Filomat 37:30 (2023), 10181–10197
https://doi.org/10.2298/FIL2330181K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On some topological aspects in neutrosophic-2- normed spaces

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Abstract. In present paper, we introduce the concepts of neutrosophic boundedness, neutrosophic compactness in neutrosophic 2-normed spaces and analyze some of their topological properties. We show that if the space is finite dimensional then any two neutrosophic 2-norms are equivalent. Finally, we defined neutrosophic boundedness and neutrosophic continuity of linear operators and study some of their properties.

1. Introduction

Fuzzy set theory was introduced by Lotfi Zadeh in 1965 [27] as a mathematical framework to reduce the uncertainty and vagueness which the crisp sets could not addressed. These sets have meaningful real life applications: in population dynamics [25] to model the uncertain behaviour of populations, in nonlinear dynamical systems [26] to study the behaviour of complex systems with imprecise or uncertain inputs; in chaos control [4] to stabilize chaotic system and prevent them from becoming unstable; in computer programming [43]; in decision making [51]; in physics ([30]-[36]) to study $(e)^{\infty}$-theory which has wide application in quantum physics and many others. During these applications there were a need of certain mathematical concepts via fuzzy logic. Consequently, in last decades a new branch of mathematics came into existence which we called today as "Fuzzy Mathematics". Under this branch, fuzzy analogue of many mathematical ideas have been developed. One among these is the study of fuzzy topological spaces which have wide applications in the study of quantum physics, specifically in connections with both string and $(e)^{\infty}$-theory (see [29], [35]). An important class of fuzzy topological spaces is the study of fuzzy metric and fuzzy normed spaces. The motivation behind this was that we can not predict several measurement or the distance exactly because of huge uncertainty. Therefore, the idea of fuzzy metric space and fuzzy norms seems appropriate to treat such situations. In view of this, Kramosil and Michalek [40] defined fuzzy and statistical metric spaces. George and Veeramani [3] introduced a Hausdorff topology on a fuzzy metric space introduced by Kramosil and Michalek, and proved some known results of metric spaces. Felbin [12] introduced the concept of a fuzzy normed linear space and proved that in a finite dimensional fuzzy normed linear space fuzzy norms are the same up to fuzzy equivalence. Xiao and Zhu [18] presented a simplified definition of fuzzy normed linear space and studied some properties of compactness and completeness.

In last few decades, many authors studied fuzzy analogue of different operators on fuzzy normed spaces. Xiao and Zhu [19] gave a new definition of the fuzzy norm of a linear operator and studied the

2020 *Mathematics Subject Classification*. 40A35; 40A05
Keywords. Neutrosophic 2-norm space; $\mathbf{N_2}$ bounded, Bounded linear operator; Continuity, $\mathbf{N_2}$ compactness.
Received: 25 March 2023; Revised: 20 June 2023; Accepted: 21 June 2023
Communicated by Eberhard Malkowsky
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space of all bounded linear operators endowed with this fuzzy norm. Bag and Samanta [48] defined fuzzy norm of strongly fuzzy bounded linear operator and weakly fuzzy bounded linear operators. They also defined and studied the fuzzy dual spaces and the Hahn–Banach theorem in fuzzy setting. Hasankhani et al [5] defined fuzzy inner product and studied some properties of the corresponding fuzzy norm. later, the authors introduced the notions of fuzzy boundedness, operator norm and investigated the relationship between continuity and boundedness. Ji et al [41] investigated relations between various notions of fuzzy boundedness of linear operators in fuzzy normed linear spaces and studied the spaces of fuzzy compact operators. For more references in this direction, we refer [9], [11], [15-16], [24] and [39].

Following the idea of Atanassov's ([20]-[22]), Park [17] defined the concept of intuitionistic fuzzy metric space as a generalization of fuzzy metric spaces, which were initially given by George and Veeramani [3]. He also proved Baire's Category type Theorems in these spaces. During a study of topological completeness of intuitionistic fuzzy metric spaces, Saadati and Park [42] generalized the idea of fuzzy normed spaces, called intutionistic fuzzy normed spaces and study the boundedness of linear operators. Karkus et al [49] defined statistical summability in these spaces and gave its useful characterization. Inspired by Gähler [46], Mursaleen and Lohani [38] defined the concept of intuitionistic fuzzy 2-normed space and established some topological results. In this new set up. After their pioneer work, a progressive development on intuitionistic fuzzy normed spaces has been started. Many concepts of analysis have been developed in intuitionistic fuzzy normed spaces. For further developments on these spaces, we recommend [10], [13], [44], [45], [50] and [52-55].

One of the important generalization of fuzzy normed spaces and intuitionistic fuzzy normed spaces is the neutrosophic normed spaces. Actually, Kirisci and Simsek [28] recently used the idea of neutrosophic sets of Smarandache [14], to define the neutrosophic normed linear space. They also studied statistical summability and some of its properties in these spaces. Later, some summability methods have been studied and developed in these spaces and can be seen in [1-2], [6-8] and [23].

Nowadays, neutrosophic normed spaces are growing very rapidly and become a point of attraction in modern research. Many results of functional analysis and operator theory are being developed in intuitionistic fuzzy and neutrosophic normed spaces. Recently, Sajid et al [47] introduced the concept of neutrosophic 2-norm space and studied statistical summability in these spaces. Motivated by the works in [38] and [47], we define in present paper the concepts of neutrosophic boundedness and neutrosophic compactness in $N-2-NS$. Finally, we introduced neutrosophic continuity, neutrosophic operators and studies some of their properties.

We organize the paper as follows: The first section remains introductory. In second section, we will give some preliminaries consisting of basic terminology, definitions and results. In third section, we starts with our main results. We define the open cover, compactness in $N-2-NS$ and develop some of their topological properties. Finally, in forth section, we define boundedness and continuity of linear operators in $N-2-NS$ and obtain some interesting relationships.

2. Preliminaries

This section gives a brief introduction about $t-$norm, $t-$conorm and neutrosophic $-2-$normed spaces ($N-2-NS$).

Definition 2.1 [37] Let $I=[0,1]$. A function $\circ : I\times I\to I$ is said to be a $t-$norm for all $f,g,h,i\in I$ we have:
(i) $f\circ g=g\circ f$;
(ii)$f\circ(g\circ h)=(f\circ g)\circ h$;
(iii) $\circ$ is continuous;
(iv) $f\circ 1=f$ for every $f\in[0,\ 1]$ and
(v) $f\circ g\leq h\circ i$ whenever $f\leq h$ and $g\leq i$.

Definition 2.2 [37] Let $I=[0,1]$. A function $\diamond : I\times I\to I$ is said to be a continuous triangular conorm or $t-$conorm for all $f,g,h,i\in I$ we have:

(i) $f \diamond g = g \diamond f$;
(ii)$f \diamond (g \diamond h) = (f \diamond g) \circ h$;
(iii) $\diamond$ is continuous;
(iv) $f \diamond 0 = f$ for every $f \in [0, 1]$
(v) $f \diamond g \leq h \diamond i$ whenever $f \leq h$ and $g \leq i$.

Definition 2.3 [28] Let V is a vector space, $N = \{\langle \vartheta, G(\vartheta), B(\vartheta), Y(\vartheta)\rangle : \vartheta \in V\}$ be a normed space in which $N : F \times \mathbb{R}^+ \to [0,\ 1]$ and $\circ, \diamond$ respectively are $t-$norm and $t-$conorm. The four tuple $(V, N,\ \circ, \diamond\ )$ is called a neutrosophic normed spaces (NNS) briefly it for every $p, q \in V$, $\rho\ \mu > 0$ and for every $\varsigma \neq 0$ we have
(i) $0 \leq G(p,\ \rho) \leq 1, 0 \leq B(p,\ \rho) \leq 1, 0 \leq Y(p,\ \rho) \leq 1$ for every $\rho \in \mathbb{R}^+$;
(ii) $0 \leq G(p,\ \rho) +\ B(p,\ \rho) + Y(p,\ \rho) \leq 3$ for $\rho \in \mathbb{R}^+$;
(iii) $G(p,\ \rho) = 1$ (for $\rho > 0$) iff $p = 0$;
(iv) $G(\varsigma p, \rho) = G\left(p,\ \frac{\rho}{|\varsigma|}\right)$;
(v) $G(p,\ \mu) \circ G(q,\ \rho) \leq G(p + q,\ \mu + \rho)$;
(vi) $G(p,\ .)$ is a non-decreasing function that runs continuously;
(vii) $\lim_{\rho\to\infty} G(p,\ \rho) = 1$;
(viii) $B(p, \rho) = 0$ (for$\rho > 0$) iff $p = 0$;
(ix) $B(\varsigma p, \rho) = B\left(p,\ \frac{\rho}{|\varsigma|}\right)$;
(x) $B(p,\ \mu) \diamond B(q,\ \rho) \geq B(p + q, \rho + \mu)$;
(xi) $B(p,\ .)$ is a non-increasing function that runs continuously;
(xii) $\lim_{\lambda\to\infty} B(p, \rho) = 0$;
(xiii) $Y(p, \rho) = 0$ (for $\rho > 0$) iff $p = 0$;
(xiv) $Y(\varsigma p, \rho) = Y\left(p,\ \frac{\rho}{|\varsigma|}\right)$;
(xv) $Y(p,\ \mu) \diamond Y(q,\ \rho) \geq Y(p + q, \rho + \mu)$;
(xvi) $Y(p,\ .)$ is a non-increasing function that runs continuously;
(xvii) $\lim_{\lambda\to\infty} Y(p, \rho) = 0$;
(xviii) If $\rho \leq 0$, then $G(p, \rho) = 0$, $B(p, \rho) = 1$ and $Y(p, \rho) = 1$.
We call $N(G, B, Y)$ the neutrosophic norm.

