Filomat 37:30 (2023), 10209–10218
https://doi.org/10.2298/FIL2330209A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Matrix transforms of subspaces of summability domains of matrices defined by speed of convergence

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Abstract.
Let X, Y be two subspaces of summability domains of matrices with real or complex entries determined by speeds of the convergence, i.e.; by monotonically increasing positive sequences λ and μ. In this paper, we give necessary and sufficient conditions for a matrix M (with real or complex entries) to transform X into Y, where X is the subspace of summability domain of a λ-reversible or a normal or a λ-perfect matrix A defined by the speed λ and Y is the subspace of a lower triangular matrix B defined by the speed μ.

1. Introduction

Let X, Y be two sequence spaces and $M = (m_{nk})$ be an arbitrary matrix with real or complex entries. Throughout this paper we assume that indices and summation indices run from 0 to ∞ unless otherwise specified. If for each $x = (x_k) \in X$ the series

$$M_n x = \sum_k m_{nk} x_k$$

converge and the sequence $Mx = (M_n x)$ belongs to Y, we say that the matrix M transforms X into Y. By (X, Y) we denote the set of all matrices, which transform X into Y. Let c and m be correspondingly the spaces of all convergent and bounded sequences, and c_0 the space of all sequences converging to zero.

Let throughout this paper $\lambda = (\lambda_k)$ be a positive monotonically increasing sequence, i.e.; the speed of convergence. Following Kangro ([13], [14]), a convergent sequence $x = (x_k)$ with

$$\lim_k x_k := \xi(x) \ \text{ and } \ l_k(x) = \lambda_k (x_k - \xi(x)) \tag{1.1}$$

is called bounded with the speed λ (shortly, λ-bounded) if $l_k(x) = O_x(1)$ (or $(l_k(x)) \in m$), and convergent with the speed λ (shortly, λ-convergent) if the finite limit

$$\lim_k l_k(x) := b(x)$$

2020 *Mathematics Subject Classification*. Primary 40C05; Secondary 40H05, 41A25.
Keywords. Matrix transforms; Normal; λ-reversible and λ-perfect matrices; Summability and boundedness with speed.
Received: 05 June 2023; Revised 25 June 2023; Accepted: 26 June 2023
Communicated by Eberhard Malkowsky
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exists (or $(l_k(x)) \in c$). We denote the set of all λ-bounded sequences by m^λ, and the set of all λ-convergent sequences by c^λ. It is not difficult to see that $c^\lambda \subset m^\lambda \subset c$. In addition to it, for unbounded sequence λ these inclusions are strict. For $\lambda_k = O(1)$ we get $c^\lambda = m^\lambda = c$.

Let $A = (a_{nk})$ be a matrix with real or complex entries. A sequence $x = (x_k)$ is said to be A^λ-bounded (A^λ-summable), if $Ax \in m^\lambda$ ($Ax \in c^\lambda$, respectively). The set of all A^λ-bounded sequences will be denoted by m_A^λ, and the set of all A^λ-summable sequences by c_A^λ. Let c_A be the summability domain of A, i.e.; the set of sequences x (with real or complex entries), for which the finite limit $\lim_n A_n x$ exists. It is easy to see that $c_A^\lambda \subset m_A^\lambda \subset c_A$, and, if λ is a bounded sequence, then $m_A^\lambda = c_A^\lambda = c_A$.

Let $\mu = (\mu_n)$ be another speed of convergence and $B = (b_{nk})$ a lower triangular matrix,

$$m_0^\mu = \{x = (x_k) \,:\, x \in m^\mu \cap c_0\},$$

$$c_0^\mu := \{x = (x_n) \,:\, x \in c^\mu \text{ and } \lim_n \mu_n(x_n - \xi(x)) = 0\},$$

and

$$(c_0)_B^\mu := \{x \in c_B^\mu \,:\, Bx \in c_0^\mu\}.$$

A matrix A is said to be normal if it is lower triangular and $a_{nn} \neq 0$ for every n, and λ-reversible, if the infinite system of equations $z_n = A_n x$ has a unique solution, for each sequence $(z_n) \in c^\lambda$. It is not difficult to see that every normal matrix is λ-reversible. Let $e = (1, 1, ...)$, $e^k = (0, ..., 0, 1, 0, ...)$, where 1 is in the k-th position, and $\lambda^{-1} = (1/\lambda_k)$. A matrix A is called λ-perfect, if A is λ-conservative (i.e.; $A \in (c^\lambda, c^\lambda)$) and the set $\{e, e^k, \lambda^{-1}\}$ is fundamental in c_A^λ.

