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Hyers-Ulam stability of Hadamard fractional stochastic differential equations

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Abstract. The current article is used to investigate the Hyers-Ulam stability (HUS) of Hadamard stochastic fractional differential equations (HSFDE) by using a version of some fixed point theorem (FPT), a technical lemma and some classical stochastic calculus tools. To show the interest of our results, we present two examples. In this manner, we generalize some recent interesting results.

1. Introduction

The Hadamard fractional derivatives (HFD) explored by J. Hadamard in 1892 (see [15]). The kernel of the integrand in the definition of fractional Hadamard derivative includes a logarithmic function with arbitrary exponent unlike the Riemann-Liouville fractional derivatives.

The field of fractional Hadamard differential equations has attracted much attention by many scientists. Numerous varieties of fractional Hadamard differential equations have been the subject of thorough study in the literature (see [1, 2, 6, 7, 9, 10, 12, 16–18, 20–22, 24]), including stability theory and associated issues.

Stochastic fractional differential equations (SFDE) are a powerful tool used to model complex real-world phenomena. For recent results on the SFDE, we refer the reader to some works (see [5, 11, 13, 14, 19, 25]). The stability analysis (SA) is a qualitative theory of differential equations. Then, the stability analysis has received necessary attention in various research domains due to their applications. In particular, existence and uniqueness results of solutions of SFDE have obtained a great perusal (see [2, 11]). Many mathematicians have studied the HUS and its varied applications in various deterministic and SFDE. For more details on this axis, see [3, 4, 13, 14, 19, 23].

In the literature, there is a few work on the HUS of HSFDE. In [8], the authors have investigated the HUS of Caputo-HSFDE using the fixed point theorem. In this sense, our paper extend the work in [8] on the case of HFD. The main advantages of our papers are as follows:
(i) investigate the HUS of HSFDE using the FPT.
(ii) generalize the work in [8].

The form of the paper are as follows: Section 2 is devoted to the basic notations and notions of HFD. In section 3, we show the HUS of HFSDE. In Section 4, we give two theoretical examples to illustrate our results. Section 5 is used to conclude our work.

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2. Basic notations

We consider a closed bounded interval $I_{\mathbf{a}} = [1, \mathbf{a}]$, $\mathbf{a} > 1$. Let $\{\Omega, \mathcal{F}, \mathbb{F}_{\mathbf{a}}, \mathbb{P}\}$, where $\mathbb{F}_{\mathbf{a}} = \{\mathbb{F}_{\varkappa}\}_{\varkappa \in I_{\mathbf{a}}}$, be a complete probability space and $W(\varkappa)$ is a standard Brownian motion.

For each $\varkappa \in I_{\mathbf{a}}$, we denote by $\mathcal{X}_{\varkappa} = L^2(\Omega, \mathbb{F}_{\varkappa}, \mathbb{P})$ the family of all $\mathbb{F}_{\varkappa}$-measurable and mean square integrable functions $\upsilon = (\upsilon_1, \cdots, \upsilon_d)^T : \Omega \to \mathbb{R}^d$ endowed with the following norm:

$$\|\upsilon\|_{ms} = \sqrt{\sum_{l=1}^{d} \mathbb{E}\left(|\upsilon_l|^2\right)} = \sqrt{\mathbb{E}\,\|\upsilon\|^2}.$$

Definition 2.1. *[17]* For some function α, the Hadamard fractional integral of order ι is given by

$$I^{\iota}\alpha(\varkappa) = \frac{1}{\Gamma(\iota)} \int_1^{\varkappa} \left(ln\frac{\varkappa}{\nu}\right)^{\iota-1} \frac{\alpha(\nu)}{\nu} d\nu, \quad \iota > 0.$$

Definition 2.2. *[17]* The HFD with order $\iota \in (0, 1)$ for a function $\alpha : [1, \infty) \to \mathbb{R}$ is given by

$${}^{H}D_1^{\iota}\alpha(\varkappa) = \frac{1}{\Gamma(1-\iota)} \left(\varkappa \frac{d}{d\varkappa}\right) \int_1^{\varkappa} \left(ln\frac{\varkappa}{\nu}\right)^{-\iota} \frac{\alpha(\nu)}{\nu} d\nu.$$

Consider the HSFDE:

$${}^{H}D_1^{\iota}\varrho(\varkappa) = \zeta_1\left(\varkappa, \varrho(\varkappa)\right) + \zeta_2\left(\varkappa, \varrho(\varkappa)\right) \frac{dW(\varkappa)}{d\varkappa}, \tag{1}$$

where the initial condition is $I^{1-\iota}\varrho(1) = \phi$, for $\phi \in \mathbb{R}^d$ and some measurable functions ζ_1, ζ_2 defined by

$$\zeta_1, \zeta_2 : I_{\mathbf{a}} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d.$$

Let us introduce the following assumptions that will be very useful later:

$\mathcal{H}_1$: There exists $\mathcal{K} > 0$:

$$\|\zeta_1(\varkappa, r_1) - \zeta_1(\varkappa, r_2)\| + \|\zeta_2(\varkappa, r_1) - \zeta_2(\varkappa, r_2)\| \le \mathcal{K}\,\|r_1 - r_2\|,$$

for all $(\varkappa, r_1, r_2) \in I_{\mathbf{a}} \times \mathbb{R}^d \times \mathbb{R}^d$.