We now recall the concept of 2-norm given in [46].
Definition 2.4 [46] Let V be a $d-$dimensional real vector space, where $2 \leq d < \infty$. A $2-$norm on F is a function $\|.,.\| : V \times V \to \mathbb{R}$ fulfilling the below listed requirements:
For all $p, q \in F$, and scalar α, we have
(i) $\|p, q\| = 0$ iff p and q are linearly dependent;
(ii) $\|p, q\| = \|q, p\|$;
(iii)$\|\alpha p, q\| = |\alpha|\|p, q\|$ and
(iv) $\|p, q + r\| \leq \|p, q\| + \|p, r\|$.
The pair $(V, \|.,.\|)$ is known as $2-$normed space in this case.
Let $V = \mathbb{R}^2$ and for $p = (p_0, p_2)$ and $q = (q_1, q_2)$ we define $\|p, q\| = |p_0 q_2 - p_2 q_1|$, then $\|p, q\|$ is a $2-$ norm on $V = \mathbb{R}^2$.

The idea of neutrosophic 2− normed spaces ($N - 2 - NS$) is given in [47].
Definition 2.5 [47] A six tuple $(V, G, B, Y, \circ, \diamond)$ is said to be a neutrosophic $2-$norm spaces (briefly $N-2-NS$) if V is a vector space, $\circ$ is a t-norm, $\diamond$ is a t-conorm, G, B, Y are fuzzy sets on $V \times V \times (0, \infty)$ satisfying the following conditions.
For every $p, q, w \in V$, $\rho, \mu \geq 0$ and $\varsigma \neq 0$,
(i) $0 \leq G(p, q; \rho) \leq 1$, $0 \leq B(p, q; \rho) \leq 1$ and $0 \leq Y(p, q; \rho) \leq 1$;
(ii) $0 \leq G(p, q; \rho) + B(p, q; \rho) + Y(p, q; \rho) \leq 3$;
(iii) $G(p, q; \rho) = 1$ iff p, q are linearly dependent;
(iv) $G(\varsigma p,\ q;\ \rho) = G(p, q; \frac{\rho}{|\varsigma|})$;
(v) $G(p, q; \rho) \circ_1 G(p, w; \mu) \leq G(p, q + w; \rho + \mu)$;

(vi) $G(p,q;.) : (0,\infty) \to [0,1]$ is a non-decreasing continuous function;
(vii) $\lim\limits_{\rho\to\infty} G(p,q;\rho) = 1$ and $\lim\limits_{\rho\to 0} G(p,q;\rho) = 0$;
(viii) $G(p,q;\rho) = G(q,p;\rho)$;
(ix) $B(p,q;\rho) = 0$ iff p,q are linearly dependent;
(x) $B(\varsigma p,q;\rho) = B(p,q;\frac{\rho}{|\varsigma|})$;
(xi) $B(p,q;\rho) \diamond_1 B(p,w;\mu) \geq B(p,q+w;\rho+\mu)$;
(xii) $B(p,q;.) : (0,\infty) \to [0,1]$ is a non-increasing continuous function;
(xiii) $\lim\limits_{\rho\to\infty} B(p,q;\rho) = 0$ and $\lim\limits_{\rho\to 0} B(p,q;\rho) = 1$;
(xiv) $B(p,q;\rho) = B(q,p;\rho)$
(xvi) $Y(p,q;\rho) = 0$ iff p,q are linearly dependent;
(xv)$Y(\varsigma p,\ q;\ \rho) = Y(p,\ q;\ \frac{\rho}{|\varsigma|})$;
(xvi) $Y(p,q;\rho) \diamond_1 Y(p,w;\mu) \geq Y(p,q+w;\rho+\mu)$;
(xvii) $Y(p,q;.) : (0,\infty) \to [0,1]$ is a non-increasing continuous function;
(xviii) $\lim\limits_{\rho\to\infty} Y(p,q;\rho) = 0$ and $\lim\limits_{\rho\to 0} Y(p,q;\rho) = 1$;
(xix) $Y(p,q;\rho) = Y(q,p;\rho)$
(xx) if $\rho \leq 0$, then $G(p,q;\rho) = 0$, $B(p,q;\rho) = 1$, $Y(p,q\rho) = 1$.

In this case, we call $(V,G,B,Y,\circ,\diamond)$ a neutrosophic 2-norm space and $N_2(G,B,Y)$ or simply N_2 a neutrosophic 2−norm.

For $\epsilon \in (0,1)$, $\rho > 0$ and $p \in V$, the open ball with center at p and of radius ϵ w.r.t. ρ is given by

$$\begin{aligned} W(p,\epsilon,\rho) := \Big\{ & p \in V : G_1(p-p,q;\rho) > 1-\epsilon \text{ and} \\ & B_1(p-p,q;\rho) < \epsilon, Y_1(p-p,q;\rho) < \epsilon, \text{for all } w \in V \Big\}. \end{aligned}$$

A set $W \subset V$ is said to be N_2-open set if for each point p in W there exists an open ball of some radius ϵ contained in W. Moreover, if define $\mathcal{T}(N_2) := \{W \subset V : N_2 - \text{open set}\}$. Then $\mathcal{T}(N_2)$ is a topology on $(V,G_1,B_1,Y_1,\circ_1,\diamond_1)$.

A subset $W \subseteq V$ is said to be N_2-bounded if $\exists\ \rho > 0$ and $\epsilon \in (0,1)$ s.t. $G_1(q-p,w;\rho) > 1-\epsilon$ and $B_1(q-p,w;\rho) < \epsilon$, $Y_1(q-p,w;\rho) < \epsilon$, for every $p,q \in W$ and $\forall\ w \in V$.

We now give the convergence structure and concept of Cauchy sequence in $(V,G_1,B_1,Y_1,\circ_1,\diamond_1)$.

A sequence (p_n) in V is said to be convergent to p w.r.t N_2 if for $\epsilon > 0$ and $\rho > 0$, $\exists\ n_0 \in \mathbb{N}$ s.t. $G_1(p_n-p,w;\rho) > 1-\epsilon$ and $B_1(p_n-p,w;\rho) < \epsilon$, $Y_1(p_n-p,w;\rho) < \epsilon$, $\forall\ n \geq n_0$ and $\forall\ w \in V$. We write in this case $N_2 - \lim\limits_{n\to\infty} p_n = p$ or $p_n \xrightarrow{N_2} p$ as $n \to \infty$.

A sequence (p_n) in V is said to be Cauchy w.r.t N_2 if for each $\epsilon > 0$ and $\rho > 0$, $\exists\, n_0 \in \mathbb{N}$ s.t. $G_1(p_n-p_m,w,\rho) > 1-\epsilon$ and $B_1(p_n-p_m,w,\rho) < \epsilon$, $Y_1(p_n-p_m,w,\rho) < \epsilon\ \forall\ n,m \geq n_0$ and for all $w \in V$.

A $N-2-NS$: $(V,G_1,B_1,Y_1,\circ_1,\diamond_1)$ is said to be complete if every Cauchy sequence is convergent in V w.r.t. $\mathcal{T}(N_2)$.