The sets (m^λ, m^μ) and (c^λ, c^μ) have been described correspondingly in [14] and [13], and the set (c^λ, m^μ) in [12] and [15]. The sets $\left(m^\lambda, c^\mu\right)$, $\left(m^\lambda, m_0^\mu\right)$, $\left(m^\lambda, c_0^\mu\right)$, $\left(c^\lambda, m_0^\mu\right)$, $\left(c^\lambda, c_0^\mu\right)$, $\left(m_0^\lambda, m^\mu\right)$, $\left(m_0^\lambda, m_0^\mu\right)$, $\left(m_0^\lambda, c^\mu\right)$, $\left(m_0^\lambda, c_0^\mu\right)$, $\left(c_0^\lambda, m^\mu\right)$, $\left(c_0^\lambda, m_0^\mu\right)$, $\left(c_0^\lambda, c^\mu\right)$ and $\left(c_0^\lambda, c_0^\mu\right)$ have beeb characterized in [2].

The notion of absolute convergence of sequences wirh speed λ has been introduced in [1]. Matrix transforms of the set of absolutely λ-convergent sequences have been studied also in [1].

For a normal matrix A necessary and sufficient conditions for $M \in \left(m_A^\lambda, m_B^\mu\right)$ have been proved in [7], and for $M \in \left(m_A^\lambda, c_B^\mu\right)$ or $M \in \left(m_A^\lambda, (c_0)_B^\mu\right)$ in [3]. For a λ-reversible or λ-perfect matrix A necessary and sufficient conditions for $M \in \left(c_A^\lambda, c_B^\mu\right)$ and $M \in \left(c_A^\lambda, m_B^\mu\right)$ have been found in [8]. A short overview on matrix transforms of sequence spaces and subspaces of summability domains of matrices determined by speeds of convergence has been presented in [4] and [15].

We note that the results connected with convergence, absolute convergence, boundedness, A^λ-boundedness and A^λ-summability with speed can be used in several applications, for example in the approximation theory. Besides, one author of the present paper used such results for the estimation of the order of approximation of Fourier expansions in Banach spaces ([5] - [7], [9]).

In this paper we extend the studies stated in [7], [8], [3]. We prove necessary and sufficient conditions for $M \in \left(c_A^\lambda, (c_0)_B^\mu\right)$, $M \in \left((c_0)_A^\lambda, m_B^\mu\right)$, $M \in \left((c_0)_A^\lambda, c_B^\mu\right)$ and $M \in \left((c_0)_A^\lambda, (c_0)_B^\mu\right)$, if A is a normal or λ-reversible matrix. For $M \in \left(c_A^\lambda, (c_0)_B^\mu\right)$ we give necessary and sufficient conditions also for a λ-perfect matrix A.

2. Auxiliary results

For the proof of the main results we need some auxiliary results.

Lemma 2.1 ([10], p. 44, see also [16], Proposition 12). *A matrix $A = (a_{nk}) \in (c_0, c)$ if and only if conditions*

$$\textit{there exist the finite limits } \lim_n a_{nk} := a_k \textit{ for all } k \tag{2.1}$$

and

$$\sum_k |a_{nk}| = O(1) \tag{2.2}$$

are satisfied. Moreover,

$$\lim_n A_n x = \sum_k a_k x_k. \tag{2.3}$$

Lemma 2.2 ([10], pp. 44-45, see also [16], Proposition 23). *We have $A = (a_{nk}) \in (c_0, c_0)$ if and only if conditions* (2.1) *and* (2.2) *with $a_k = 0$ are satisfied.*

Lemma 2.3 ([10], p. 42, see also [16], Proposition 1). *We have $A = (a_{nk}) \in (c_0, m)$ if and only if condition* (2.2) *is satisfied.*

A sequence space X is called an FK-space, if X is an F-space (i.e., a complete space with countable system of half-norms separating points in X), where coordinate-wise convergence holds.

