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Threshold dynamics of an age–space structured brucellosis model with nonlinear incidence rate on a heterogeneous environment

Eric Avila-Vales[a], Angel G. C. Pérez[a]

[a]*Facultad de Matemáticas, Universidad Autónoma de Yucatán, Anillo Periférico Norte, Tablaje Catastral 13615, C.P. 97119, Mérida, Yucatán, Mexico*

Abstract. We propose an age–space structured brucellosis model that includes diffusion with heterogeneous coefficients and a general nonlinear incidence rate. The renewal process is used to calculate the next generation operator, and the basic reproduction number $\mathcal{R}_0$ is defined by the spectral radius of the next generation operator. We prove that $\mathcal{R}_0$ governs the threshold dynamics of the brucellosis model: when $\mathcal{R}_0 < 1$ the disease dies out, and when $\mathcal{R}_0 > 1$ the disease persists.

1. Introduction

Brucellosis is a zoonotic disease caused by one of several species of the Gram-negative coccobacillus *Brucella*; it is endemic to the Middle East, sub-Saharan Africa, and Central America [4]. Brucellosis affects primarily domestic animals, although humans are often infected due to direct contact with animals or ingestion of contaminated dairy products [3]. This disease causes a huge burden in society due to long-term treatment of the infected people and losses in livestock, also causing abortion and infertility in productive animals [1, 3]. Hence, many researchers have tried to study the prevalence of brucellosis and determine the best strategies to control its spread with the aid of mathematical modelling. For instance, an ordinary differential equation model for bovine brucellosis with four classes was proposed in [7]. Liang et al. [10] studied an SI model of animal brucellosis with transport, while Hou et al. [9] evaluated the effect of vaccination on brucellosis prevention.

More recent models have included spatial heterogeneity to account for the different contact rates in each geographic location, as well as the movement of animals in their living space, which can be described with systems of partial differential equations. A two-patch model was proposed in [14] to analyse the spatial and seasonal variations in the transmission of brucellosis. On the other hand, Yang et al. [16] proposed an age-structured model with spatial diffusion and studied its threshold dynamics with respect to the basic reproduction number. Further, the dynamics of a periodic SIV brucellosis model with nonlocal infection and heterogeneous diffusion rates was studied in [17].

In this paper, we propose a brucellosis transmission model with infection age and space structure that generalizes the model studied in [16] by including heterogeneous coefficients and a general force of

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Email addresses: avila@correo.uady.mx (Eric Avila-Vales), angel.cervantes@correo.uady.mx (Angel G. C. Pérez)

infection. Our model is given by

$$
\begin{aligned}
\frac{\partial S(t,x)}{\partial t} &= \nabla\cdot[D_S(x)\nabla S(t,x)] + \Lambda(x) - \mu(x)S(t,x) \\
&\quad - S(t,x)\left[\int_0^\infty \beta(a,x)g(i(t,a,x))\,\mathrm{d}a + \beta_v(x)v(t,x)\right], && x\in\Omega; \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)i(t,a,x) &= \nabla\cdot[D_i(a,x)\nabla i(t,a,x)] - [\mu(x)+\alpha(a,x)]i(t,a,x), && a\geq 0,\ x\in\Omega; \\
\frac{\partial v(t,x)}{\partial t} &= \nabla\cdot[D_v(x)\nabla v(t,x)] + \int_0^\infty p(a,x)i(t,a,x)\,\mathrm{d}a \\
&\quad - c(x)v(t,x), && x\in\Omega; \\
i(t,0,x) &= S(t,x)\left[\int_0^\infty \beta(a,x)g(i(t,a,x))\,\mathrm{d}a + \beta_v(x)v(t,x)\right], && x\in\Omega; \\
[D_S(x)\nabla S(t,x)]\cdot\mathbf{n} &= [D_i(x)\nabla i(t,a,x)]\cdot\mathbf{n} = [D_v(x)\nabla v(t,x)]\cdot\mathbf{n} = 0, && a\geq 0,\ x\in\partial\Omega,
\end{aligned}
\tag{1}
$$

with the initial conditions

$$
S(0,x)=\phi_S(x)\geq 0,\quad i(0,a,x)=\phi_i(a,x)\in L^1_+\big(\mathbb{R}_+;\,C(\Omega)\big),\quad v(0,x)=\phi_v(x)\geq 0,\qquad x\in\overline{\Omega}. \tag{2}
$$

The variables $S(t,x)$ and $v(t,x)$ denote the densities of susceptible animals and brucellosis virions, respectively, at time $t\geq 0$ and location $x\in\Omega$, where Ω is a bounded, connected subset of $\mathbb{R}^n$, called the *habitat*. We denote by $i(t,a,x)$ the density of infected animals with infection age a at time t and location x.