Definition 2.6 The six-tuple $(\mathbb{R}^n,\phi_1,\psi_1,\varphi_1,\circ_1,\diamond_1)$ where $\mathbb{R}^n = \mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R}$ (n-time), $\circ_1$ a t-norm, $\diamond_1$ a t-conorm and $(\phi_1,\psi_1,\varphi_1)_2$ a neutrosophic Euclidean 2-norm defined by

$$\begin{aligned} \phi_1(p,w;\rho) &= \prod_{\circ} G_1(p_i,w;\rho), \\ \psi_1(p,w;\rho) &= \prod_{\diamond} B_1(p_i,w;\rho), \quad \text{and} \\ \varphi_1(p,w;\rho) &= \prod_{\diamond} Y_1(p_i,w;\rho) \qquad i = 1,2,3\cdots n, \end{aligned}$$

where $\rho > 0$, $p = (p_0, p_2, \cdots, p_n)$, $w = (w_1, w_2, \cdots, w_n)$, $N_2(G_1, B_1, Y_1)$ a neutrosophic 2-norm and for $i = 1, 2, 3 \cdots n$,

$$\prod_{\circ} \alpha_i = \alpha_1 \circ_1 \alpha_2 \circ_1 \cdots \circ_1 \alpha_n, \prod_{\diamond} \alpha_i = \alpha_1 \diamond_1 \alpha_2 \diamond_1 \cdots \diamond_1 \alpha_n$$

is called a neutrosophic Euclidean 2-norm spaces and $(G, B, Y)_2$ is a neutrosophic 2-norm.

For any two $N-2-NS$, $(V, G_1, B_1, Y_1, \circ_1, \diamond_1)$ and $(V, G_2, B_2, Y_2, \circ_2, \diamond_2)$, neutrosophic 2-norms $N_2(G_1, B_1, Y_1)_2$ and $N_2(G_2, B_2, Y_2)_2$ are said to be equivalent provided that $p_k \to p$ in $(V, G_1, B_1, Y_1, \circ_1, \diamond_1)$ iff $p_k \to p$ in $(V, G_2, B_2, Y_2, \circ_2, \diamond_2)$.

Further, A linear operator $T : (V, G_1, B_1, Y_1, \circ_1, \diamond_1) \to (V, G_2, B_2, Y_2, \circ_2, \diamond_2)$ is said to be neutrosophic 2-topological isomorphism if it is bijective and bicontinious. If such an operator exists then the neutrosophic 2-normed space $(V, G_1, B_1, Y_1, \circ_1, \diamond_1)$, $(V, G_2, B_2, Y_2, \circ_2, \diamond_2)$ are said to be neutrosophic 2-topological isomorphic.

3. Some elementary properties on neutrosophic 2-normed spaces.

This section begins with the following definition of compactness in a $N-2-NS$.

Definition 3.1 Let $(V, G_1, B_1, Y_1, \circ_1, \diamond_1)$ be a $N-2-NS$ and $W \subseteq V$. A collection $\{W_\alpha : \alpha \in \Lambda\}$ of N_2 open sets is said to be an N_2-open cover of W if $W \subseteq \cup_{\alpha \in \Lambda} W_\alpha$.

Definition 3.2 Let $(V, G_1, B_1, Y_1, \circ_1, \diamond_1)$ be a $N-2-NS$ and $W \subseteq V$. The set W is said to be N_2-compact if every N_2-open cover $\{W_\alpha\}$ of W (i.e., $W \subseteq \cup_{\alpha \in \Lambda} W_\alpha$, W_α are N_2-open sets for some index set Λ), there exist a finite sub-cover $\{W_{\alpha_1}, W_{\alpha_2}, \cdots W_{\alpha_n}\}$ s.t $W \subseteq \cup_{i=1}^{n} W_{\alpha_i}$.

Theorem 3.1. Every N_2-compact set in a $N-2-NS$, $(V, G_1, B_1, Y_1, \circ_1, \diamond_1)$ is N_2-closed and N_2-bounded.

Proof. Proof of the Theorem is straightforward so omitted. □

Theorem 3.2. Any subset W of $\mathbb{R}$ is N_2-bounded with respect to the $N-2-NS$ $(\mathbb{R}, G_1, B_1, Y_1, \circ_1, \diamond_1)$ if and only if it is bounded as a subset of $\mathbb{R}$.

Proof. Let $W \subseteq \mathbb{R}$ be N_2-bounded in $(\mathbb{R}, G_1, B_1, Y_1, \circ_1, \diamond_1)$. Then, $\exists\ \rho > 0$ and $\epsilon \in (0, 1)$ s.t. $G_1(q-p, w; \rho) > 1-\epsilon$ and $B_1(q-p, w; \rho) < \epsilon$, $Y_1(q-p, w; \rho) < \epsilon$, for every $p, q \in W$ and $\forall\ w \in \mathbb{R}$. Thus, for every non zero $p, q \in W$ i.e. $q-p \neq 0$ and $\forall\ w \in \mathbb{R}$

$$1-\epsilon < G_1(q-p, w; \rho) = G_1\Big(1, w; \frac{\rho}{|q-p|}\Big) \text{ and}$$
$$\epsilon > B_1(q-p, w; \rho) = B_1\Big(1, w; \frac{\rho}{|q-p|}\Big)$$
$$\epsilon > Y_1(q-p, w; \rho) = Y_1\Big(1, w; \frac{\rho}{|q-p|}\Big),$$

and therefore, $\exists\ M \in \mathbb{R}^+$ s.t. $|q-p| \leq M$. This shows that W is bounded in $\mathbb{R}$. The converse part may be obtained similarly. □

Theorem 3.3. A sequence $p = (p_n)$ is N_2-convergent in the neutrosophic 2− normed spaces $(N-2-NS)$, $(\mathbb{R}, G_1, B_1, Y_1, \circ_1, \diamond_1)$ if and only if it is convergent in $(\mathbb{R}, |.|)$.

Proof. Suppose that $p = (p_n)$ is convergent to p in $(\mathbb{R}, |.|)$, then $|p_n - p| \to 0$ as $n \to \infty$. Now for any $w \in \mathbb{R}$,

$$\lim_{n\to\infty} G_1(p_n - p, w; \rho) = \lim_{n\to\infty} G_1\Big(1, w; \frac{\rho}{|p_n - p|}\Big) = G_1(1, w; \infty) = 1,$$
$$\lim_{n\to\infty} B_1(p_n - p, w; \rho) = \lim_{n\to\infty} B_1\Big(1, w; \frac{\rho}{|p_n - p|}\Big) = B_1(1, w; \infty) = 0,$$
$$\lim_{n\to\infty} Y_1(p_n - p, w; \rho) = \lim_{n\to\infty} Y_1\Big(1, w; \frac{\rho}{|p_n - p|}\Big) = Y_1(1, w; \infty) = 0.$$

This shows that $p_n \xrightarrow{N_2} p$ in $(\mathbb{R}, G_1, B_1, Y_1, \circ_1, \diamond_1)$.

Conversely, suppose that $p = (p_n)$ is N_2-convergent to p in the neutrosophic 2− normed spaces $(N{-}2{-}NS)$, $(\mathbb{R}, G_1, B_1, Y_1, \circ_1, \diamond_1)$, then for any $w \in \mathbb{R}$ and $\rho > 0$, $\lim_{n\to\infty} G_1(p_n - p, w; \rho) = 1$ and $\lim_{n\to\infty} B_1(p_n - p, w; \rho) = 0$, $\lim_{n\to\infty} Y_1(p_n - p, w; \rho) = 0\ \forall\ w \in \mathbb{R}$. Now we shall prove that $p = (p_n)$ is convergent to p in $(\mathbb{R}, |.|)$.

Case1: Let, $\liminf(p_n - p) = r$ and $\limsup(p_n - p) = s$ s.t. r and s are not $+\infty$ or $-\infty$ then we can find the subsequence $(p_{n_j} - p)$ and $(p_{m_j} - p)$ converges to r and s respectively. Moreover, by assumption $G_1(r, w; \rho) = G_1(s, w; \rho) = 1$. This implies that r, w and s, w are linearly independent $\forall w \in \mathbb{R}$. Thus, there exists scalars $\alpha_1, \beta_1, \alpha_2$ and β_2 s.t. for all $w \in \mathbb{R}$, $\alpha_1 r + \beta_1 w = 0$ and $\alpha_2 s + \beta_2 w = 0$ where at least one of $\alpha_1, \beta_1 \neq 0$ and $\alpha_2, \beta_2 \neq 0$. As this holds for all $w \in \mathbb{R}$ so we have $\beta_1 = \beta_2 = 0$ and α_1 and $\alpha_2 \neq 0$. This implies that $\alpha_1 r = \alpha_2 r = 0$ and therefore $r = 0$. Similarly, $s = 0$. This shows that $\lim(p_n - p) = 0$. i.e., $p = (p_n)$ is convergent to $p = 0$ in $(\mathbb{R}, |.|)$.