Lemma 2.4 ([11], see also [15], [18]). *The domain c_A^λ is an FK-space for an arbitrary matrix A.*

Let $A = (a_{nk})$ be a λ-reversible matrix. For $x \in c_A^\lambda$, let

$$\phi := \lim_n A_n x, \; d_n = \lambda_n \left(A_n x - \phi\right), \; d := \lim_n d_n,$$

and $\eta := (\eta_k)$, $\varphi := (\varphi_k)$ and $\eta_j := (\eta_{kj})$, for each fixed j, are the solutions of the system $y = Ax$ corresponding to $y = (\delta_{nn})$, $y = (\delta_{nn}/\lambda_n)$ and $y = (y_n) = (\delta_{nj})$ (where $\delta_{nj} = 1$ if $n = j$, and $\delta_{nj} = 0$ if $n \neq j$).

Lemma 2.5 ([4], Corollary 9.1). *Let $A = (a_{nk})$ be a λ-reversible matrix. Then every coordinate x_k of a sequence $x := (x_k) \in c_A^\lambda$ can be represented in the form*

$$x_k = \phi\eta_k + d\varphi_k + \sum_n \frac{\eta_{kn}}{\lambda_n}(d_n - d), \; \sum_n \left|\frac{\eta_{kn}}{\lambda_n}\right| < \infty \textit{ for every fixed } k. \tag{2.4}$$

Remark 2.1. If $x \in (c_0)_A^\lambda$, then $d = 0$. Hence every coordinate x_k of a sequence $x = (x_k) \in (c_0)_A^\lambda$ can be represented in the form

$$x_k = \phi\eta_k + \sum_n \frac{\eta_{kn}}{\lambda_n} d_n, \; \sum_n \left|\frac{\eta_{kn}}{\lambda_n}\right| < \infty \textit{ for every fixed } k. \tag{2.5}$$

Lemma 2.6 ([12], Theorems 7.3 and 7.8, see also [15], p. 138-140). *Let A be an arbitrary matrix. Then $c^\lambda \subset m_A^\mu$ if and only if $m^\lambda \subset m_A^\mu$.*

3. Matrix transforms for λ-reversible matrices

In this section we characterize the sets $\left(c_A^\lambda, (c_0)_B^\mu\right)$, $\left((c_0)_A^\lambda, m_B^\mu\right)$, $\left((c_0)_A^\lambda, c_B^\mu\right)$ and $\left((c_0)_A^\lambda, (c_0)_B^\mu\right)$ if A is a λ-reversible matrix. First we present necessary and sufficient conditions for existence of the transformation $y = Mx$ for every $x \in (c_0)_A^\lambda$ and for every $x \in c_A^\lambda$.

Proposition 3.1. *Let $A = (a_{nk})$ be a λ-reversible matrix and $M = (m_{nk})$ an arbitrary matrix. Then the transformation $y = Mx$ exists for every $x \in (c_0)_A^\lambda$ if and only if*

$$\textit{there exist finite limits } \lim_j h_{jl}^n := h_{nl} \textit{ for every fixed } l \textit{ and } n, \tag{3.1}$$

$$\sum_l \frac{\left|h_{jl}^n\right|}{\lambda_l} = O_n(1) \textit{ for every fixed } n, \tag{3.2}$$

and

$$\sum_k m_{nk}\eta_k < \infty \; for\, every\, fixed\, n, \tag{3.3}$$

where

$$h_{jl}^n := \sum_{k=0}^{j} m_{nk}\eta_{kl}.$$

Proof. Necessity. Assume that the transformation $y = Mx$ exists for every $x \in (c_0)_A^\lambda$. By Remark 2.1, every coordinate x_k of a sequence $x := (x_k) \in (c_0)_A^\lambda$ can be represented in the form (2.5) for every fixed k. Hence we can write

$$\sum_{k=0}^{j} m_{nk}x_k = \phi \sum_{k=0}^{j} m_{nk}\eta_k + \sum_l \frac{h_{jl}^n}{\lambda_l} d_l \tag{3.4}$$

for every sequence $x := (x_k) \in (c_0)_A^\lambda$. It is easy to see that $\eta \in (c_0)_A^\lambda$ since $e \in (c_0)_A^\lambda$ and A is λ-reversible. Consequently condition (3.3) holds.