$\mathcal{H}_2$: On the interval $I_{\mathbf{a}}$, the functions $\zeta_1(\cdot, 0)$, $\zeta_2(\cdot, 0)$ satisfy:

$$\int_1^{\mathbf{a}} \|\zeta_1(l, 0)\|^2\, dl < \infty, \quad \|\zeta_2(\cdot, 0)\|_{\infty} = \operatorname{ess\,sup}_{l \in I_{\mathbf{a}}} \|\zeta_2(l, 0)\| < \infty.$$

Theorem 2.3. *[10] Given (C, ϑ) as a complete metric space and a contraction $Q : C \to C$ (with $s \in [0, 1)$). Assume that $g \in \mathcal{T}$, $\vartheta(g, Q(g)) \le \sigma$ and $\sigma > 0$. So, there is a unique $\xi \in \mathcal{T}$ satisfies $Q(\xi) = \xi$. Moreover, we have the following identity:*

$$\vartheta(g, \xi) \le \frac{\sigma}{1-s}.$$

3. Stability results

Let $\mathbb{H}^2(I_{\mathbf{a}})$ be the family of all the processes ω which are $\mathbb{F}_{\mathbf{a}}$-adapted and measurable satisfying $\sup_{\ell \in I_{\mathbf{a}}} \left\|(ln\ell)^{1-\iota}\omega(\ell)\right\|_{ms} < \infty$. Let $\|\cdot\|_{\mathbb{H}^2}$ be the norm on $\mathbb{H}^2(I_{\mathbf{a}})$ given by:

$$\|\omega\|_{\mathbb{H}^2} = \sup_{\ell \in I_{\mathbf{a}}} \left\|(ln\ell)^{1-\iota}\omega(\ell)\right\|_{ms}.$$

Consequently, $(\mathbb{H}^2(I_{\mathbf{a}}), \|\cdot\|_{\mathbb{H}^2})$ is a Banach space. For $\phi \in \mathcal{X}_1$, consider $\mathcal{R}_\phi : \mathbb{H}^2(I_{\mathbf{a}}) \to \mathbb{H}^2(I_{\mathbf{a}})$ the operator defined as follows:

$$\begin{aligned}\mathcal{R}_\phi y(\varkappa) &= (ln\varkappa)^{\iota-1}\frac{\phi}{\Gamma(\iota)} + \frac{1}{\Gamma(\iota)}\int_1^{\varkappa}\left(ln\frac{\varkappa}{\vartheta}\right)^{\iota-1}\frac{\zeta_1(\vartheta, y(\vartheta))}{\vartheta}d\vartheta \\ &\quad + \frac{1}{\Gamma(\iota)}\int_1^{\varkappa}\left(ln\frac{\varkappa}{\vartheta}\right)^{\iota-1}\frac{\zeta_2(\vartheta, y(\vartheta))}{\vartheta}dW(\vartheta), \quad \forall y \in \mathbb{H}^2(I_{\mathbf{a}}).\end{aligned} \tag{2}$$

Lemma 3.1. *The operator $\mathcal{R}_\phi$ is well defined for all $\phi \in \mathcal{X}_1$.*

Proof. Let $y \in \mathbb{H}^2(I_{\mathbf{a}})$. Using (2), we can derive that:

$$\begin{aligned}\left\|(ln\varkappa)^{1-\iota}\mathcal{R}_\phi y(\varkappa)\right\|_{ms}^2 &\leq 3\frac{\|\phi\|_{ms}^2}{\Gamma(\iota)^2} + \frac{3}{\Gamma(\iota)^2}\mathbb{E}\left(\left\|\int_1^{\varkappa}(ln\varkappa)^{1-\iota}\left(ln\frac{\varkappa}{\delta}\right)^{\iota-1}\frac{\zeta_1(\delta, y(\delta))}{\delta}d\delta\right\|^2\right) \\ &\quad + \frac{3}{\Gamma(\iota)^2}\mathbb{E}\left(\left\|\int_1^{\varkappa}(ln\varkappa)^{1-\iota}\left(ln\frac{\varkappa}{\delta}\right)^{\iota-1}\frac{\zeta_2(\delta, y(\delta))}{\delta}dW(\delta)\right\|^2\right).\end{aligned} \tag{3}$$

Now, applying the Cauchy-Schwarz inequality and Fubini's theorem we get:

$$\begin{aligned}&\mathbb{E}\left(\left\|\int_1^{\varkappa}(ln\varkappa)^{1-\iota}(ln\vartheta)^{1-\iota}(ln\vartheta)^{\iota-1}\left(ln\frac{\varkappa}{\vartheta}\right)^{\iota-1}\frac{\zeta_1(\vartheta, y(\vartheta))}{\vartheta}d\vartheta\right\|^2\right) \\ &\leq \left(\int_1^{\varkappa}(ln\varkappa)^{2-2\iota}(ln\vartheta)^{2\iota-2}\frac{\left(ln\frac{\varkappa}{\vartheta}\right)^{2\iota-2}}{\vartheta}d\vartheta\right)\mathbb{E}\left(\int_1^{\varkappa}\left\|(ln\vartheta)^{1-\iota}\zeta_1(\vartheta, y(\vartheta))\right\|^2 d\vartheta\right).\end{aligned} \tag{4}$$