The diffusion coefficients of susceptible animals, infected animals, and brucellosis virions are $D_S(x)$, $D_i(a,x)$, and $D_v(x)$, respectively. The parameters $\Lambda(x)$, $\mu(x)$, and $\alpha(a,x)$ represent the birth rate, natural death rate, and disease-induced death rate of animal population, respectively; $c(x)$ is the clearance rate of virions. Susceptible animals can be infected by infected animals with infection age a at rate $\beta(a,x)g(i(t,a,x))$, and by brucellosis virions at rate $\beta_v(x)v(t,x)$. Each infected animal with infection age a produces new virions at rate $p(a,x)$; $\partial\Omega$ is the boundary of Ω and $\mathbf{n}$ denotes the outward unit normal vector.

The organization of this paper is as follows: in Section 2, we present the assumptions of the model and prove the existence and uniqueness of positive solutions. In Section 3, we compute the basic reproduction number. The threshold dynamics, stability of the disease-free equilibrium, and uniform persistence of the model are studied in Section 4. Finally, Section 5 presents some conclusions for our work.

2. Preliminaries

Based on [15], we make the following assumptions for the model parameters.

Assumption 2.1.

(i) *For each $x\in\Omega$, $\Lambda(x)$ is strictly positive.*

(ii) *There exist positive constants d_j $(j=S,i,v)$ such that $d_j<D_j(\cdot)$ $(j=S,i,v)$ for all $x\in\Omega$.*

(iii) *For $\psi=\alpha,\beta,p$, there exist $a_1,a_2\geq 0$, $\underline{\psi}\in L^1(\mathbb{R}_+)$ and $\psi^+\in L^\infty(\mathbb{R}_+)$ such that $\underline{\psi}(a)\leq\psi(a,x)\leq\psi^+(a)$ for $a\in[a_1,a_2]$, $x\in\Omega$. We will use the notation*

$$
\overline{\psi} := \operatorname*{ess\,sup}_{a\in\mathbb{R}_+}\psi^+(a).
$$

The total infectivity due to contact with infected population $S(t,x)\int_0^\infty\beta(a,x)g(i(t,a,x))\,\mathrm{d}a$ is inspired by that in [5] and depends on a general nonlinear function g, which satisfies the following assumptions.

Assumption 2.2. *For $y \in \mathbb{R}_+$, $g(y) \geq 0$ with equality if and only if $y = 0$; $g'(y) \geq 0$ and $g''(y) \leq 0$.*

The above assumption implies that $g'(y)$ is bounded above by the constant $K := g'(0) > 0$ for $y \in \mathbb{R}_+$. Thus, by the Mean Value Theorem, we can see that

$$|g(y_1) - g(y_2)| \leq K|y_1 - y_2|, \qquad \text{for all } y_1, y_2 \geq 0. \tag{3}$$

Define the functional spaces $X = C(\overline{\Omega}, \mathbb{R})$ and $Y = L^1(\mathbb{R}_+, X)$ with the norms

$$\|\phi\|_X = \sup_{x\in\overline{\Omega}} |\phi(x)|, \qquad \|\varphi\|_Y = \int_0^\infty \|\varphi(a)\|_X \,\mathrm{d}a$$

for $\phi \in X$, $\varphi \in Y$. Let X_+ and Y_+ denote the positive cones of X and Y, respectively. Then, by [11, Theorem 1.5], the operator $\nabla \cdot [D_j(x)\nabla]$ ($j = S, i, v$) with the no-flux boundary condition generates the semigroup

$$\big(T_j(t)[\phi]\big)(x) = \int_\Omega \Gamma_j(t, x, y)\phi(y)\,\mathrm{d}y, \quad j = 1, 2, 3,$$

where Γ_j ($j = 1, 2, 3$) are Green's functions and $\phi \in X$. Furthermore, by [12, Corollary 7.2.3], $T_j(t) : X \to X$ ($j = 1, 2, 3$) are strongly positive and compact for any $t > 0$.