Using (3.4), we obtain that the matrix $H_\lambda^n := \left(h_{jl}^n/\lambda_l\right)$ for every n transforms this sequence $(d_l) \in c_0$ into c. We show that H_λ^n transforms every sequence $(d_l) \in c_0$ into c. Indeed, for every sequence $(d_l) \in c_0$, the sequence $(d_l/\lambda_l) \in c_0$. But, for (d_l/λ_l), there exists a convergent sequence $z := (z_l)$ with $\phi := \lim_l z_l$, such that $d_l/\lambda_l = z_l - \phi$. Due to λ-reversibility of A for every convergent sequence $z := (z_l)$ with $\phi := \lim_l z_l$ there exists a convergent sequence x, such that $z_l = A_l x$. Thus, we have proved that, for every sequence $(d_l) \in c_0$ there exists a sequence $(x_k) \in (c_0)_A^\lambda$ such that $d_l = \lambda_l \left(A_l x - \phi\right)$. Hence $H_\lambda^n \in (c_0, c)$. Therefore, by Lemma 2.1, conditions (3.1) and (3.2) are satisfied.

Sufficiency. Let all conditions of the present proposition be satisfied. Then conditions (3.1) and (3.2) imply, by Lemma 2.1, that $H^n \in (c_0, c)$. Consequently, from (3.4), we can conclude, by (3.3) that the transformation $y = Mx$ exists for every $x \in (c_0)_A^\lambda$.

As the validity of (3.1) and (3.2) implies the validity of

$$\sum_l \frac{|h_{nl}|}{\lambda_l} = O_n(1) \; for\, every\, fixed\, n, \tag{3.5}$$

then from Proposition 3.1 we immediately obtain the following result.

Corollary 3.2. *Let $A = (a_{nk})$ be a λ-reversible matrix and $M = (m_{nk})$ an arbitrary matrix. If the transformation $y = Mx$ exists for every $x \in (c_0)_A^\lambda$, then condition* (3.5) holds.

Proposition 3.3. ([8], Lemma 2, see also [4], Proposition 9.5). *Let $A = (a_{nk})$ be a λ-reversible matrix and $M = (m_{nk})$ an arbitrary matrix. Then the transformation $y = Mx$ exists for every $x \in c_A^\lambda$ if and only if conditions (3.1)-(3.3) are satisfied and*

$$\sum_k m_{nk}\varphi_k < \infty \; for\, every\, fixed\, n. \tag{3.6}$$

Now we are able to prove the main results. Let $B = (b_{nk})$ be a triangular matrix and $M = (m_{nk})$ an arbitrary matrix, and $G = (g_{nk}) := BM$; i.e.,

$$g_{nk} = \sum_{l=0}^{n} b_{nl}m_{lk}.$$

Let $\Gamma^n := \left(\gamma_{nl}^j\right)$ be the lower triangular matrix for every fixed n with

$$\gamma_{nl}^j := \sum_{k=0}^{j} g_{nk}\eta_{kl}.$$

If the matrix transform $y = Mx$ exists for every $x \in c_A^\lambda$ or $x \in (c_0)_A^\lambda$, then the finite limits

$$\gamma_{nl} := \lim_j \gamma_{nl}^j$$

exist.

Theorem 3.4. *Let $A = (a_{nk})$ be a λ-reversible matrix, $B = (b_{nk})$ a triangular matrix and $M = (m_{nk})$ an arbitrary matrix. Then $M \in (c_A^\lambda, (c_0)_B^\mu)$ if and only if conditions* (3.1)-(3.3), (3.6) *are satisfied, and*

$$\sum_l \frac{\left|\gamma_{nl}\right|}{\lambda_l} = O(1), \tag{3.7}$$

$$\mu_n \sum_l \frac{\left|\gamma_{nl} - \gamma_l\right|}{\lambda_l} = O(1), \tag{3.8}$$

$$\eta, \varphi \in (c_0)_G^\mu, \tag{3.9}$$

$$e^k \in (c_0)_\Gamma^\mu;\ \Gamma := (\gamma_{nk}), \tag{3.10}$$

where

$$\gamma_l := \lim_n \gamma_{nl}.$$

Proof. Necessity. Assume that $M \in (c_A^\lambda, (c_0)_B^\mu)$. Then the transformation $y = Mx$ exists for every $x \in c_A^\lambda$. Hence conditions (3.1) - (3.3) and (3.6) hold by Proposition 3.3, and