Let us denote by $T_1(\varkappa)$ the first term of the second member of the inequality (4):

$$T_1(\varkappa) = \left(\int_1^{\varkappa}(ln\varkappa)^{2-2\iota}(ln\vartheta)^{2\iota-2}\frac{\left(ln\frac{\varkappa}{\vartheta}\right)^{2\iota-2}}{\vartheta}d\vartheta\right).$$

Thanks to the change of variable $u = \frac{(ln\vartheta)}{(ln\varkappa)}$, for $\varkappa > 1$, we get:

$$T_1(\varkappa) = (ln\varkappa)^{2\iota-1}\left(\int_0^1 u^{2\iota-2}(1-u)^{2\iota-2}du\right) \leq (ln\mathbf{a})^{2\iota-1}B(2\iota-1, 2\iota-1), \tag{5}$$

where $B(\cdot,\cdot)$ is the beta function. Now using hypothesis $\mathcal{H}_1$, for the second term of the second member of the inequality (4), we have:

$$\left\|(lnl)^{1-\iota}\zeta_1(l, y(l))\right\|^2 \leq 2\mathcal{K}^2\left\|(lnl)^{1-\iota}y(l)\right\|^2 + 2(lnl)^{2-2\iota}\|\zeta_1(l,0)\|^2.$$

Therefore, we can deduce that:

$$\mathbb{E}\left(\int_1^{\varkappa}\left\|(lnl)^{1-\iota}\zeta_1(l, y(l))\right\|^2 dl\right) \leq 2\mathcal{K}^2(\mathbf{a}-1)\|y\|_{\mathbb{H}^2} + 2(ln\mathbf{a})^{2-2\iota}\int_1^{\mathbf{a}}\|\zeta_1(l,0)\|^2 dl. \tag{6}$$

Thus, using (4), (5) and (6), we have:

$$\begin{aligned}&\mathbb{E}\left(\left\|\int_1^{\varkappa}(ln\varkappa)^{1-\iota}(ln\vartheta)^{1-\iota}(ln\vartheta)^{\iota-1}\left(ln\frac{\varkappa}{\vartheta}\right)^{\iota-1}\frac{\zeta_1(\vartheta, y(\vartheta))}{\vartheta}d\vartheta\right\|^2\right) \\ &\leq 2(ln\mathbf{a})^{2\iota-1}B(2\iota-1, 2\iota-1)\left(\mathcal{K}^2(\mathbf{a}-1)\|y\|_{\mathbb{H}^2} + (ln\mathbf{a})^{2-2\iota}\int_1^{\mathbf{a}}\|\zeta_1(l,0)\|^2 dl\right).\end{aligned} \tag{7}$$

Applying the Itô's isometry formula for the third term of the second member of inequality (3), we obtain:

$$\mathbb{E}\left(\left\|\int_1^{\varkappa}(ln\varkappa)^{1-\iota}(ln\vartheta)^{1-\iota}(ln\vartheta)^{\iota-1}\left(ln\frac{\varkappa}{\vartheta}\right)^{\iota-1}\frac{\zeta_2(\vartheta,y(\vartheta))}{\vartheta}dW(\vartheta)\right\|^2\right)$$

$$=\mathbb{E}\left(\int_1^{\varkappa}(ln\varkappa)^{2-2\iota}(ln\vartheta)^{2\iota-2}\left(ln\frac{\varkappa}{\vartheta}\right)^{2\iota-2}\frac{\left\|(ln\vartheta)^{1-\iota}\zeta_2(\vartheta,y(\vartheta))\right\|^2}{\vartheta^2}d\vartheta\right). \tag{8}$$

Moreover, applying hypothesis $\mathcal{H}_1$ for inequality (8), we obtain:

$$\left\|(ln\vartheta)^{1-\iota}\zeta_2(\vartheta,y(\vartheta))\right\|^2\leq 2\mathcal{K}^2\left\|(ln\vartheta)^{1-\iota}y(\vartheta)\right\|^2+2(ln\vartheta)^{2-2\iota}\|\zeta_2(\cdot,0)\|_\infty^2. \tag{9}$$

Therefore, plugging (9) into (8), it yields that:

$$\mathbb{E}\left(\left\|\int_1^{\varkappa}(ln\varkappa)^{1-\iota}(ln\vartheta)^{1-\iota}(ln\vartheta)^{\iota-1}\left(ln\frac{\varkappa}{\vartheta}\right)^{\iota-1}\frac{\zeta_2(\vartheta,y(\vartheta))}{\vartheta}dW(\vartheta)\right\|^2\right)$$

$$\leq 2\mathcal{K}^2\left\|y\right\|_{\mathbb{H}^2}\left(\int_1^{\varkappa}(ln\varkappa)^{2-2\iota}(ln\vartheta)^{2\iota-2}\left(ln\frac{\varkappa}{\vartheta}\right)^{2\iota-2}\frac{d\vartheta}{\vartheta^2}\right)$$

$$+2\|\zeta_2(\cdot,0)\|_\infty^2\left(\int_1^{\varkappa}(ln\varkappa)^{2-2\iota}\left(ln\frac{\varkappa}{\vartheta}\right)^{2\iota-2}\frac{d\vartheta}{\vartheta^2}\right).$$

It is not hard to see that:

$$\left(\int_1^{\varkappa}(ln\varkappa)^{2-2\iota}(ln\vartheta)^{2\iota-2}\left(ln\frac{\varkappa}{\vartheta}\right)^{2\iota-2}\frac{d\vartheta}{\vartheta^2}\right)\leq T_1(\varkappa).$$