Let $B(t, x) := i(t, 0, x)$ for $(t, x) \in \mathbb{R}_+ \times \Omega$. The second equation of (1) can be solved by the method of integration along the characteristic lines $t - a =$ constant. The solution is given by

$$i(t, a, x) = \begin{cases} \pi(a, x) \displaystyle\int_\Omega \Gamma_2(a, x, y)B(t - a, y)\,\mathrm{d}y, & a \leq t,\ x \in \Omega; \\ \dfrac{\pi(a, x)}{\pi(a - t, x)} \displaystyle\int_\Omega \Gamma_2(a, x, y)\phi_i(a - t, y)\,\mathrm{d}y, & a > t,\ x \in \Omega, \end{cases} \tag{4}$$

where $\pi(a, x) = \exp\left(-\mu(x)a - \int_0^a \alpha(s, x)\,\mathrm{d}s\right)$. Then, we can reformulate system (1) as

$$\begin{aligned} \frac{\partial S(t, x)}{\partial t} &= \nabla \cdot [D_S(x)\nabla S(t, x)] + \Lambda(x) - \mu(x)S(t, x) - B(t, x), && x \in \Omega; \\ B(t, x) &= S(t, x) \int_0^t \beta(a, x)\, g\left(\pi(a, x) \int_\Omega \Gamma_2(a, x, y)B(t - a, y)\,\mathrm{d}y\right) \mathrm{d}a \\ &\quad + S(t, x)\beta_v(x)v(t, x) + S(t, x)F(t, x), && x \in \Omega; \\ F(t, x) &= \int_0^\infty \beta(a + t, x)\, g\left(\frac{\pi(a + t, x)}{\pi(a, x)} \int_\Omega \Gamma_2(a + t, x, y)\phi_i(a, y)\,\mathrm{d}y\right) \mathrm{d}a, && x \in \Omega; \\ \frac{\partial v(t, x)}{\partial t} &= \nabla \cdot [D_v(x)\nabla v(t, x)] + \int_0^t p(a, x)\pi(a, x) \int_\Omega \Gamma_2(a, x, y)B(t - a, y)\,\mathrm{d}y\,\mathrm{d}a \\ &\quad - c(x)v(t, x) + G(t, x), && x \in \Omega; \\ G(t, x) &= \int_0^\infty p(a + t, x)\frac{\pi(a + t, x)}{\pi(a, x)} \int_\Omega \Gamma_2(a + t, x, y)\phi_i(a, y)\,\mathrm{d}y\,\mathrm{d}a, && x \in \Omega; \\ &[D_S(x)\nabla S(t, x)] \cdot \mathbf{n} = [D_v(x)\nabla v(t, x)] \cdot \mathbf{n} = 0, && x \in \partial\Omega. \end{aligned} \tag{5}$$

The existence and uniqueness of positive solutions to system (1) can be obtained via the Banach–Picard fixed point theorem, as follows.

Theorem 2.3. *Let $\phi = (\phi_S, \phi_i, \phi_v)^T \in X\times Y\times X$. Then, system (1) has a unique solution defined on $[0, T]\times(X\times Y\times X)$.*

Proof. Define a functional space $Y_T = C([0, T], X)$ with the norm

$$\|\phi\|_{Y_T} = \sup_{0\leq t\leq T} \|\phi(t, \cdot)\|_X, \qquad \phi \in Y_T.$$

Solving the first equation of (1), we have

$$S(t,x) = F_S(t,x) + \int_0^t e^{-\mu(x)(t-a)} \int_\Omega \Gamma_1(t-a,x,y)\Big[\Lambda(x) - B(a,y)\Big]\,\mathrm{d}y\,\mathrm{d}a, \quad (t,x) \in [0,T]\times\Omega,$$

where $F_S(t,x) = e^{-\mu(x)t}\int_\Omega \Gamma_1(t,x,y)\phi_S(y)\,\mathrm{d}y$. For $(t,x) \in [0,T]\times\Omega$, by (4) and the definition of B, we get

$$B(t,x) = S(t,x)\left[F(t,x) + \int_0^t \beta(a,x)\, g\left(\pi(a,x)\int_\Omega \Gamma_2(a,x,y)B(t-a,y)\,\mathrm{d}y\right)\mathrm{d}a + \beta_v(x)v(t,x)\right], \tag{6}$$