$$B_n y = G_n x \tag{3.11}$$

for every $x \in c_A^\lambda$ because the change of the order of summation is allowed by the lower triangularity of B. This implies that $c_A^\lambda \subset (c_0)_G^\mu$. Hence condition (3.9) is satisfied because $\eta, \varphi \in c_A^\lambda$. Since every element x_k of a sequence $x := (x_k) \in c_A^\lambda$ may be written in the form (2.4) by Lemma 2.5, we can write

$$\sum_{k=0}^{j} g_{nk}x_k = \phi \sum_{k=0}^{j} g_{nk}\eta_k + d \sum_{k=0}^{j} g_{nk}\varphi_k + \sum_l \frac{\gamma_{nl}^j}{\lambda_l}(d_l - d) \tag{3.12}$$

for every $x \in c_A^\lambda$. From (3.12) it follows, by (3.9), that $\Gamma_\lambda^n := \left(\gamma_{nl}^j / \lambda_l\right) \in (c_0, c)$ for every n because A is λ-reversible (see a proof of the necessity of Proposition 3.1). Moreover, from conditions (3.1) - (3.3) and (3.6), we can conclude that the series $G_n x$ are convergent for every $x \in c_A^\lambda$. Therefore there exist the finite limits γ_{nl}, and from (3.12) we obtain

$$G_n x = \phi G_n \eta + d G_n \varphi + \sum_l \frac{\gamma_{nl}}{\lambda_l}(d_l - d) \tag{3.13}$$

for every $x \in c_A^\lambda$. From (3.13) we see, with the help of (3.9) that $\Gamma_\lambda := (\gamma_{nl}/\lambda_l) \in (c_0, c)$. Consequently, condition (3.7) holds, the finite limits γ_l exist and for every $x \in c_A^\lambda$ we obtain due by Lemma 2.1 that

$$\lim_n G_n x = \phi\gamma + d\psi + \sum_l \frac{\gamma_l}{\lambda_l}(d_l - d), \tag{3.14}$$

where

$$\gamma := \lim_n G_n\eta;\ \ \psi := \lim_n G_n\varphi.$$

Therefore we can write

$$\mu_n(G_nx - \lim_n G_nx) = \phi\mu_n(G_n\eta - \gamma) + d\mu_n(G_n\varphi - \psi) + \mu_n \sum_l \frac{\gamma_{nl} - \gamma_l}{\lambda_l}(d_l - d) \tag{3.15}$$

for every $x \in c_A^\lambda$. With the help of (3.9), it follows from (3.15) that the matrix $\Gamma_{\lambda,\mu} := (\mu_n(\gamma_{nl} - \gamma_l)/\lambda_l) \in (c_0, c_0)$. Hence, using Lemma 2.2, we conclude that conditions (3.8) and (3.10) are satisfied.

Sufficiency. We assume that all of the conditions of the present theorem hold. Then the matrix transformation $y = Mx$ exists for every $x \in c_A^\lambda$ by Proposition 3.3. This implies that relations (3.11) and (3.12) hold for every $x \in c_A^\lambda$ (see the proof of the necessity). Using (3.7) and (3.10), we conclude, with the help of Lemma 2.1, that $\Gamma_\lambda^n \in (c_0, c)$ for every n, one can take the limit under the summation sign in the last summand of (3.12). Then, from (3.12), we obtain by (3.9), the validity of (3.13) for every $x \in c_A^\lambda$. Conditions (3.7), (3.9) and (3.10) imply that (3.14) holds for every $x \in c_A^\lambda$, due to Lemma 2.1. Then relation (3.15) also holds for every $x \in c_A^\lambda$. Moreover, $\Gamma_{\lambda,\mu} \in (c_0, c_0)$, by Lemma 2.2. Therefore $M \in (c_A^\lambda, (c_0)_B^\mu)$ by (3.9).

The proofs of the next results (Theorems 3.5-3.7) are similar to the proof of Theorem 3.4. Therefore we only give a short description of the proofs. In these results always every element x_k of a sequence $x := (x_k) \in (c_0)_A^\lambda$ may be presented in the form (2.5) by Remark 2.1. Hence the relations (3.12) - (3.15) hold with $d = 0$. In addition, always $\Gamma_\lambda^n \in (c_0, c)$ and $\Gamma_\lambda \in (c_0, c)$. But the role of the matrix $\Gamma_{\lambda,\mu}$ is different: in the proof of Theorem 3.5, $\Gamma_{\lambda,\mu} \in (c_0, m)$, in the proof of Theorem 3.6, $\Gamma_{\lambda,\mu} \in (c_0, c)$, and in the proof of Theorem 3.7, $\Gamma_{\lambda,\mu} \in (c_0, c_0)$. Therefore, for completing the proof of Theorem 3.5 it is necessary to use Lemmas 2.1 and 2.3, to conclude the proof of Theorem 3.6 - Lemma 2.1, and for completing the proof of Theorem 3.7 - Lemmas 2.1 and 2.2.