Moreover, we have:

$$\begin{aligned}\int_1^{\varkappa}(ln\varkappa)^{2-2\iota}\left(ln\frac{\varkappa}{\vartheta}\right)^{2\iota-2}\frac{1}{\vartheta^2}d\vartheta &\leq (ln\mathbf{a})^{2-2\iota}\int_1^{\varkappa}\left(ln\frac{\varkappa}{\vartheta}\right)^{2\iota-2}\frac{1}{\vartheta}d\vartheta,\\ &\leq \frac{(ln\mathbf{a})}{2\iota-1}.\end{aligned}$$

Hence, we can easily deduce that:

$$\mathbb{E}\left(\left\|\int_1^{\varkappa}(ln\varkappa)^{1-\iota}(ln\delta)^{1-\iota}(ln\delta)^{\iota-1}\left(ln\frac{\varkappa}{\delta}\right)^{\iota-1}\frac{\zeta_2(\delta,y(\delta))}{\delta}dW(\delta)\right\|^2\right)$$

$$\leq 2\mathcal{K}^2(ln\mathbf{a})^{2\iota-1}B(2\iota-1,2\iota-1)\left\|y\right\|_{\mathbb{H}^2}+2\frac{(ln\mathbf{a})}{2\iota-1}\|\zeta_2(\cdot,0)\|_\infty^2. \tag{10}$$

Therefore, by (3), (7) and (10), we can prove that the operator $\mathcal{R}_\phi$ is well defined. □

Let us introduce the following notation:

$$\mathcal{U}(\zeta,\varkappa,\delta,z(\delta))=\frac{1}{\Gamma(\iota)}\left(ln\frac{\varkappa}{\delta}\right)^{\iota-1}\frac{\zeta(\delta,z(\delta))}{\delta},\quad\forall\delta\in[1,\varkappa].$$

Definition 3.2. *Equation (1) is Hyers-Ulam Stable with respect to ϵ (HUSwrϵ), if there is a number $\mathcal{M}>0$ satisfying: for each solution $z\in\mathbb{H}^2(I_{\mathbf{a}})$ of the system (11), for each $\epsilon>0$ and for all $\varkappa\in I_{\mathbf{a}}$, we have*

$$\begin{cases}\mathbb{E}\left\|(ln\varkappa)^{1-\iota}\left(z(\varkappa)-(ln\varkappa)^{\iota-1}\frac{\phi}{\Gamma(\iota)}-\left(\int_1^{\varkappa}\mathcal{U}(\zeta_1,\varkappa,\vartheta,z(\vartheta))d\vartheta+\int_1^{\varkappa}\mathcal{U}(\zeta_2,\varkappa,\vartheta,z(\vartheta))dW(\vartheta)\right)\right)\right\|^2\leq\epsilon,\\ I^{1-\iota}z(1)=\phi,\ (initial\ condition),\end{cases} \tag{11}$$

there exists a solution $\varrho\in\mathbb{H}^2(I_{\mathbf{a}})$ of (1), with initial condition $I^{1-\iota}\varrho(1)=\phi$, such that:

$$\mathbb{E}\left\|(ln\varkappa)^{1-\iota}(z(\varkappa)-\varrho(\varkappa))\right\|^2\leq\mathcal{M}\epsilon,\ \forall\varkappa\in I_{\mathbf{a}}.$$

We now state the following Lemma which plays crucial role in this paper.

Lemma 3.3. *For $\beta \in (0,1)$, there exists $\mathcal{M}_\beta > 0$ such that, for all $\mu > 0$,*

$$t^{1-\beta} e^{-\mu t} \int_0^t (t-s)^{\beta-1} s^{\beta-1} e^{\mu s} ds \leq \frac{1}{\mu^\beta}\left(\mathcal{M}_\beta + \frac{\Gamma(\beta)}{2^{\beta-1}}\right), \quad \forall t \geq 0.$$

Proof. We start by the following decomposition:

$$\mathcal{J}(t) = t^{1-\beta} e^{-\mu t} \int_0^t (t-s)^{\beta-1} s^{\beta-1} e^{\mu s} ds = \mathcal{J}_1(t) + \mathcal{J}_2(t),$$

where $\mathcal{J}_1$ and $\mathcal{J}_2$ are defined by:

$$\begin{aligned} \mathcal{J}_1(t) &= t^{1-\beta} e^{-\mu t} \int_0^{t/2} (t-s)^{\beta-1} s^{\beta-1} e^{\mu s} ds, \\ \mathcal{J}_2(t) &= t^{1-\beta} e^{-\mu t} \int_{t/2}^t (t-s)^{\beta-1} s^{\beta-1} e^{\mu s} ds. \end{aligned}$$

1. *Estimation of $\mathcal{J}_1(t)$:* For $0 \leq s \leq t/2$, we have $(t-s)^{\beta-1} \leq (t/2)^{\beta-1}$. Then,

$$\mathcal{J}_1(t) \leq \frac{e^{-\mu t}}{2^{\beta-1}} \int_0^{t/2} s^{\beta-1} e^{\mu s} ds \leq \frac{e^{-\mu t}}{2^{\beta-1}} \int_0^t s^{\beta-1} e^{\mu s} ds.$$