Case2: If either r or s or both are ∞. Since $G_1(p, q; .)$ is non-decreasing and $G_1(p, w; \rho) = G_1\Big(1, w; \frac{\rho}{|p|}\Big)$, it follows that

$$\begin{aligned}\limsup G_1\Big(1, w; \frac{\rho}{|p_n - p|}\Big) &\leq G_1(p_n - p, w; \rho)\\ &\leq \liminf G_1\Big(1, w; \frac{\rho}{|p_n - p|}\Big).\end{aligned}$$

If $\liminf(p_n - p) = -\infty$, then

$$\begin{aligned}\lim G_1(p_n - p, w; \rho) &\leq \liminf G_1(p_n - p, w; \rho)\\ &\leq \liminf G_1\Big(1, w; \frac{\rho}{|p_n - p|}\Big),\end{aligned}$$

and therefore we have $1 < 0$ (not possible). Further, if $\limsup(p_n - p) = +\infty$, then $\liminf(p - p_n) = -\infty$, and we have again $1 < 0$. Hence, $p = (p_n)$ is convergent to p in $(\mathbb{R}, |.|)$. □

Corollary 3.1. If the real sequence (α_n) is N_2-bounded, then it has at least one limit point.

Lemma 3.1. If $\circ = \min$ and $\diamond = \max$, then $(\mathbb{R}^n, \phi_1, \psi_1, \varphi_1, \circ_1, \diamond_1)$ is an neutrosophic 2-normed spaces.
Proof. The proof of the Lemma is straightforward so we omit here. □

Corollary 3.2. The neutrosophic Euclidean 2-normed space $(\mathbb{R}^n, \phi_1, \psi_1, \varphi_1, \circ_1, \diamond_1)$ is complete.

Theorem 3.4. Let $\{p_0, p_2, \cdots, p_n\}$ be a linearly independent set of vectors in V and $(V, G_1, B_1, Y_1, \circ_1, \diamond_1)$ be an neutrosophic 2-normed space. Then $\exists$ numbers α, β and $\gamma \neq 0$ and an neutrosophic 2-normed space $(\mathbb{R}, G_2, B_2, Y_2, \circ_1, \diamond_1)$ such that $\forall \alpha_i, 1 \leq i \leq n$, for all $w \in V$ and $\rho > 0$ we have

$$G_1\Big(\sum_{i=1}^{n} \alpha_i p_i, w; \rho\Big) \leq G_2\Big(\alpha \sum_{i=1}^{n} |\alpha_i|, w; \rho\Big) \tag{1}$$

$$B_1\Big(\sum_{i=1}^{n} \alpha_i p_i, w; \rho\Big) \geq B_2\Big(\beta \sum_{i=1}^{n} |\alpha_i|, w; \rho\Big) \text{ and} \tag{2}$$

$$Y_1\Big(\sum_{i=1}^{n} \alpha_i p_i, w; \rho\Big) \geq Y_2\Big(\gamma \sum_{i=1}^{n} |\alpha_i|, w; \rho\Big) \tag{3}$$

Proof. Let $\sum_{i=1}^{n} |\alpha_i| = L$. If $L = 0$, then $\alpha_i = 0$ for all $1 \leq i \leq n$, and therefore (1), (2) and (3) holds. Let $L > 0$. If we define $\beta_i = \frac{\alpha_i}{L}$, then $\sum_{i=1}^{n} |\beta_i| = 1$ and therefore we have from (1) to (3)

$$G_1\Big(\sum_{i=1}^{n} \beta_i p_i, w; \frac{\rho}{L}\Big) \leq G_2\Big(\alpha, w; \frac{\rho}{L}\Big) \tag{4}$$

$$B_1\Big(\sum_{i=1}^{n}\beta_i p_i, w; \frac{\rho}{L}\Big) \geq B_2\Big(\beta, w; \frac{\rho}{L}\Big) \tag{5}$$

$$Y_1\Big(\sum_{i=1}^{n}\beta_i p_i, w; \frac{\rho}{L}\Big) \geq Y_2\Big(\gamma, w; \frac{\rho}{L}\Big) \tag{6}$$

for all $w \in V$ and $\rho > 0$. Thus, to prove the existence of $\alpha, \beta, \gamma \neq 0$ and the neutrosophic 2-normed space $(G_2, B_2, Y_2)_2$ satisfying (1), (2) and (3) it is sufficient to prove their existence satisfying (4) to (6). Suppose, this is not hold, then we can find out a sequence $q = (q_m)$ of vectors defined by

$$q_m = \sum_{i=1}^{n}\beta_{i,m}p_i \text{ where } \Big(\sum_{i=1}^{n}|\beta_{i,m}| = 1\Big)$$

such that $G_1(q_m, w, \rho) \to 1$, $B_1(q_m, w, \rho) \to 0$ and $Y_1(q_m, w, \rho) \to 0$ as $m \to \infty$ for every $\rho > 0$ and $\forall\ w \in V$. Since $\Big(\sum_{i=1}^{n}|\beta_{i,m}| = 1\Big)$ therefore $|\beta_{i,m}| \leq 1$. Thus, by Theorem 3.2, $(\beta_{i,m})$ is N_2-bounded. So by corollary 3.1, $(\beta_{1,m})$ has a convergent subsequences. Let β_1 be the limit of the subsequence and let $(q_{1,m})$ denote the corresponding subsequence of (q_m). Similarly, let $(q_{1,m})$ has a subsequence $(q_{2,m})$ for which the corresponding sequence $(\beta_{2,m})$ in $\mathbb{R}$ converges to β_2. Continuing in this way, after n-steps we can obtain a subsequence $(q_{n,m})$ of (q_m) s.t.

$$q_{n,m} = \sum_{i=1}^{n}\gamma_{i,m}p_i, \Big(\sum_{i=1}^{n}|\gamma_{i,m}| = 1\Big) \text{ and } \gamma_{i,m} \to \beta_i \text{ as } m \to \infty.$$

Since $\forall\ w \in V$

$$\lim_{m\to\infty} G_1\Big(q_{n,m} - \sum_{i=1}^{n}\beta_i p_i, w; \rho\Big) = \lim_{m\to\infty} G_1\Big(\sum_{i=1}^{n}(\gamma_{i,m} - \beta_i)p_i, w; \rho\Big)$$

$$\geq \lim_{m\to\infty}\Big[G_1\Big((\gamma_{1,m} - \beta_1)p_0, w; \frac{\rho}{n}\Big) \circ_1 \cdots \circ_1 G_1\Big((\gamma_{n,m} - \beta_n)p_n, w; \frac{\rho}{n}\Big)\Big] = 1$$

and

$$\lim_{m\to\infty} B_1\Big(q_{n,m} - \sum_{i=1}^{n}\beta_i p_i, w; \rho\Big) = \lim_{\to\infty} B_1\Big(\sum_{i=1}^{n}(\gamma_{i,m} - \beta_i)p_i, w; \rho\Big)$$

$$\leq \lim_{m\to\infty}\Big[B_1\Big((\gamma_{1,m} - \beta_1)p_0, w; \frac{\rho}{n}\Big) \diamond_1 \cdots \diamond_1 B_1\Big((\gamma_{n,m} - \beta_n)p_n, w; \frac{\rho}{n}\Big)\Big] = 0,$$

$$\lim_{m\to\infty} Y_1\Big(q_{n,m} - \sum_{i=1}^{n}\beta_i p_i, w; \rho\Big) = \lim_{\to\infty} Y_1\Big(\sum_{i=1}^{n}(\gamma_{i,m} - \beta_i)p_i, w; \rho\Big)$$

$$\leq \lim_{m\to\infty}\Big[Y_1\Big((\gamma_{1,m} - \beta_1)p_0, w; \frac{\rho}{n}\Big) \diamond_1 \cdots \diamond_1 Y_1\Big((\gamma_{n,m} - \beta_n)p_n, w; \frac{\rho}{n}\Big)\Big] = 0$$

so we have, $\lim_{m\to\infty} q_{n,m} = \sum_{i=1}^{n}\beta_i p_i, \Big(\sum_{i=1}^{n}|\beta_i| = 1\Big)$, with not all β_i can be zero. Put $q = \sum_{i=1}^{n}\beta_i p_i$. Further, $\{p_0, \cdots, p_n\}$ is linearly independent, so we have $q \neq 0$. Moreover, for every $\rho > 0$ and $\forall\ w \in V$, we have

$$G_1(q_m, w; \rho) \to 1,\ \ B_1(q_m, w; \rho) \to 0 \text{ and } Y_1(q_m, w; \rho) \to 0 \text{ as } m \to \infty,$$

therefore

$$G_1(q_{n,m}, w; \rho) \to 1,\ \ B_1(q_{n,m}, w; \rho) \to 0 \text{ and } Y_1(q_{n,m}, w; \rho) \to 0 \text{ as } m \to \infty.$$