Theorem 3.5. *Let $A = (a_{nk})$ be a λ-reversible matrix, $B = (b_{nk})$ a triangular matrix and $M = (m_{nk})$ an arbitrary matrix. Then $M \in ((c_0)_A^\lambda, m_B^\mu)$ if and only if conditions* (3.1)-(3.3), (3.7), (3.8) *are satisfied, and*

$$\textit{there exist the finite limits } \gamma_l := \lim_n \gamma_{nl}, \tag{3.16}$$

$$\eta \in m_G^\mu. \tag{3.17}$$

Theorem 3.6. *Let $A = (a_{nk})$ be a λ-reversible matrix, $B = (b_{nk})$ a triangular matrix and $M = (m_{nk})$ an arbitrary matrix. Then $M \in ((c_0)_A^\lambda, c_B^\mu)$ if and only if conditions* (3.1)-(3.3), (3.7), (3.8), (3.16) *are satisfied, and*

$$\eta \in c_G^\mu, \tag{3.18}$$

$$e^k \in c_\Gamma^\mu;\ \Gamma := (\gamma_{nk}). \tag{3.19}$$

Theorem 3.7. *Let $A = (a_{nk})$ be a λ-reversible matrix, $B = (b_{nk})$ a triangular matrix and $M = (m_{nk})$ an arbitrary matrix. Then $M \in ((c_0)_A^\lambda, (c_0)_B^\mu)$ if and only if conditions* (3.1)-(3.3), (3.7), (3.8), (3.10), (3.16) *are satisfied, and*

$$\eta \in (c_0)_G^\mu. \tag{3.20}$$

Corollary 3.8. *Condition* (3.7) *can be replaced by condition*

$$\sum_l \frac{|\gamma_l|}{\lambda_l} < \infty \tag{3.21}$$

in Theorems 3.1-3.4.

Proof. It is easy to see that condition (3.21) follows from (3.7) and (3.10). In the same way, conditions (3.8), (3.10) and (3.21) imply the validity of (3.7). Indeed, first from condition (3.8) we obtain that

$$\sum_l \frac{|\gamma_{nl} - \gamma_l|}{\lambda_l} = O(1), \tag{3.22}$$

since (μ_n) is bounded from below. Since

$$\frac{\gamma_{nl}}{\lambda_l} = \frac{\gamma_{nl} - \gamma_l}{\lambda_l} + \frac{\gamma_l}{\lambda_l},$$

then

$$\sum_l \frac{|\gamma_{nl}|}{\lambda_l} \leq \sum_l \frac{|\gamma_{nl} - \gamma_l|}{\lambda_l} + \sum_l \frac{|\gamma_l|}{\lambda_l}.$$

Moreover, the finite limits γ_l exist by (3.10). Hence the condition (3.7) is satisfied by (3.21) and (3.22).

4. Corollaries for normal matrices

In this section we characterize the sets $\left(c_A^\lambda, (c_0)_B^\mu\right)$, $\left((c_0)_A^\lambda, m_B^\mu\right)$, $\left((c_0)_A^\lambda, c_B^\mu\right)$ and $\left((c_0)_A^\lambda, (c_0)_B^\mu\right)$ if A is a normal matrix. It is known (see [4], p. 90) that for a normal matrix A, $A^{-1} := (\eta_{kl})$ is the inverse matrix of A, and between η_k and A^{-1} the relationship

$$\eta_k = \sum_{l=0}^{k} \eta_{kl} \tag{4.1}$$

holds (in this case A^{-1} is also normal). Therefore in the present case the matrices $H^n := \left(h_{jl}^n\right)$ and $\Gamma^n := \left(\gamma_{nl}^j\right)$ are lower triangular with

$$h_{jl}^n = \sum_{k=l}^{j} m_{nk}\eta_{kl} \text{ and } \gamma_{nl}^j = \sum_{k=l}^{j} g_{nk}\eta_{kl}, \; l \leq j.$$

Also for a normal matrix A we have (see [4], p. 196)

$$\varphi_k = \sum_{l=0}^{k} \frac{\eta_{kl}}{\lambda_l}. \tag{4.2}$$

From Theorem 3.4 we obtain the following corollary.