So, we can establish that: for all $t > 0$

$$\begin{aligned} \int_0^t s^{\beta-1} e^{\mu s} ds &= \sum_{\tau \geq 0} \frac{\mu^\tau}{\tau!} \int_0^t s^{\tau+\beta-1} ds = \sum_{\tau \geq 0} \frac{\mu^\tau}{\tau!} \frac{t^{\tau+\beta}}{\tau+\beta}, \\ &= \frac{t^{\beta-1}}{\mu} \sum_{\tau \geq 0} \frac{\mu^{\tau+1} t^{\tau+1}}{\tau!(\tau+\beta)} = \frac{t^{\beta-1}}{\mu} \sum_{\tau \geq 0} \frac{(\mu t)^{\tau+1}}{(\tau+1)!} (\frac{\tau+1}{\tau+\beta}), \\ &\leq \frac{1}{\beta} \frac{t^{\beta-1}}{\mu} \sum_{\tau \geq 0} \frac{(\mu t)^{\tau+1}}{(\tau+1)!}. \end{aligned}$$

It's easy to show that:

$$\begin{aligned} \frac{1}{\beta} \frac{t^{\beta-1}}{\mu} \sum_{\tau \geq 0} \frac{(\mu t)^{\tau+1}}{(\tau+1)!} &= \frac{1}{\beta} \frac{t^{\beta-1}}{\mu} \left(e^{\mu t} - 1\right) = \frac{1}{\beta} t^\beta \left(\frac{e^{\mu t} - 1}{\mu t}\right), \\ &= \frac{1}{\beta \mu^\beta} (\mu t)^\beta \left(\frac{e^{\mu t} - 1}{\mu t}\right). \end{aligned}$$

Hence, we obtain:

$$\mathcal{J}_1(t) \leq \frac{1}{\mu^\beta \beta 2^{\beta-1}} \left((\mu t)^\beta \frac{1 - e^{-\mu t}}{\mu t}\right) \leq \frac{1}{\mu^\beta \beta 2^{\beta-1}} H(\mu t),$$

where the positive function H is defined by:

$$H(l) = l^\beta \frac{1 - e^{-l}}{l}.$$

Consequently, we can conclude that:

$$\mathcal{J}_1(t) \leq \frac{\mathcal{M}_\beta}{\mu^\beta}, \tag{12}$$

where the constant $\mathcal{M}_\beta$ is given by:

$$\mathcal{M}_\beta = \frac{1}{\beta 2^{\beta-1}} \sup_{x>0} (H(x)). \tag{13}$$

2. *Estimation of $\mathcal{J}_2(t)$:*

$$\begin{aligned}\mathcal{J}_2(t) &= t^{1-\beta}e^{-\mu t}\int_{\frac{t}{2}}^{t}(t-s)^{\beta-1}s^{\beta-1}e^{\mu s}ds,\\ &\leq \left(\frac{t}{2}\right)^{\beta-1}t^{1-\beta}e^{-\mu t}\int_{\frac{t}{2}}^{t}(t-s)^{\beta-1}e^{\mu s}ds.\end{aligned}$$

According to the change of variable $l = t - s$, we get:

$$\mathcal{J}_2(t) \leq \frac{1}{2^{\beta-1}}\int_0^{\frac{t}{2}} l^{\beta-1}e^{-\mu l}dl \leq \frac{\Gamma(\beta)}{2^{\beta-1}\mu^{\beta}}. \tag{14}$$

Therefore, by (12) and (14), we obtain:

$$\mathcal{J}(t) \leq \frac{1}{\mu^{\beta}}\left(\mathcal{M}_{\beta} + \frac{\Gamma(\beta)}{2^{\beta-1}}\right).$$

□

Let $\mathbf{a} > 1$ and $\mu > 0$ such that:

$$c_\mu = \frac{2\mathbf{a}\mathcal{K}^2}{\mu^{2\iota-1}\Gamma(\iota)^2}\left(\mathcal{M}_{2\iota-1} + \frac{\Gamma(2\iota-1)}{2^{2\iota-2}}\right) < 1,$$

where the constant $\mathcal{M}_{2\iota-1}$ is given as in (13). Then, we define a norm $\|\cdot\|_\mu$ on the space $\mathbb{H}^2(I_{\mathbf{a}})$ by:

$$\|\varrho\|_\mu = \sqrt{\sup_{\varkappa\in I_{\mathbf{a}}}\frac{\mathbb{E}\left(\left\|(ln\varkappa)^{1-\iota}\varrho(\varkappa)\right\|^2\right)}{\varkappa^{\mu}}}, \quad \forall\varrho\in\mathbb{H}^2(I_{\mathbf{a}}). \tag{15}$$

It is not hard to show that $\|\cdot\|_{\mathbb{H}^2}$ and $\|\cdot\|_\mu$ are equivalent. Hence, $(\mathbb{H}^2(I_{\mathbf{a}}), \|\cdot\|_\mu)$ is a Banach space.

