u+d_v-2)^\alpha\left(\frac{1}{d_u+d_v}\right)^\alpha \geq Pl_\alpha\left(\frac{1}{2\Delta}\right)^\alpha.$$

If $\alpha < 1$, then the same argument gives

$$Pl_\alpha\left(\frac{1}{2\delta}\right)^\alpha \leq ABS_\alpha(G) \leq Pl_\alpha\left(\frac{1}{2\Delta}\right)^\alpha.$$

If the graph is regular, then the lower and upper bounds are the same, and they are equal to $ABS_\alpha(G)$. □

Theorem 3.10. *Let G be a nontrivial connected graph with m edges, $\alpha \in \mathbb{R}$ and $\beta > 0$. Then*

$$ABS_\alpha(G) \geq m^{1+\frac{1}{\beta}}\left(ABS_{-\alpha\beta}(G)\right)^{\frac{-1}{\beta}}.$$

Proof. Since $f(x) = x^{-\beta}$ is a convex function in $\mathbb{R}_+$ for each $\beta > 0$, Lemma 2.2 gives

$$\left(\frac{m}{\sum_{uv\in E(G)}\left(\frac{d_u+d_v-2}{d_u+d_v}\right)^\alpha}\right)^\beta \leq \frac{1}{m}\sum_{uv\in E(G)}\left(\frac{d_u+d_v-2}{d_u+d_v}\right)^{-\alpha\beta}$$
$$\frac{m}{ABS_\alpha(G)} \leq \sqrt[\beta]{\frac{ABS_{-\alpha\beta}(G)}{m}}.$$

This completes the proof.

□

Theorem 3.11. *Let G be a nontrivial connected graph with m edges and integer number $\alpha \geq 1$. Then*

$$ABS_\alpha(G) \leq (m-1)^{\alpha-1}\left(ABS_{\frac{1}{\alpha}}(G)\right)^\alpha.$$

Proof. Since, we have $\frac{d_u+d_v-2}{d_u+d_v} = 1 - \frac{2}{d_u+d_v} \leq 1 - \frac{1}{\Delta} \leq m-1$. Hence, Theorem 2.3 gives for any $uv \in E(G)$

$$(m-1)^{1-\alpha}\sum_{uv\in E(G)}\left(\frac{d_u+d_v-2}{d_u+d_v}\right)^\alpha \leq \left(\sum_{uv\in E(G)}\left(\frac{d_u+d_v-2}{d_u+d_v}\right)^{\frac{1}{\alpha}}\right)^\alpha.$$

The above inequality leads to the desired bound. □

In the next theorem, we determine a lower bound on the general atom-bond sum-connectivity index, the general sum-connectivity index, and the general Platt index of graphs in terms of minimum and maximum degree.

Theorem 3.12. *Let G be a nontrivial connected graph with m edges, maximum degree Δ and minimum degree δ, and $\alpha \geq 0$. Then*

$$ABS_\alpha(G) \geq \frac{2\sqrt{Pl_{2\alpha}G)\chi_{-2\alpha}(G)}}{\left(\sqrt{\frac{(2\Delta-2)^\alpha(2\delta)^{-\alpha}}{(2\delta-2)^\alpha(2\Delta)^{-\alpha}}}+\sqrt{\frac{(2\delta-2)^\alpha(2\Delta)^{-\alpha}}{(2\Delta-2)^\alpha(2\delta)^{-\alpha}}}\right)}.$$

Equality holds if and only if G is a regular graph.

Proof. We have

$$(2\delta-2)^\alpha \leq (d_u+d_v-2)^\alpha \leq (2\Delta-2)^\alpha, \qquad (2\Delta)^{-\alpha} \leq (d_u+d_v)^{-\alpha} \leq (2\delta)^{-\alpha} \quad \text{if } \alpha \geq 0.$$

For $x_i = (d_u+d_v-2)^\alpha$, $a = (2\delta-2)^\alpha$, $A = (2\Delta-2)^\alpha$ $y_i = \frac{1}{(d_u+d_v)^\alpha}$, $b = \frac{1}{(2\Delta)^\alpha}$, and $B = \frac{1}{(2\delta)^\alpha}$ for $\alpha \geq 0$, where summation goes over all adjacent vertices of G, i.e., over all edges, the inequality (2) becomes

$$\left(\sum_{uv\in E(G)}\frac{(d_u+d_v-2)^\alpha}{(d_u+d_v)^\alpha}\right) \geq \frac{\left(\sum_{uv\in E(G)}(d_u+d_v-2)^{2\alpha}\right)^{\frac{1}{2}}\left(\sum_{uv\in E(G)}(d_u+d_v)^{-2\alpha}\right)^{\frac{1}{2}}}{2\left(\sqrt{\frac{(2\Delta-2)^\alpha(2\delta)^{-\alpha}}{(2\delta-2)^\alpha(2\Delta)^{-\alpha}}}+\sqrt{\frac{(2\delta-2)^\alpha(2\Delta)^{-\alpha}}{(2\Delta-2)^\alpha(2\delta)^{-\alpha}}}\right)}.$$

This completes the proof. □

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