Corollary 4.1. *Let $A = (a_{nk})$ be a normal matrix, $B = (b_{nk})$ a triangular matrix and $M = (m_{nk})$ an arbitrary matrix. Then $M \in (c_A^\lambda, (c_0)_B^\mu)$ if and only if conditions* (3.1), (3.2), (3.7), (3.8), (3.10) *are satisfied, and*

$$\lim_j \sum_{l=0}^{j} h_{jl}^n \text{ exists and is finite}, \tag{4.3}$$

$$(\rho_n) \in c_0^\mu, \tag{4.4}$$

$$\left(\gamma_\lambda^n\right) \in c_0^\mu, \tag{4.5}$$

where

$$\rho_n := \lim_j \sum_{l=0}^{j} \gamma_{nl}^j, \; \gamma_\lambda^n := \lim_j \sum_{l=0}^{j} \frac{\gamma_{nl}^j}{\lambda_l}.$$

Proof. Necessity. Using relations (4.1) and (4.2) we obtain

$$\sum_{l=0}^{j} m_{nl}\eta_l = \sum_{l=0}^{j} h_{jl}^n, \ \sum_{l=0}^{j} m_{nl}\varphi_l = \sum_{l=0}^{j} \frac{h_{jl}^n}{\lambda_l},$$

$$\sum_{l=0}^{j} g_{nl}\eta_l = \sum_{l=0}^{j} \gamma_{nl}^j, \ \sum_{l=0}^{j} g_{nl}\varphi_l = \sum_{l=0}^{j} \frac{\gamma_{nl}^j}{\lambda_l}.$$

Moreover, it follows from (4.3) that

$$\lim_j \sum_{l=0}^{j} \frac{h_{jl}^n}{\lambda_l}$$

exists and is finite, by the well-known theorem of Dedekind, since the sequence λ^{-1} is monotonically decreasing and bounded. Therefore condition (4.3) is equivalent to (3.3) and (3.6), and conditions (4.4) and (4.5) are equivalent to (3.9). Thus, $M \in (c_A^\lambda, (c_0)_B^\mu)$ by Theorem 3.4.

As we have seen in Corollary 4.1 for a normal matrix A it is possible to obtain from Theorems 3.5-3.7 corollaries correspondingly for the sets $\left((c_0)_A^\lambda, m_B^\mu\right)$, $\left((c_0)_A^\lambda, c_B^\mu\right)$ and $\left((c_0)_A^\lambda, (c_0)_B^\mu\right)$. As the proofs of these corollaries are similar to the proof of Corollary 4.1, we omit them.

Corollary 4.2. *Let $A = (a_{nk})$ be a normal matrix, $B = (b_{nk})$ a triangular matrix and $M = (m_{nk})$ an arbitrary matrix. Then $M \in (c_0)_A^\lambda, m_B^\mu)$ if and only if conditions* (3.1), (3.2), (3.7), (3.8), (3.16), (4.3) *are satisfied, and*

$$(\rho_n) \in m^\mu. \tag{4.6}$$

Corollary 4.3. *Let $A = (a_{nk})$ be a normal matrix, $B = (b_{nk})$ a triangular matrix and $M = (m_{nk})$ an arbitrary matrix. Then $M \in (c_0)_A^\lambda, c_B^\mu)$ if and only if conditions* (3.1), (3.2), (3.7), (3.8), (3.16), (3.19), (4.3) *are satisfied, and*

$$(\rho_n) \in c^\mu. \tag{4.7}$$

Corollary 4.4. *Let $A = (a_{nk})$ be a normal matrix, $B = (b_{nk})$ a triangular matrix and $M = (m_{nk})$ an arbitrary matrix. Then $M \in (c_0)_A^\lambda, c_0)_B^\mu)$ if and only if conditions* (3.1), (3.2), (3.7), (3.8), (3.10), (3.16), (4.3) *and* (4.4) *are satisfied.*

Let now $A = (a_{nk})$ be a normal matrix satisfying the property $a_{n0} = 1$ for every n. Then (see [4], Lemma 7.3) $\eta_k = \delta_{k0}$. Hence, by (4.1) we obtain

$$\sum_{l=0}^{j} h_{jl}^n = m_{n0} \text{ and } \sum_{l=0}^{j} \gamma_{nl}^j = g_{n0}.$$

Therefore we immediately get the following result.