Proposition 3.4. *The operator $\mathcal{R}_\phi : \mathbb{H}^2(I_{\mathbf{a}}) \to \mathbb{H}^2(I_{\mathbf{a}})$ defined in (2) is contractive. Moreover, we have*

$$\left\|\mathcal{R}_\phi u - \mathcal{R}_\phi v\right\|_\mu \leq \sqrt{c_\mu}\,\|u-v\|_\mu, \quad \forall u, v \in \mathbb{H}^2(I_{\mathbf{a}}).$$

Proof. Let $u, v \in \mathbb{H}^2(I_{\mathbf{a}})$, by (2), we have $\forall\varkappa \in I_{\mathbf{a}}$,

$$\begin{aligned}&\mathbb{E}\left(\left\|(ln\varkappa)^{1-\iota}\mathcal{R}_\phi u(\varkappa) - (ln\varkappa)^{1-\iota}\mathcal{R}_\phi v(\varkappa)\right\|^2\right)\\ &\leq \frac{2}{\Gamma(\iota)^2}\mathbb{E}\left(\left\|\int_1^{\varkappa}(ln\varkappa)^{1-\iota}\left(ln\frac{\varkappa}{\vartheta}\right)^{\iota-1}\frac{(\zeta_1(\vartheta,u(\vartheta))-\zeta_1(\vartheta,v(\vartheta)))}{\vartheta}d\vartheta\right\|^2\right),\\ &\quad+\frac{2}{\Gamma(\iota)^2}\mathbb{E}\left(\left\|\int_1^{\varkappa}(ln\varkappa)^{1-\iota}\left(ln\frac{\varkappa}{\vartheta}\right)^{\iota-1}\frac{(\zeta_2(\vartheta,u(\vartheta))-\zeta_2(\vartheta,v(\vartheta)))}{\vartheta}dW(\vartheta)\right\|^2\right).\end{aligned}$$

Using Cauchy-Schwarz inequality and Fubini's Theorem, we have:

$$\begin{aligned}&\mathbb{E}\left(\left\|\int_1^{\varkappa}(ln\varkappa)^{1-\iota}(ln\omega)^{\iota-1}\left(ln\frac{\varkappa}{\omega}\right)^{\iota-1}\frac{(ln\omega)^{1-\iota}(\zeta_1(\omega,u(\omega))-\zeta_1(\omega,v(\omega)))}{\omega}d\omega\right\|^2\right)\\ &\leq \mathcal{K}^2(\varkappa-1)\int_1^{\varkappa}(ln\varkappa)^{2-2\iota}(ln\vartheta)^{2\iota-2}\left(ln\frac{\varkappa}{\vartheta}\right)^{2\iota-2}\frac{\mathbb{E}\left(\left\|(ln\vartheta)^{1-\iota}(u(\vartheta)-v(\vartheta))\right\|^2\right)}{\vartheta^2}d\vartheta.\end{aligned}$$

Moreover, by Itô isometry formula and Fubini's Theorem, we get:

$$\mathbb{E}\left(\left\|\int_1^{\varkappa}(ln\varkappa)^{1-\iota}(ln\delta)^{\iota-1}\left(ln\frac{\varkappa}{\delta}\right)^{\iota-1}\frac{(ln\delta)^{1-\iota}\left(\zeta_2(\delta,u(\delta))-\zeta_2(\delta,v(\delta))\right)}{\delta}dW(\delta)\right\|^2\right)$$
$$=\mathbb{E}\left(\int_1^{\varkappa}(ln\varkappa)^{2-2\iota}(ln\delta)^{2\iota-2}\left(ln\frac{\varkappa}{\delta}\right)^{2\iota-2}\frac{\left\|(ln\delta)^{1-\iota}\left(\zeta_2(\delta,u(\delta))-\zeta_2(\delta,v(\delta))\right)\right\|^2}{\delta^2}d\delta\right),$$
$$\leq\mathcal{K}^2\int_1^{\varkappa}(ln\varkappa)^{2-2\iota}(ln\vartheta)^{2\iota-2}\left(ln\frac{\varkappa}{\vartheta}\right)^{2\iota-2}\frac{\mathbb{E}\left(\left\|(ln\vartheta)^{1-\iota}(u(\vartheta)-v(\vartheta))\right\|^2\right)}{\vartheta^2}d\vartheta.$$

Thus,

$$\mathbb{E}\left(\left\|(ln\varkappa)^{1-\iota}\mathcal{R}_\phi u(\varkappa)-(ln\varkappa)^{1-\iota}\mathcal{R}_\phi v(\varkappa)\right\|^2\right)$$
$$\leq\frac{2\mathcal{K}^2\mathbf{a}}{\Gamma(\iota)^2}\int_1^{\varkappa}(ln\varkappa)^{2-2\iota}(ln\vartheta)^{2\iota-2}\frac{\left(ln\frac{\varkappa}{\vartheta}\right)^{2\iota-2}}{\vartheta}\mathbb{E}\left(\left\|(ln\vartheta)^{1-\iota}(u(\vartheta)-v(\vartheta))\right\|^2\right)d\vartheta,$$
$$\leq\frac{2\mathcal{K}^2\mathbf{a}}{\Gamma(\iota)^2}\|u-v\|_\mu^2\int_1^{\varkappa}(ln\varkappa)^{2-2\iota}(ln\vartheta)^{2\iota-2}\frac{\left(ln\frac{\varkappa}{\vartheta}\right)^{2\iota-2}}{\vartheta}\vartheta^\mu d\vartheta. \tag{16}$$

Now, let:

$$\hbar(t)=\int_1^t(lnt)^{2-2\iota}(ln\vartheta)^{2\iota-2}(lnt-ln\vartheta)^{2\iota-2}\vartheta^\mu\frac{d\vartheta}{\vartheta}.$$

Thanks to the change of variable $u=ln\vartheta$, we can derive that

$$\hbar(t)=(lnt)^{2-2\iota}\int_0^{lnt}u^{2\iota-2}(lnt-u)^{2\iota-2}e^{\mu u}du.$$