Hence, for every $\rho > 0$ and $\forall\, w \in V$

$$G_1(q, w; \rho) = G_1((q - q_{n,m}) + q_{n,m}, w; \rho) \geq$$
$$G_1\Big(q - q_{n,m}, w; \frac{\rho}{2}\Big) \circ_1 G_1\Big(q_{n,m}, w; \frac{\rho}{2}\Big) \to 1$$

and

$$B_1(q, w; \rho) = B_1((q - q_{n,m}) + q_{n,m}, w; \rho) \leq$$
$$B_1\Big(q - q_{n,m}, w; \frac{\rho}{2}\Big) \diamond_1 B_1\Big(q_{n,m}, w; \frac{\rho}{2}\Big) \to 0,$$

$$Y_1(q, w; \rho) = Y_1((q - q_{n,m}) + q_{n,m}, w; \rho) \leq$$
$$Y_1\Big(q - q_{n,m}, w; \frac{\rho}{2}\Big) \diamond_1 Y_1\Big(q_{n,m}, w; \frac{\rho}{2}\Big) \to 0.$$

This shows that q, w are linearly dependent, for all $w \in V$. Hence $\alpha q + \beta w = 0$ implies $\alpha = 0$ or $\beta = 0$. But this holds for all $w \in V$ if and only if $\beta = 0$. Hence, $\alpha q = 0$ where $\alpha \neq 0$, gives $q = 0$. So we obtained a contradiction and therefore the theorem is proved. □

Theorem 3.5. Any two neutrosophic 2-norm $(G_1, B_1, Y_1)_2$ and $(G_2, B_2, Y_2)_2$ are equivalent on a finite dimensional vector space V.
Proof. Since V is finite dimensional so let dim $V = n$ and $\{v_1, \cdots, v_n\}$ be a basis for V. So every $p \in V$ has a unique representation $p = \sum_{i=1}^{n} \alpha_i v_i$. Let $p_m \to p$ w.r.t. $(G_1, B_1, Y_1)_2$. Further, $\forall\, m$, p_m can be written uniquely as

$$p_m = \sum_{i=1}^{n} \alpha_{i,m} v_i.$$

Further, By theorem 3.4 there are $\alpha, \beta, \gamma \neq 0$ and an neutrosophic 2-norm $(G_2, B_2, Y_2)_2$ satisfying (1), (2) and (3) and therefore,

$$G_1(p_m - p, w; \rho) \leq G_2\Big(\alpha \sum_{i=1}^{n} |\alpha_{i,m} - \alpha_i|, w; \rho\Big) \leq G_2(\alpha|\alpha_{i,m} - \alpha_i|, w; \rho) \;\; \forall\, w \in V \text{ and}$$
$$B_1(p_m - p, w; \rho) \geq B_2\Big(\beta \sum_{i=1}^{n} |\alpha_{i,m} - \alpha_i|, w; \rho\Big) \geq B_2(\beta|\alpha_{i,m} - \alpha_i|, w; \rho) \;\; \forall\, w \in V,$$
$$Y_1(p_m - p, w; \rho) \geq Y_2\Big(\gamma \sum_{i=1}^{n} |\alpha_{i,m} - \alpha_i|, w; \rho\Big) \geq Y_2(\gamma|\alpha_{i,m} - \alpha_i|, w; \rho) \;\; \forall\, w \in V.$$

Now, $m \to \infty$ then we have, $G_1(p_m - p, w; \rho) \to 1$ and $B_1(p_m - p, w; \rho) \to 0$, $Y_1(p_m - p, w; \rho) \to 0$ for every $\rho > 0$, $w \in V$ and hence $|\alpha_{i,m} - \alpha_i| \to 0$ in $\mathbb{R}$. We have,

$$G_2(p_m - p, w; \rho) \geq G_2\Big((\alpha_{1,m} - \alpha_1)v_1, w; \frac{\rho}{n}\Big) \circ_2 \cdots \circ_2 G_2\Big((\alpha_{n,m} - \alpha_n)v_n, w; \frac{\rho}{n}\Big)$$
$$= G_2\Big(v_1, w; \frac{\rho}{n(\alpha_{1,m} - \alpha_1)}\Big) \circ_2 \cdots \circ_2 G_2\Big(v_n, w; \frac{\rho}{n(\alpha_{n,m} - \alpha_n)}\Big) \text{ and}$$
$$B_2(p_m - p, w; \rho) \leq B_2\Big((\alpha_{1,m} - \alpha_1)v_1, w; \frac{\rho}{n}\Big) \diamond_2 \cdots \diamond_2 B_2\Big((\alpha_{n,m} - \alpha_n)v_n, w; \frac{\rho}{n}\Big)$$
$$= B_2\Big(v_1, w; \frac{\rho}{n(\alpha_{1,m} - \alpha_1)}\Big) \diamond_2 \cdots \diamond_2 B_2\Big(v_n, w; \frac{\rho}{n(\alpha_{n,m} - \alpha_n)}\Big),$$
$$Y_2(p_m - p, w; \rho) \leq Y_2\Big((\alpha_{1,m} - \alpha_1)v_1, w; \frac{\rho}{n}\Big) \diamond_2 \cdots \diamond_2 Y_2\Big((\alpha_{n,m} - \alpha_n)v_n, w; \frac{\rho}{n}\Big)$$
$$= Y_2\Big(v_1, w; \frac{\rho}{n(\alpha_{1,m} - \alpha_1)}\Big) \diamond_2 \cdots \diamond_2 Y_2\Big(v_n, w; \frac{\rho}{n(\alpha_{n,m} - \alpha_n)}\Big).$$

Since $|\alpha_{i,m} - \alpha_i| \to 0$, $\frac{\rho}{n(\alpha_{i,m} - \alpha_i)} \to \infty$ and then we have

$$G_2\Big(v_i, w; \frac{\rho}{n(\alpha_{i,m} - \alpha_i)}\Big) \to 1 \text{ and } B_2\Big(v_i, w; \frac{\rho}{n(\alpha_{i,m} - \alpha_i)}\Big) \to 0,$$
$$Y_2\Big(v_i, w; \frac{\rho}{n(\alpha_{i,m} - \alpha_i)}\Big) \to 0. \ \forall\, w \in V.$$

So, we have $p_m \xrightarrow{(G_2,B_2,Y_2)_2} p$ in $(V, G_2, B_2, Y_2, \circ_2, \diamond_2)$. Analogously $p_m \xrightarrow{(G_2,B_2,Y_2)_2} p$ in $(V, G_2, B_2, Y_2, \circ_2, \diamond_2) \Rightarrow p_m \xrightarrow{(G_1,B_1,Y_1)_2} p$ in $(V, G_1, B_1, Y_1, \circ_1, \diamond_1)$. □

4. Bounded linear operator

Definition 4.1 A mapping of linear operator $T : (V, G_1, B_1, Y_1, \circ_1, \diamond_1) \to (V, G_2, B_2, Y_2, \circ_2, \diamond_2)$ is said to be neutrosophic 2-bounded $\exists$ constants $l, m, n \in \mathbb{R} - \{0\}$ s.t for nonzero $p, w \in V$ and $\rho > 0$,

$$G_2(Tp, w, \rho) \geq G_1(lp, w, \rho) \text{ and}$$
$$B_2(Tp, w, \rho) \leq B_1(mp, w, \rho),\ Y_2(Tp, w, \rho) \leq Y_1(np, w, \rho).$$

Definition 4.2 T is said to be neutrosophic 2-continuous at $p_0 \in V$, if given $\epsilon > 0$, $\exists$, $\xi = \xi(\epsilon) > 0$ s.t $p \in V$ and $\forall\ 0 \neq w \in V$.

$$G_2(Tp - Tp_0, w; \epsilon) \geq G_1(p - p_0, w; \xi), \text{ and}$$
$$B_2(Tp - Tp_0, w; \epsilon) \leq B_1(p - p_0, w; \xi),\ Y_2(Tp - Tp_0, w; \epsilon) \leq Y_1(p - p_0, w; \xi).$$

Definition 4.3 A linear operator $T : (V, G_1, B_1, Y_1, \circ_1, \diamond_1) \to (V, G_2, B_2, Y_2, \circ_2, \diamond_2)$ is said to be neutrosophic 2-topological isomorphism if it is bijective and bicontinious. If such an operator T exists then, we call neutrosophic 2-normed space $(V, G_1, B_1, Y_1, \circ_1, \diamond_1)$, $(V, G_2, B_2, Y_2, \circ_2, \diamond_2)$ as neutrosophic 2-topological isomorphic spaces.

Theorem 4.1 Every neutrosophic 2-bounded linear operator is neutrosophic 2-continuous.