where $F(t,x) = \int_0^\infty \beta(a+t,x)\, g\left(\frac{\pi(a+t,x)}{\pi(a,x)}\int_\Omega \Gamma_2(a+t,x,y)\phi_i(a,y)\,\mathrm{d}y\right)\mathrm{d}a$. Using a similar argument, we have

$$v(t,a) = F_v(t,x) + \int_0^t e^{-c(x)(t-a)}\int_\Omega \Gamma_3(t-a,x,y)\int_0^\infty p(b,x)i(a,b,y)\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}a, \quad (t,x) \in [0,T]\times\Omega,$$

where $F_v(t,x) = e^{-c(x)t}\int_\Omega \Gamma_3(t,x,y)\phi_v(y)\,\mathrm{d}y$. Again, according to (4), it follows that

$$\int_0^\infty p(a,x)i(t,a,x)\,\mathrm{d}a = \int_0^t p(a,x)\pi(a,x)\int_\Omega \Gamma_2(a,x,y)B(t-a,y)\,\mathrm{d}y\,\mathrm{d}a + F_p(t,x),$$

where $F_p(t,x) = \int_0^\infty p(a+t,x)\frac{\pi(a+t,x)}{\pi(a,x)}\int_\Omega \Gamma_2(t,x,y)i_0(a,y)\,\mathrm{d}y\,\mathrm{d}a$. Substituting S and v into (6) leads to

$$\begin{aligned}
B(t,x) = &\left[F_S(t,x) + \int_0^t e^{-\mu(x)(t-a)}\int_\Omega \Gamma_1(t-a,x,y)\Big[\Lambda(x) - B(a,y)\Big]\,\mathrm{d}y\,\mathrm{d}a\right] \\
&\times\Bigg[F_i(t,x) + \beta_v(x)\int_0^t e^{-c(x)(t-a)}\int_\Omega \Gamma_3(t-a,x,y)F_p(a,y)\,\mathrm{d}y\,\mathrm{d}a \\
&+ \int_0^t \beta(a,x)\, g\left(\pi(a,x)\int_\Omega \Gamma_2(a,x,y)B(t-a,y)\,\mathrm{d}y\right)\mathrm{d}a \\
&+ \beta_v(x)\int_0^t e^{-c(x)(t-a)}\int_\Omega \Gamma_3(t-a,x,y)\int_0^\infty p(b,x)\pi(b,x)\int_\Omega \Gamma_2(b,y,z)B(a-b,z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}a\Bigg] \\
:= &\,\mathcal{F}[B](t,x),
\end{aligned} \tag{7}$$

where $\mathcal{F} : Y_T \to Y_T$ is a nonlinear operator.

Next, we will show that the operator $\mathcal{F}$ has a unique fixed point in Y_T, which will guarantee the existence of a unique solution to system (1). For convenience, we define

$$\begin{aligned}
F_{ip}(t,x) &= F_i(t,x) + \beta_v(x)\int_0^t e^{-c(x)(t-a)}\int_\Omega \Gamma_3(t-a,x,y)F_p(a,y)\,\mathrm{d}y\,\mathrm{d}a, \\
\Theta_1(B) &= \int_0^t e^{-\mu(x)(t-a)}\int_\Omega \Gamma_1(t-a,x,y)\Big[\Lambda(x) - B(a,y)\Big]\,\mathrm{d}y\,\mathrm{d}a, \\
\Theta_2(B) &= \int_0^t \beta(a,x)\, g\left(\pi(a,x)\int_\Omega \Gamma_2(a,x,y)B(t-a,y)\,\mathrm{d}y\right)\mathrm{d}a, \\
\Theta_3(B) &= \beta_v(x)\int_0^t e^{-c(x)(t-a)}\int_\Omega \Gamma_3(t-a,x,y)\int_0^\infty p(b,x)\pi(b,x)\int_\Omega \Gamma_2(b,y,z)B(a-b,z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}a
\end{aligned}$$

for each $(t,x) \in [0,T]\times\Omega$. Then

$$\mathcal{F}(B) = \Big(F_S + \Theta_1(B)\Big)\Big(F_{ip} + \Theta_2(B) + \Theta_3(B)\Big).$$