Using Lemma 3.3, we get:

$$\hbar(t)\leq\frac{t^\mu}{\mu^{2\iota-1}}\left(\mathcal{M}_{2\iota-1}+\frac{\Gamma(2\iota-1)}{2^{2\iota-2}}\right).$$

Then by (16), it is easy to deduce that (after some algebraic manipulations)

$$\left\|\mathcal{R}_\phi u-\mathcal{R}_\phi v\right\|_\mu\leq\sqrt{c_\mu}\,\|u-v\|_\mu\,.$$

Then, the operator $\mathcal{R}_\phi$ is contractive. □

Theorem 3.5. *Assume that $\mathcal{H}_1$-$\mathcal{H}_2$ hold. Then, the HSFDE (1) is HUSwrϵ on $I_{\mathbf{a}}$.*

Proof. It follows from (11) that: for all $\varkappa\in I_{\mathbf{a}}$

$$\frac{\mathbb{E}\left\|(ln\varkappa)^{1-\iota}\left(z(\varkappa)-\mathcal{R}_\phi z(\varkappa)\right)\right\|^2}{\varkappa^\mu}\leq\epsilon.$$

Then, using the definition of the norm $\|\cdot\|_\mu$ given in (15), we have:

$$\left\|\mathcal{R}_\phi z-z\right\|_\mu\leq\sqrt{\epsilon}.$$

By Theorem 2.3 and Proposition 3.4, there is a unique solution ϱ such that $I^{1-\iota}\varrho(1)=I^{1-\iota}z(1)$ and:

$$\left\|\varrho-z\right\|_\mu\leq\frac{\sqrt{\epsilon}}{1-\sqrt{c_\mu}}.$$

Consequently, there exists $C > 0$ such that:

$$\mathbb{E}\left\|(ln\varkappa)^{1-\iota}\left(z(\varkappa)-\varrho(\varkappa)\right)\right\|^{2}\leq C\epsilon,\ \forall\varkappa\in I_{\mathbf{a}}.$$

□

4. Illustrative examples

In this section, we illustrate two examples to prove our results.

Example 1: Let the following HSFDE, for $\varkappa \in [1,5]$ and for each $\epsilon > 0$, given by

$$\begin{cases} {}^{H}D_{1}^{\frac{2}{3}}\varrho(\varkappa)=\zeta_1\left(\varkappa,\varrho(\varkappa)\right)+\zeta_2\left(\varkappa,\varrho(\varkappa)\right)\frac{dW(\varkappa)}{d\varkappa}, \\ \mathbb{E}\Big\|(ln\varkappa)^{1-\iota}\Big(z(\varkappa)-(ln\varkappa)^{\iota-1}\frac{\phi}{\Gamma(\iota)} \\ \qquad \Big(\int_1^{\varkappa}\mathcal{U}(\zeta_1,\varkappa,s,z(s))ds+\int_1^{\varkappa}\mathcal{U}(\zeta_2,\varkappa,s,z(s))dW(s)\Big)\Big)\Big\|^{2}\leq\epsilon, \\ I^{1-\iota}z(1)=\phi, \end{cases} \tag{17}$$

where

$$\begin{array}{rcl} \xi(\varkappa) & \in & \mathbb{H}^{2}([1,5],\mathbb{R}) \\ \zeta_1(\varkappa,\xi(\varkappa)) & = & (\arctan(\xi(\varkappa))+\cos(\xi(\varkappa))) \\ \zeta_2(\varkappa,\xi(\varkappa)) & = & \varkappa\cos(\xi(\varkappa)). \end{array}$$

We will prove that equation (17) is HUSwrϵ.
Let $(\varkappa,\xi_1,\xi_2)\in[1,5]\times\mathbb{R}^{p}\times\mathbb{R}^{p}$, thus

$$\|\zeta_1(\varkappa,\xi_1)-\zeta_1(\varkappa,\xi_2)\|\leq 2\|\xi_1-\xi_2\|,$$

and

$$\|\zeta_2(\varkappa,\xi_1)-\zeta_2(\varkappa,\xi_2)\|\leq 5\|\xi_1-\xi_2\|.$$

Hence, assumption $\mathcal{H}_1$ fulfilled. Moreover,

$$\|\zeta_2(\cdot,0)\|_{\infty}=\text{ess}\sup_{\varkappa\in[1,5]}\|\zeta_2(\varkappa,0)\|\leq 5\quad\text{and}\quad\int_1^5\|\zeta_1(\varkappa,0)\|^{2}d\varkappa\leq 5.$$

Thus, Assumptions $\mathcal{H}_1$-$\mathcal{H}_2$ fulfilled. Hence, applying Theorem 3.5, Equation (17) is HUSwrϵ on $[1,5]$.