Corollary 4.5. *Let $A = (a_{nk})$ be a normal matrix satisfying the property $a_{n0} = 1$ for every n. Then condition* (4.3) *is redundant in Corollaries* 4.1-4.4, *condition* (4.4) *in Corollaries* 4.1 *and* 4.4 *can be replaced by condition $e^0 \in (c_0)_G^\mu$, and condition* (4.6) *in Corollary* 4.2 *can be replaced by condition $e^0 \in m_G^\mu$ and in Corollary* 4.3 *by condition $e^0 \in c_G^\mu$.*

Remark 4.1. If M is triangular, then conditions (3.1) - (3.6) are redundant in Theorems 3.1-3.4 and in Corollaries 4.1-4.4.

5. Matrix transforms for λ-perfect matrices

In this section we characterize the set $\left(c_A^\lambda, (c_0)_B^\mu\right)$ for a λ-perfect matrix A and a triangular matrix B.

Theorem 5.1. *Let $A = (a_{nk})$ be a λ-perfect matrix, $B = (b_{nk})$ a triangular matrix and $M = (m_{nk})$ an arbitrary matrix. Then $M \in (c_A^\lambda, c_0)_B^\mu)$ if and only if*

$$c_A^\lambda \subset m_G^\mu, \tag{5.1}$$

$$e^k, e, \lambda^{-1} \in c_0)_G^\mu. \tag{5.2}$$

Proof. Necessity. Let $M \in (c_A^\lambda, c_0)_B^\mu)$. This implies that $M \in (c_A^\lambda, m_B^\mu)$, and relation (3.11) holds for every $x \in c_A^\lambda$ since B is triangular. Hence conditions (5.1) and (5.2) are satisfied since $e^k, e, \lambda^{-1} \in c_A^\lambda$.
Sufficiency. Assume that conditions (5.1) and (5.2) are satisfied. Then f_n, defined by

$$f_n(x) := \mu_n(G_n x - \lim_n G_n x),$$

is a continuous and linear functional on c_A^λ. Moreover, the sequence (f_n) is bounded for every $x \in c_A^\lambda$. As, in addition, c_A^λ and c_B^λ are FK-spaces by Lemma 2.4, then (see [17], p.2 or [15], Corollary 4.22) the set

$$L := \{x \in c_A^\lambda : \lim_n f_n(x) = 0\}$$

is closed on c_A^λ. It follows from condition (5.2) that

$$\mathrm{lin}\{e^k, e, \lambda^{-1}\} \subset L.$$

Hence due to the λ-perfectness of A, we obtain

$$c_A^\lambda = cl(\mathrm{lin}\{e^k, e, \lambda^{-1}\}).$$

We conclude $c_A^\lambda \subset clL = L \subset c_0)_G^\mu$ and $M \in (c_A^\lambda, c_0)_B^\mu)$.

Corollary 5.2. *If A is a normal matrix, and B and M are triangular matrices, then condition* (5.1) *in Theorem* 5.1 *can be replaced by the condition $m_A^\lambda \in m_G^\mu$.*
Proof. In this case we have $\Gamma = GA^{-1}$. So we can write

$$B_n y = G_n x = \Gamma_n z$$

for every $x \in c_A^\lambda$, where the transformation $y = Mx$ exists and $z = Ax \in m^\lambda$. For each $z \in m^\lambda$ ($z \in c^\lambda$) there exists an $x \in m_A^\lambda$ ($x \in c_A^\lambda$, respectively), such that $z = Ax$ because the normal matrix A is also λ-reversible, $c^\lambda \subset m^\lambda \subset c$ and $c_A^\lambda \subset m_A^\lambda \subset c_A$. Hence $M \in (c_A^\lambda, m_B^\mu)$ is equivalent to $c^\lambda \subset m_\Gamma^\mu$, and $M \in (m_A^\lambda, m_B^\mu)$ is equivalent to $m^\lambda \subset m_\Gamma^\mu$. By Lemma 2.6 we conclude that $c^\lambda \subset m_\Gamma^\mu$ if and only if $m^\lambda \subset m_\Gamma^\mu$.

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