For any $B_1, B_2 \in Y_T$, we set $\tilde{B} = B_1 - B_2$. Then, (3) implies that $\Theta_2(B_1) - \Theta_2(B_2) \le K\Theta_2(\tilde{B})$. Hence,

$$\begin{aligned}\mathcal{F}(B_1) - \mathcal{F}(B_2) &\le F_S\Big[K\Theta_2(\tilde{B}) + \Theta_3(\tilde{B})\Big] + F_{ip}\Theta_1(\tilde{B})\\ &\quad + \Big[\Theta_2(B_1) + \Theta_3(B_1)\Big]\Theta_1(B_1) - \Big[\Theta_2(B_2) + \Theta_3(B_2)\Big]\Theta_1(B_2)\\ &\le \Big|(F_S + \Theta_1)(K\hat{\Theta}_2 + \hat{\Theta}_3) + (F_{ip} + K\Theta_2 + \Theta_3)\Theta_1\Big|\tilde{B},\end{aligned}$$

where

$$\begin{aligned}\hat{\Theta}_1 &= \int_0^t e^{-\mu(x)(t-a)} \int_\Omega \Gamma_1(t-a,x,y)\,\mathrm{d}y\,\mathrm{d}a,\\ \hat{\Theta}_2 &= \int_0^t \beta(a,x)\pi(a,x) \int_\Omega \Gamma_2(a,x,y)\,\mathrm{d}y\,\mathrm{d}a,\\ \hat{\Theta}_3 &= \beta_v(x)\int_0^t e^{-c(x)(t-a)} \int_\Omega \Gamma_3(t-a,x,y)\int_0^\infty p(b,x)\pi(b,x)\int_\Omega \Gamma_2(b,y,z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}a.\end{aligned}$$

Denote

$$m(T) = \Big\|\Big[F_S(T,\cdot) + \Theta_1(T,\cdot)\Big](K\hat{\Theta}_2 + \hat{\Theta}_3) + \Big[F_{ip}(T,\cdot) + K\Theta_2(T,\cdot) + \Theta_3(T,\cdot)\Big]\Theta_1(T,\cdot)\Big\|_X.$$

Then

$$\|\mathcal{F}(B_1) - \mathcal{F}(B_2)\|_{Y_T} \le m(T)\,\|B_1 - B_2\|_{Y_T}.$$

Clearly, we can choose T small enough such that $m(T) < 1$. Hence, applying the contraction operator theorem [18], we conclude that $\mathcal{F}$ has a fixed point and that it is unique. This completes the proof. □

Next, we will prove the positiveness of the solutions.

Theorem 2.4. *Let (S,i,v) be a solution of (1) corresponding to $\phi_S, \phi_i, \phi_v \in X_+ \times Y_+ \times X_+$. Then, $S(t,x) > 0$, $B(t,x) > 0$ and $v(t,x) > 0$ for all $(t,x) \in [0,T]\times\Omega$.*

Proof. From the first equation of (1), we have

$$S(t,x) = \hat{F}_S(t,x) + \int_0^t e^{-\int_a^t [\mu(x)+B(b,y)]\,\mathrm{d}b}\Lambda(x)\int_\Omega \Gamma_1(t-a,x,y)\,\mathrm{d}y\,\mathrm{d}a,$$

where $\hat{F}_S(t,x) = \int_t^\infty e^{-\int_{a-t}^a [\mu(x)+B(b,y)]\,\mathrm{d}b}\int_\Omega \Gamma_1(t,x,y)\phi_S(y)\,\mathrm{d}y\,\mathrm{d}a$. The positivity of Λ and ϕ_S ensures $S(t,x) > 0$ for each $(t,x) \in [0,T]\times\Omega$.

The positivity of B is established by constructing Picard sequences as follows.