Lemma 4.1 A map $T : (V, G_1, B_1, Y_1, \circ_1, \diamond_1) \to (V, G_2, B_2, Y_2, \circ_2, \diamond_2)$ is neutrosophic 2-topological isomorphism if T is onto and $\exists$ nonzero constants $\alpha, \beta,\ \alpha^{'}, \beta^{'},\ \alpha^{''}, \beta^{''} \neq 0$ s.t.

$$G_1(\alpha p, w, \rho) \leq G_2(Tp, w, \rho) \leq G_1(\beta p, w, \rho) \text{ and}$$
$$B_1(\alpha^{'} p, w, \rho) \leq B_2(Tp, w, \rho) \leq B_1(\beta^{'} p, w, \rho)$$
$$Y_1(\alpha^{''} p, w, \rho) \leq Y_2(Tp, w, \rho) \leq Y_1(\beta^{''} p, w, \rho).$$

Proof. By hypothesis, it is clear that T is neutrosophic 2-bounded, and by Definition 4.2, T is continuous. Since $Tp = 0$ so $1 = G_2(Tp, w, \rho) \leq G_1\Big(p, w, \frac{\rho}{|\beta|}\Big)$ and therefore $p = 0$, then T is one-to-one and therefore T^{-1} will exists. Since

$$G_2(Tp, w, \rho) \leq G_1(\beta p, w, \rho) \text{ and}$$
$$B_2(Tp, w, \rho) \leq B_1(\beta^{'} p, w, \rho), Y_2(Tp, w, \rho) \leq Y_1(\beta^{''} p, w, \rho)$$

are equivalent to

$$G_2(q, w, \rho) \leq G_1(\beta T^{-1} q, w, \rho) = G_1\Big(T^{-1} q, w, \frac{\rho}{|\beta|}\Big) \text{ and}$$
$$B_2(q, w, \rho) \leq B_1(\beta^{'} T^{-1} q, w, \rho) = B_1\Big(T^{-1} q, w, \frac{\rho}{|\beta^{'}|}\Big),$$
$$Y_2(q, w, \rho) \leq Y_1(\beta^{''} T^{-1} q, w, \rho) = Y_1\Big(T^{-1} q, w, \frac{\rho}{|\beta^{''}|}\Big)$$

or

$$G_2\Big(\frac{1}{\beta} q, w, \rho\Big) \leq G_1(T^{-1} q, w, \rho) \text{ and}$$
$$B_2\Big(\frac{1}{\beta^{'}} q, w, \rho\Big) \leq B_1(T^{-1} q, w, \rho),$$
$$Y_2\Big(\frac{1}{\beta^{''}} q, w, \rho\Big) \leq Y_1(T^{-1} q, w, \rho);$$

where $q = Tp$, so T^{-1} is neutrosophic 2-bounded by definition 4.2 is continuous. This show that T is a neutrosophic 2-topologically isomorphism. □

Corollary 4.1 Neutrosophic 2-topologically isomorphism is preserves completeness.

Theorem 4.2 A linear operator $T : (V, G_1, B_1, Y_1, \circ, \diamond) \to (V, G_2, B_2, Y_2, \circ, \diamond)$ where $\circ = \max$, $\diamond = \min$ and $\dim V < \infty$ not necessarily finite dimensional is continuous.
Proof First prove that if we define G_3, B_3, Y_3 s.t

$$G_3(p, w, \rho_1) = G_1(p, w, \rho_1) \circ G_2(Tp, w, \rho_1) \text{ and} \tag{7}$$
$$B_3(p, w, \rho_1) = B_1(p, w, \rho_1) \diamond B_2(Tp, w, \rho_1), \tag{8}$$
$$Y_3(p, w, \rho_1) = Y_1(p, w, \rho_1) \diamond Y_2(Tp, w, \rho_1) \tag{9}$$

then $(V, G_3, B_3, Y_3, \circ, \diamond)$ becomes a neutrosophic 2-normed space. As it easy to show the properties because $(ii) - (iv), (vi) - (ix), (xii), (xiii) - (xvi), (xvii), (xviii)$ are immediate from definition 2.1. So we only prove the properties (v) $(xi)$and (xvi)

$$\begin{aligned} G_3(w, p, \rho_1) \circ G_3(w, q, \rho_2) &= G_3(p, w, \rho_1) \circ G_3(q, w, \rho_2) \\ &= [G_1(p, w, \rho_1) \circ G_2(Tp, w, \rho_1)] \circ [G_1(q, w, \rho_2) \circ G_2(Tq, w, \rho_2)] \\ &= [G_1(p, w, \rho_1) \circ G_1(q, w, \rho_2)] \circ [G_2(Tp, w, \rho_1) \circ G_2(Tq, w, \rho_2)] \\ &\leq G_1(p + q, w, \rho_1 + \rho_2) \circ G_2(T(p + q), w, \rho_1 + \rho_2) \\ &= G_3(p + q, w, \rho_1 + \rho_2) = G_3(w, p + q, \rho_1 + \rho_2). \end{aligned}$$

Similarly, we can prove (xi). Let $(p_n) \xrightarrow{(G_1,B_1,Y_1)_2} p$ (by Theorem 3.5) $p_n \xrightarrow{(G_3,B_3,Y_3)_2} p$ but since by (7), (8) and (9) and te choice of $\circ, \diamond$

$$G_2(Tp, w, \rho_1) \geq G_3(p, w, \rho_1) \text{ and}$$
$$B_2(Tp, w, \rho_1) \geq B_3(p, w, \rho_1),\ Y_2(Tp, w, \rho_1) \geq Y_3(p, w, \rho_1)\ \forall\, w \in V.$$

Then $T(p_n) \xrightarrow{(G_2,B_2,Y_2)_2} Tp$. Hence T is continuous. □

Corollary 4.2 Every linear isomorphism between finite dimensional neutrosophic 2-normed space is an neutrosophic 2-topological isomorphism.

Corollary 4.3 Every finite dimensional neutrosophic 2-normed spaces $(V, G_1, B_1, Y_1, \circ, \diamond)$, where $\circ$ = max and $\diamond$ = min, is complete.
Proof $\circ$ = max and $\diamond$ = min, (by corollary 4.2) $(V, G_1, B_1, Y_1, \circ, \diamond)$ is neutrosophic 2-topologically isomorphic to $(\mathbb{R}^n, G_1, B_1, Y_1, \circ, \diamond)$. Since $(\mathbb{R}^n, G_1, B_1, Y_1, \circ, \diamond)$ is complete and neutrosophic 2-topological isomorphism preserve completeness, $(V, G_1, B_1, Y_1, \circ, \diamond)$ is complete. □

Theorem 4.3 Let $(V, G_1, B_1, Y_1, \circ, \diamond)$ be an neutrosophic 2-normed space where $\circ_l$, $\diamond_l$ be s.t $\circ \geq \circ_l$, $\diamond \leq \diamond_l$ and $a \circ_l b = \max(a+b-1, 0)$, $a \diamond_l b = \min(a+b, 1)$. Then the family $\nu = V(\rho, \epsilon, F) : \rho > 0,\ \epsilon \in (0,1)$ for every $F \in f$. If f denoted the family of all finite and non-empty subsets of vector space V, is a base system of neighborhood of zero in V.
Proof Let $V(\rho_n, \epsilon_n, F_n)$, $n = 1, 2$ be in ν. We consider $F = F_1 \cup F_2$, $\rho = \min\{\rho_1, \rho_2\}$, $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ then $V(\rho, \epsilon, F) \subset V(\rho_1, \epsilon_1, F_1) \cap V(\rho_2, \epsilon_2, F_2)$. Let $\beta \in \mathbb{R}$ s.t $0 < \beta \leq 1$ and $p \in \beta V(\rho, \epsilon, F)$, then $p = \beta q$, where $q \in V(\rho, \epsilon, F)$. For every $q \in F$ we have,

$$G_1(p, q; \rho) = G_1(\beta q, q; \rho) = G_1\Big(q, q; \frac{\rho}{\beta}\Big) = 1 > 1 - \epsilon \text{ and}$$
$$B_1(p, q; \rho) = B_1(\beta q, q; \rho) = B_1\Big(q, q; \frac{\rho}{\beta}\Big) = 0 < \epsilon,$$
$$Y_1(p, q; \rho) = Y_1(\beta q, q; \rho) = Y_1\Big(q, q; \frac{\rho}{\beta}\Big) = 0 < \epsilon.$$

This implies that $p \in V(\rho, \epsilon, F)$, hence $\beta V(\rho, \epsilon, F) \subset V(\rho, \epsilon, F)$. Now, we Show that for every $A \subset \nu$ and $p \in A$, $\exists\, \gamma \in \mathbb{R}$, $\gamma \neq 0$ s.t $\gamma p \in A$. If $A \in \nu$, $\exists\, \rho > 0$, $\epsilon \in (0,1)$ and $F \in f$ s.t $A = V(\rho, \epsilon, F)$. Let p be fixed in V and $\beta \in \mathbb{R}$, $\beta \neq 0$, then

$$G_1(\beta p, q, \rho) = G_1\Big(p, q; \frac{\rho}{|\beta|}\Big) \text{ and}$$
$$B_1(\beta p, q, \rho) = B_1\Big(p, q; \frac{\rho}{|\beta|}\Big), Y_1(\beta p, q, \rho) = Y_1\Big(p, q; \frac{\rho}{|\beta|}\Big).$$