Example 2:

Let the following HSFDE, for $\varkappa \in [1,4]$ and for each $\epsilon > 0$, given by

$$\begin{cases} {}^{H}D_{1}^{\frac{3}{4}}\varrho(\varkappa)=\zeta_1\left(\varkappa,\varrho(\varkappa)\right)+\zeta_2\left(\varkappa,\varrho(\varkappa)\right)\frac{dW(\varkappa)}{d\varkappa}, \\ \mathbb{E}\Big\|(ln\varkappa)^{1-\iota}\Big(z(\varkappa)-(ln\varkappa)^{\iota-1}\frac{\phi}{\Gamma(\iota)} \\ \qquad -\Big(\int_1^{\varkappa}\mathcal{U}(\zeta_1,\varkappa,s,z(s))ds+\int_1^{\varkappa}\mathcal{U}(\zeta_2,\varkappa,s,z(s))dW(s)\Big)\Big)\Big\|^{2}\leq\epsilon, \\ I^{1-\iota}z(1)=\phi, \end{cases} \tag{18}$$

where

$$\begin{aligned} \xi(\varkappa) &\in \mathbb{H}^2([1,4],\mathbb{R}) \\ \zeta_1(\varkappa,\xi(\varkappa)) &= \frac{e^{-\varkappa}}{1+e^{-\varkappa}}(1+\cos\xi(\varkappa)) \\ \zeta_2(\varkappa,\xi(\varkappa)) &= \frac{1+\xi(\varkappa)}{1+\varkappa^2}. \end{aligned}$$

We will prove that equation (18) is HUSwrϵ.
Let $(\varkappa,\xi_1,\xi_2) \in [1,4]\times\mathbb{R}^p\times\mathbb{R}^p$, then

$$\|\zeta_1(\varkappa,\xi_1)-\zeta_1(\varkappa,\xi_2)\| \le \|\xi_1-\xi_2\| \quad \text{and} \quad \|\zeta_2(\varkappa,\xi_1)-\zeta_2(\varkappa,\xi_2)\| \le \|\xi_1-\xi_2\|.$$

Thus, Assumption $\mathcal{H}_1$ hold. On the other hand,

$$\|\zeta_2(\cdot,0)\|_\infty = \operatorname{ess}\sup_{\varkappa\in[1,4]}\|\zeta_2(\varkappa,0)\| \le 1 \quad \text{and} \quad \int_1^4 \|\zeta_1(\varkappa,0)\|^2 d\varkappa \le ln\left(1+e^{-1}\right).$$

Then, Assumptions $\mathcal{H}_1$-$\mathcal{H}_2$ fulfilled. Hence, applying Theorem 3.5, Equation (18) is HUSwrϵ on $[1,4]$.

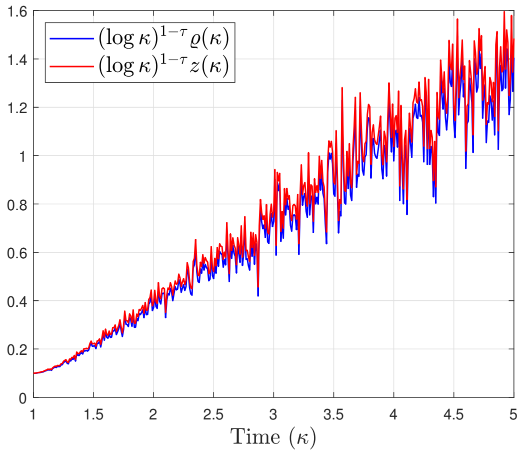

Figure 1: Trajectory simulation of solutions of System (17) on the interval [1,5].

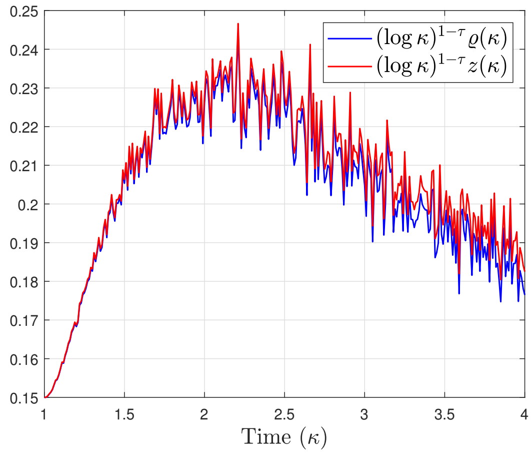

Figure 2: Trajectory simulation of solutions of System (18) on the interval [1,4].

Using MATLAB, we conduct a simulation based on Euler-Maruyama scheme with step size 10^{-8} for both examples. Then, in Figure 1 (respectively Figure 2) we give the simulation trajectory $(ln\varkappa)^{1-\iota}\varrho(\varkappa)$ and $(ln\varkappa)^{1-\iota}z(\varkappa)$ of System (17) (respectively System(18)) with the same initial condition $I^{1-\iota}\varrho(1) = I^{1-\iota}z(1) = 0.1$ (respectively $I^{1-\iota}\varrho(1) = I^{1-\iota}z(1) = 0.15$). Consequently, we can see from Figure 1, 2 that the solution trajectory of the inequations (11) almost coincides with that of System (17) (respectively (18)). It follows that the distance between $(ln\varkappa)^{1-\iota}\varrho(\varkappa)$ and $(ln\varkappa)^{1-\iota}z(\varkappa)$ is less than a constant which shows that System (17) and (18) are HUSwrϵ according to Definition 3.2.

5. Conclusion

We managed to utilize a version of some FPT and some classical stochastic calculus tools to present stability results for HSFDE. Moreover, we show that under some conditions, there are functions satisfying the equation roughly (in some way) that are close (in the HUS sense) to the exact solution.
In future work, we plan to extend these findings by generalizing our results to include Hadamard fractional

stochastic differential equations with time delay. This expansion will enhance our understanding of the dynamics and behavior of these equations and broaden the practical applications of our research.

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