It is clear that $B_0(t,x) := S(t,x)F(t,x) > 0$, where F is defined in the proof of Theorem 2.3. Now, we assume that $B_n(t,x) > 0$ ($n \in \mathbb{N}$) for $\phi_i > 0$ and $(t,x) \in [0,T]\times\Omega$. Then

$$\begin{aligned}B_{n+1}(t,x) = B_0(t,x) + S(t,x)\Big[&\int_0^t \beta(t-a,x)\, g\Big(\pi(t-a,x)\int_\Omega \Gamma_2(t-a,x,y)B_n(a,y)\,\mathrm{d}y\Big)\,\mathrm{d}a\\ &+ \beta_v(x)\int_0^t e^{-c(x)(t-a)}\int_\Omega \Gamma_3(t-a,x,y)\int_0^\infty p(a-b,x)\pi(a-b,x)\\ &\int_\Omega \Gamma_2(a-b,y,z)B_n(b,z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}a\Big].\end{aligned}$$

From the nonnegativity of β, π and p, together with the positivity of Γ_2 and Γ_3, it follows that B_{n+1} is positive.

Next, applying the contraction mapping principle, we show that the sequence $\{B_n\}_0^\infty$ converges to $B(t,x)$ for any $(t,x) \in [0,T] \times \Omega$ as n approaches infinity. For this, we define a variable

$$\hat{B}_n(t,x) = e^{-\lambda t} B_n(t,a), \qquad \lambda \in \mathbb{R}_+,\ (t,x) \in [0,T] \times \Omega.$$

By the definition of B_n, we have

$$\begin{aligned}\hat{B}_{n+1}(t,x) = e^{-\lambda t} B_0(t,x) + S(t,x)\Big[&\int_0^t \beta(a,x)\, g\left(\pi(a,x) \int_\Omega \Gamma_2(a,x,y) e^{-\lambda a} \hat{B}_n(t-a,y)\,\mathrm{d}y\right)\mathrm{d}a \\ &+ \beta_v(x) \int_0^t e^{-c(x)a} \int_\Omega \Gamma_3(a,x,y) \int_0^{t-a} p(b,x)\pi(b,x) \\ &\int_\Omega \Gamma_2(b,y,z) e^{-\lambda(a+b)} \hat{B}_n(t-a-b,z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}a\Big].\end{aligned}$$

Define $\tilde{B}(t,\tilde{x}) = \max_{x\in\Omega} B(t,x)$. For any $n \in \mathbb{N}$,

$$\begin{aligned}\left\|\tilde{B}_{n+1} - \tilde{B}_n\right\|_\infty \le \hat{S}\Big[&\int_0^\infty \beta(a,x)\pi(a,x) K e^{-\lambda a} \int_\Omega \Gamma_2(a,x,y)\,\mathrm{d}y\,\mathrm{d}a \\ &+ \beta_v(x) \int_0^\infty e^{-c(x)a} \int_\Omega \Gamma_3(a,x,y) \int_0^\infty p(b,x)\pi(b,x) \\ &\int_\Omega \Gamma_2(b,y,z) e^{-\lambda(a+b)}\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}a\Big] \left\|\tilde{B}_n - \tilde{B}_{n-1}\right\|_\infty \\ \le{}& \frac{\hat{S}\left(\overline{\beta} K \lambda + \overline{\beta_v}\overline{p}\right)}{\lambda^2} \left\|\tilde{B}_n - \tilde{B}_{n-1}\right\|_\infty,\end{aligned}$$

where $\hat{S} = \max_{t\in[0,T]} \|S(t,\cdot)\|_X$ and $\overline{\beta_v} = \max_{x\in\Omega} \beta_v(x)$. Repeating this process, we have

$$\left\|\tilde{B}_{n+1} - \tilde{B}_n\right\|_\infty \le M_\lambda \left\|\tilde{B}_n - \tilde{B}_{n-1}\right\|_\infty \le M_\lambda^n \left\|\tilde{B}_1 - \tilde{B}_0\right\|_\infty,$$

where $M_\lambda = \hat{S}\left(\overline{\beta} K \lambda + \overline{\beta_v}\overline{p}\right)/\lambda^2$. Therefore, for any $m, n \in \mathbb{N}$,

$$\left\|\tilde{B}_m - \tilde{B}_n\right\|_\infty \le \frac{M_\lambda^n}{1 - M_\lambda} \left\|\tilde{B}_1 - \tilde{B}_0\right\|_\infty.$$

If we choose λ sufficiently large such that $M_\lambda < 1$, then $\left\|\tilde{B}_m - \tilde{B}_n\right\|_\infty \to 0$ as $n \to \infty$, which implies that $\tilde{B}_n \to B(t)$ as $n \to \infty$.

By (4), together with the positivity of i_0 and B, we conclude that $i(t,a,x)$ is positive. For the positivity of v, we will proceed by contradiction.