Since

$$\lim_{|\beta| \to 0} G_1\Big(p, q; \frac{\rho}{|\beta|}\Big) = 1 \text{ and}$$
$$\lim_{|\beta| \to 0} B_1\Big(p, q; \frac{\rho}{|\beta|}\Big) = 0,\ \lim_{|\beta| \to 0} Y_1\Big(p, q; \frac{\rho}{|\beta|}\Big) = 0.$$

so forall $q \in F\ \exists\, \beta(q) \in \mathbb{R}$ s.t

$$G_1\Big(p, q; \frac{\rho}{|\beta(q)|}\Big) > 1 - \epsilon \text{ and } B_1\Big(p, q; \frac{\rho}{|\beta(q)|}\Big) < \epsilon,\ Y_1\Big(p, q; \frac{\rho}{|\beta(q)|}\Big) < \epsilon.$$

Choose $\gamma = \min\{|\beta(q)| : q \in F\}$, then we have

$$G_1(\gamma p, q; \rho) = G_1\Big(p, q; \frac{\rho}{|\gamma|}\Big) \geq G_1\Big(p, q; \frac{\rho}{|\beta(q)|}\Big) > 1 - \epsilon \text{ and}$$

$$B_1(\gamma p, q; \rho) = B_1\Big(p, q; \frac{\rho}{|\gamma|}\Big) \leq B_1\Big(p, q; \frac{\rho}{|\beta(q)|}\Big) < \epsilon,$$

$$Y_1(\gamma p, q; \rho) = Y_1\Big(p, q; \frac{\rho}{|\gamma|}\Big) \leq Y_1\Big(p, q; \frac{\rho}{|\beta(q)|}\Big) < \epsilon. \ \forall\ q \in F,$$

Hence $\gamma p \in A$.

Now we prove that for any $A \in \nu$, $\exists\, A_0 \in \nu$ s.t $A_0 + A_0 \subset A$. If $A = V(\rho, \epsilon, F)$ and $p \in V(\rho, \epsilon, F)$, then $\exists$, $\zeta > 0$ s.t

$$G_1(p, q; \rho) > 1 - \zeta > 1 - \epsilon \text{ and } B_1(p, q; \rho) < \zeta < \epsilon,\ Y_1(p, q; \rho) < \zeta < \epsilon,$$

for every $q \in F$. If $A_0 = V\Big(\frac{\rho}{2}, \frac{\zeta}{2}, F\Big)$ and $p, w \in A_0$, $q \in F$ then by the inequality (iv) and (xii), we have

$$\begin{aligned}
G_1(p + w, q, \rho) = G_1(q, p + w, \rho) &\geq G_1\Big(q, p, \frac{\rho}{2}\Big) \circ G_1\Big(q, w, \frac{\rho}{2}\Big) \\
&= G_1\Big(p, q, \frac{\rho}{2}\Big) \circ G_1\Big(w, q, \frac{\rho}{2}\Big) \geq \Big(1 - \frac{\zeta}{2}\Big) \circ \Big(1 - \frac{\zeta}{2}\Big) \\
&\geq \Big(1 - \frac{\zeta}{2}\Big) \circ_l \Big(1 - \frac{\zeta}{2}\Big) > 1 - \zeta > 1 - \epsilon, \\
B_1(p + w, q, \rho) = B_1(q, p + w, \rho) &\leq B_1\Big(q, p, \frac{\rho}{2}\Big) \diamond B_1\Big(q, w, \frac{\rho}{2}\Big) \\
&= B_1\Big(p, q, \frac{\rho}{2}\Big) \diamond B_1\Big(w, q, \frac{\rho}{2}\Big) \leq \Big(\frac{\zeta}{2}\Big) \diamond \Big(\frac{\zeta}{2}\Big) \\
&\leq \Big(\frac{\zeta}{2}\Big) \diamond_l \Big(\frac{\zeta}{2}\Big) < \zeta < \epsilon, \\
Y_1(p + w, q, \rho) = Y_1(q, p + w, \rho) &\leq Y_1\Big(q, p, \frac{\rho}{2}\Big) \diamond Y_1\Big(q, w, \frac{\rho}{2}\Big) \\
&= Y_1\Big(p, q, \frac{\rho}{2}\Big) \diamond Y_1\Big(w, q, \frac{\rho}{2}\Big) \leq \Big(\frac{\zeta}{2}\Big) \diamond \Big(\frac{\zeta}{2}\Big) \\
&\leq \Big(\frac{\zeta}{2}\Big) \diamond_l \Big(\frac{\zeta}{2}\Big) < \zeta < \epsilon.
\end{aligned}$$

These inequalities show that $A_0 + A_0 \subset A$.In what follows, we show that $A \subset \nu$ and $\beta \in \mathbb{R}$, $\beta \neq 0$ implies $\beta A \subset \nu$.

Further we also remark that $\beta A = \beta \nu(\rho, \epsilon, F) = \{\beta p : G_1(\beta p, q; \rho) > 1 - \epsilon$ and $B_1(\beta p, q; \rho) < \epsilon,\ Y_1(\beta p, q; \rho) < \epsilon,\ \forall\ q \in F\}$. We also observe that

$$G_1(p, q; \rho) > 1 - \epsilon \text{ iff } G_1\Big(p, q; \frac{|\beta|\rho}{|\beta|}\Big) = G_1(\beta p, q; |\beta|\rho) > 1 - \epsilon, \text{ and}$$

$$B_1(p, q; \rho) < \epsilon \text{ iff } B_1\Big(p, q; \frac{|\beta|\rho}{|\beta|}\Big) = B_1(\beta p, q; |\beta|\rho) < \epsilon,$$

$$Y_1(p, q; \rho) < \epsilon \text{ iff } Y_1\Big(p, q; \frac{|\beta|\rho}{|\beta|}\Big) = Y_1(\beta p, q; |\beta|\rho) < \epsilon.$$

This shows that $\beta A = A(|\beta|\rho, \epsilon, F)$, hence $\beta A \in \nu$.

Remark 4.1 The topologically generated by this system i.e. the system ν on the vector space V is named $\mathbf{N_2}-$ topology on V. The above statement show that ν is base for a system neighbourhood of the origin.

Theorem 4.4 Let V be the vector space on $\mathbb{R}$, $\circ = \min$, $\diamond = \max$ and $T : V \times V \to [0, \infty)$ be a map on $V \times V$. If G_1, B_1, Y_1 are functions from $V \times V \times [0, 1]$ to $[0, 1]$ defined by

$$G_1(p, q, \rho) = \frac{\rho}{\rho + T(p, q)} \text{ and } B_1(p, q, \rho) = \frac{T(p, q)}{\rho + T(p, q)},\ Y_1(p, q, \rho) = \frac{T(p, q)}{\rho}.$$

Then:
(i) (V, T) is a 2−normed space if and only if $(V, G_1, B_1, Y_1, \circ, \diamond)$ is a neutrosophic 2− normed space.
(ii) Topologies generated by T and $(G_1, B_1, Y_1)_2$ on V are equivalent.
Proof. (**i**) We first assume that (V, T) be a 2-norm. Let $(p, q) \in V \times V,\ \rho > 0$ and $\beta \in \mathbb{R} - \{0\}$ then we have,

$$\begin{aligned}
G_1(\beta p, q; \rho) &= \frac{\rho}{\rho + T(\beta p, q)} = \frac{\rho}{\rho + |\beta| T(p, q)} = \frac{\frac{\rho}{|\beta|}}{\frac{\rho}{|\beta|} + T(p, q)} = G_1\Big(p, q; \frac{\rho}{|\beta|}\Big)\\
B_1(\beta p, q; \rho) &= \frac{T(\beta p, q)}{\rho + T(\beta p, q)} = \frac{|\beta| T(p, q)}{\rho + |\beta| T(p, q)} = \frac{T(p, q)}{\frac{\rho}{|\beta|} + T(p, q)} = B_1\Big(p, q; \frac{\rho}{|\beta|}\Big)\\
Y_1(\beta p, q; \rho) &= \frac{T(\beta p, q)}{\rho} = \frac{|\beta| T(p, q)}{\rho} = \frac{T(p, q)}{\frac{\rho}{|\beta|}} = Y_1\Big(p, q; \frac{\rho}{|\beta|}\Big).
\end{aligned}$$

Similarly,

$$G_1(p, \beta q; \rho) = G_1\Big(p, q; \frac{\rho}{|\beta|}\Big) \text{ and}$$
$$B_1(p, \beta q; \rho) = B_1\Big(p, q; \frac{\rho}{|\beta|}\Big), Y_1(p, \beta q; \rho) = Y_1\Big(p, q; \frac{\rho}{|\beta|}\Big).$$