Suppose that there exists a $t_v = \inf\{t \in \mathbb{R}_+ : v(t,x_1) = 0 \text{ for some } x_1 \in \Omega\}$ such that $v(t_v,x_1) = 0$. Then $\frac{\partial v(t_0,x_1)}{\partial t} < 0$ and $v(t,x_1) > 0$ for $t \in [0,t_v)$. By the third equation of (1), we can easily obtain

$$\frac{\partial v(t_0,x_1)}{\partial t} = F_p(t_0,x_1) + \int_0^{t_v} p(a,x_1) \int_\Omega \int_0^a \Gamma_2(b,x_1,y) B(a-b,y) \pi(b,x_1)\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}a \ge 0.$$

This leads to a contradiction. Hence, $v(t,x) > 0$ for all $(t,x) \in [0,T] \times \Omega$. □

3. Basic reproduction number

Lemma 2.2 in [8] ensures that system (1) admits a unique disease-free steady state $E_0 = (S^0(x),\ 0,\ 0)$. Linearizing (1) at E_0, we obtain

$$\begin{aligned}\hat{B}(t,x) = S^0(x)\Big[&F(t,x) + \beta_v(x) F_v(t,x) + \int_0^t \beta(a,x)\pi(a,x) \int_\Omega \Gamma_2(a,x,y) g'(0) \hat{B}(t-a,y)\,\mathrm{d}y\,\mathrm{d}a \\ &+ \beta_v(x) \int_0^t e^{-c(x)(t-a)} \int_\Omega \Gamma_3(t-a,x,y) \int_0^a p(b,x)\pi(b,x) \int_\Omega \Gamma_2(b,y,z) \hat{B}(a-b,z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}a\Big],\end{aligned} \tag{8}$$

where $F(t,x)$ is defined in system (5) and

$$F_v(t,x) = \int_0^t e^{-c(x)(t-a)} \int_\Omega \Gamma_3(t-a,x,y) \int_a^\infty p(b,x) \frac{\pi(b,x)}{\pi(a-b,x)} \int_\Omega \Gamma_2(b,y,z)\phi_i(b-a,z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}a.$$

Note that (8) is a renewal equation, so the next generation operator $\mathcal{R}$ is calculated as follows:

$$\begin{aligned}
\mathcal{R}[\psi](x) &= S^0(x)\Big[g'(0)\int_0^\infty \beta(a,x)\pi(a,x)\int_\Omega \Gamma_2(a,x,y)\psi(y)\,\mathrm{d}y\,\mathrm{d}a \\
&\quad + \beta_v(x)\int_0^\infty \int_0^t e^{-c(x)(t-a)}\int_\Omega \Gamma_3(t-a,x,y)\int_0^a p(b,x)\pi(b,x) \\
&\quad \int_\Omega \Gamma_2(b,y,z)\psi(z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}a\,\mathrm{d}t\Big] \\
&= S^0(x)\Big[g'(0)\int_0^\infty \beta(a,x)\pi(a,x)\int_\Omega \Gamma_2(a,x,y)\psi(y)\,\mathrm{d}y\,\mathrm{d}a \\
&\quad + \beta_v(x)\int_0^\infty \int_a^\infty e^{-c(x)a}\int_\Omega \Gamma_3(a,x,y)\int_0^{t-a} p(b,x)\pi(b,x) \\
&\quad \int_\Omega \Gamma_2(b,y,z)\psi(z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}a\Big] \\
&= S^0(x)\Big[g'(0)\int_0^\infty \beta(a,x)\pi(a,x)\int_\Omega \Gamma_2(a,x,y)\psi(y)\,\mathrm{d}y\,\mathrm{d}a \\
&\quad + \beta_v(x)\int_0^\infty e^{-c(x)a}\int_\Omega \Gamma_3(a,x,y)\int_0^\infty p(b,x)\pi(b,x)\int_\Omega \Gamma_2(b,y,z)\psi(z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}a\Big].
\end{aligned} \tag{9}$$

Hence, following the definition in [6, 13], the basic reproduction number $\mathcal{R}_0$ is given by

$$\mathcal{R}_0 = r(\mathcal{R}),$$

where $r(\cdot)$ is the spectral radius of $\mathcal{R}$. Then, we can obtain the following results.