We only prove the properties (v), (xi) and (xvi) as other properties of a neutrosophic 2-normed space can be easily obtained. Let $\exists\ \rho_1,\ \rho_2 > 0$ and $p, q, w \in V$ s.t

$$G_1(p, q + w; \rho_1 + \rho_2) < G_1(p, q; \rho_1) \circ G_1(p, w; \rho_2) = \min\Big\{\frac{\rho_1}{\rho_1 + T(p, q)}, \frac{\rho_2}{\rho_2 + T(p, w)}\Big\}.$$

It follows:

$$\frac{\rho_1 + \rho_2}{\rho_1 + \rho_2 + T(p, q + w)} < \frac{\rho_1}{\rho_1 + T(p, q)};$$
$$\frac{\rho_1 + \rho_2}{\rho_1 + \rho_2 + T(p, q + w)} < \frac{\rho_2}{\rho_2 + T(p, w)}.$$

$$\text{Hence} \qquad \begin{aligned} &(\rho_1 + \rho_2) T(p, q) < \rho_1 T(p, q + w);\\ &(\rho_1 + \rho_2) T(p, w) < \rho_2 T(p, q + w). \end{aligned}$$

By addition
$\Rightarrow (\rho_1 + \rho_2)\ (T(p, q) + T(p, w)) < (\rho_1 + \rho_2)\ T(p, q + w)$
$\Rightarrow T(p, q) + T(p, w) < T(p, q + w)$.
Which is contradiction as T is 2-norm. Thus,

$$G_1(p, q + w; \rho_1 + \rho_2) \geq \min\Big\{G_1(p, q; \rho_1), G_1(p, w; \rho_2)\ \forall\ p, q, w \in V, \rho_1, \rho_2 > 0\Big\}.$$

Further,

$$B_1(p,q+w;\rho_1+\rho_2) > B_1(p,q;\rho_1) \diamond B_1(p,w;\rho_2) = \max\left\{\frac{\rho_1}{\rho_1+T(p,q)}, \frac{\rho_2}{\rho_2+T(p,w)}\right\},$$

it follows,

$$\frac{T(p,q+w)}{\rho_1+\rho_2+T(p,q+w)} > \frac{T(p,q)}{\rho_1+T(p,q)};$$
$$\frac{T(p,q+w)}{\rho_1+\rho_2+T(p,q+w)} > \frac{T(p,w)}{\rho_2+T(p,w)}.$$

Hence $\quad \rho_1\, T(p,q+w) > (\rho_1+\rho_2)T(p,q);$
$\rho_2 T(p,q+w) > (\rho_1+\rho_2)T(p,w).$

By addition
$\Rightarrow (\rho_1+\rho_2)\,(T(p,q)+T(p,w)) < (\rho_1+\rho_2)\,T(p,q+w)$
$\Rightarrow T(p,q)+T(p,w) < T(p,q+w).$
Which is contradiction as T is a 2-norm. Thus, we have

$$B_1(p,q+w;\rho_1+\rho_2) \geq \max\Big\{B_1(p,q;\rho_1), B_1(p,w;\rho_2)\ \forall\ p,q,w\in V, \rho_1,\rho_2>0\Big\}.$$

Similarly,

$$Y_1(p,q+w;\rho_1+\rho_2) > Y_1(p,q;\rho_1) \diamond Y_1(p,w;\rho_2) = \max\left\{\frac{\rho_1}{\rho_1+T(p,q)}, \frac{\rho_2}{\rho_2+T(p,w)}\right\},$$

it follows,

$$\frac{T(p,q+w)}{\rho_1+\rho_2} > \frac{T(p,q)}{\rho_1};$$
$$\frac{T(p,q+w)}{\rho_1+\rho_2} > \frac{T(p,w)}{\rho_2}.$$

Hence, $\quad \rho_1\, T(p,q+w) > (\rho_1+\rho_2)T(p,q);$
$\rho_2 T(p,q+w) > (\rho_1+\rho_2)T(p,w).$

By addition
$\Rightarrow (\rho_1+\rho_2)\,(T(p,q)+T(p,w)) < (\rho_1+\rho_2)\,T(p,q+w)$
$\Rightarrow T(p,q)+T(p,w) < T(p,q+w),$
Which is contradiction as T is a 2-norm. Thus,

$$Y_1(p,q+w;\rho_1+\rho_2) \geq \max\Big\{Y_1(p,q;\rho_1), Y_1(p,w;\rho_2)\ \forall\ p,q,w\in V, \rho_1,\rho_2>0\Big\}.$$

This shows that the conditions (v), (xi) and (xvi) are verified.
Conversely, let $(G_1, B_1, Y_1)_2$ be neutrosophic 2-norm defined on $V\times V\times[0,1]$.

Since, $G_1(\beta p, q, \rho) = G_1\Big(p, q, \frac{\rho}{|\beta|}\Big)$.
Then,

$$\frac{\rho}{\rho + T(\beta p, q)} = \frac{\frac{\rho}{|\beta|}}{\frac{\rho}{|\beta|} + T(p, q)} = \frac{\rho}{\rho + |\beta| T(p, q)},$$

$$\Rightarrow T(\beta p, q) = |\beta| T(p, q).$$

Since, $G_1(p, q + w, \rho_1 + \rho_2) \geq G_1(p, q, \rho_1) \circ G_1(p, q, \rho_2)$, So

$$\frac{\rho_1 + \rho_2}{\rho_1 + \rho_2 + T(p, q + w)} \geq \min\left\{\frac{\rho_1}{\rho_1 + T(p, q)}, \ \frac{\rho_2}{\rho_2 + T(p, w)}\right\},$$
$$\frac{\rho_1 + \rho_2}{\rho_1 + \rho_2 + T(p, q + w)} \geq \frac{\rho_1}{\rho_1 + T(p, q)};$$
$$\frac{\rho_1 + \rho_2}{\rho_1 + \rho_2 + T(p, q + w)} \geq \frac{\rho_2}{\rho_2 + T(p, w)}.$$

Hence,

$$(\rho_1 + \rho_2) T(p, q) \geq \rho_1 T(p, q + w);$$
$$(\rho_1 + \rho_2) T(p, w) \geq \rho_2 T(p, q + w).$$

By addition, we have
$T(p, q + w) \geq T(p, q) + T(p, w)$.
Further,
$B_1(p, q + w, \rho_1 + \rho_2) \leq B_1(p, q, \rho_1) \diamond B_1(p, w, \rho_2)$
gives
$T(p, q + w) \leq T(p, q) + T(p, w) \ \forall \ p, q, w \in V$,
and
$Y_1(p, q + w, \rho_1 + \rho_2) \leq Y_1(p, q, \rho_1) \diamond Y_1(p, w, \rho_2)$
leads to
$T(p, q + w) \leq T(p, q) + T(p, w) \ \forall \ p, q, w \in V$. This shows that T is a 2-norm.
(**ii**). Taking the following inequalities are equivalent:

$$G_1(p, q; \rho) > 1 - \eta \text{ and } B_1(p, q; \rho) < \eta, \ Y_1(p, q; \rho) < \eta,$$

if and only if

$$\frac{\rho}{\rho + T(p, q)} > 1 - \eta \text{ and } \frac{T(p, q)}{\rho + T(p, q)} < \eta, \ \frac{T(p, q)}{\rho} < \eta$$

Now, $\qquad T(p, q) < \dfrac{\rho \eta}{1 - \eta}$.

Since,
$\rho > 0$, so for every $0 < \eta < 1$, ρ we can write in terms of η as $\rho = \frac{1}{\eta} - 1 > 0$.
Hence, $T(p, q) < \frac{1-\eta}{\eta} \times \frac{\eta}{1-\eta}$
$\Rightarrow T(p, q) < 1$. □

5. Conclusion

There exists situations where the exact distance between two points and the exact length of a point are not possible to evaluate due to huge uncertainty. If this uncertainty is due to fuzziness instead of randomness, we look forward to a new approach of metric and norm, respectively, called the fuzzy metric and the fuzzy norm. Recently, [28] presented an advanced version of fuzzy norm for the treatment of those problems of fuzzy functional analysis which can't be modeled via fuzzy norm due to indeterminacy. In present work, we define neutrosophic 2-boundedness, neutrosophic 2- compactness in neutrosophic 2-normed spaces which will provides a large framework to modeled these kind of problems. We also define neutrosophic 2-boundedness, neutrosophic 2-continuity, neutrosophic 2-topological isomorphism for operators on neutrosophic 2-normed spaces and study some of their properties in a more general setting under neutrosophic environment.

6. Acknowledgment

We would like to thanks the reviewers of the paper for their valuable comments and suggestions which improve the presentation of the paper.

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