Lemma 3.1. *Let $\mathcal{R}$ be defined by* (9)*. Then $\mathcal{R}$ is strictly positive and compact.*

Proof. The positivity of the operator $\mathcal{R}$ follows immediately from Assumption 2.1. Now, choose a bounded sequence $\{\psi_n\}_{n\in\mathbb{N}}$ in X such that $|\psi_n| < M$, $n \in \mathbb{N}$, for some $M > 0$. Denote $\overline{\Lambda} = \max_{x\in\Omega}\Lambda(x)$, $\overline{\beta_v} = \max_{x\in\Omega}\beta_v(x)$, $\underline{\mu} = \min_{x\in\Omega}\mu(x)$, $\underline{c} = \min_{x\in\Omega}c(x)$. For each $x \in \Omega$,

$$\begin{aligned}
\mathcal{R}[\psi_n](x) &\le S^0(x)M\Big[g'(0)\int_0^\infty \beta(a,x)\pi(a,x)\int_\Omega \Gamma_2(a,x,y)\,\mathrm{d}y\,\mathrm{d}a \\
&\quad + \beta_v(x)\int_0^\infty e^{-c(x)a}\int_\Omega \Gamma_3(a,x,y)\int_0^\infty p(b,x)\pi(b,x)\int_\Omega \Gamma_2(b,y,z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}a\Big] \\
&\le \frac{\overline{\Lambda}M}{\underline{\mu}}\Big[g'(0)\int_0^\infty \beta^+(a)\int_\Omega \Gamma_2(a,x,y)\,\mathrm{d}y\,\mathrm{d}a \\
&\quad + \overline{\beta_v}\int_0^\infty e^{-\underline{c}a}\int_\Omega \Gamma_3(a,x,y)\int_0^\infty p^+(b)\int_\Omega \Gamma_2(b,y,z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}a\Big].
\end{aligned}$$

Therefore, $\mathcal{R}$ is uniformly bounded.

Now, we will show that $\mathcal{R}$ is equicontinuous. For any $x, \hat{x} \in \Omega$, we have

$$\begin{aligned}
\mathcal{R}[\psi_n](x) - \mathcal{R}[\psi_n](\hat{x}) &\le \frac{\overline{\Lambda}M}{\underline{\mu}}\Big[g'(0)\int_0^\infty \beta^+(a)\int_\Omega |\Gamma_2(a,x,y)-\Gamma_2(a,\hat{x},y)|\,\mathrm{d}y\,\mathrm{d}a \\
&\quad + \overline{\beta_v}\int_0^\infty e^{-\underline{c}a}\int_\Omega |\Gamma_3(a,x,y)-\Gamma_3(a,\hat{x},y)|\int_0^\infty p^+(b)\int_\Omega \Gamma_2(b,y,z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}a\Big].
\end{aligned}$$

Define $\beta^* = \int_0^\infty \beta^+(a)\,\mathrm{d}a$, $p^* = \int_0^\infty p^+(a)\,\mathrm{d}a$ and $c^* = \int_0^\infty e^{-\underline{c}a}\,\mathrm{d}a$. From the compactness of the operator Δ, we conclude that for any $\epsilon > 0$ there exists a positive constant δ such that

$$|\Gamma_2(a,x,y) - \Gamma_2(a,\hat{x},y)| \leq \frac{\underline{\mu}\epsilon}{2\overline{\Lambda}Mg'(0)\beta^*} \quad \text{and} \quad |\Gamma_3(a,x,y) - \Gamma_3(a,\hat{x},y)| \leq \frac{\underline{\mu}c^*\epsilon}{2\overline{\Lambda}M\overline{\beta_v}p^*}$$

for all $x - \hat{x} < \delta$ and $y \in \Omega$. For such δ and ϵ, we get $\mathcal{R}[\psi_n](x) - \mathcal{R}[\psi_n](\hat{x}) < \epsilon$. Hence $\mathcal{R}$ is equicontinuous, and its compactness follows from the Arzelà–Ascoli theorem. □

Lemma 3.1, together with the Krein–Rutman theorem [2] indicate that $\mathcal{R}_0$ is the only positive eigenvalue associated with a positive eigenvector.

4. Threshold dynamics

In this section, we will study the threshold dynamics of system (1) in terms of the basic reproduction number $\mathcal{R}_0$.

4.1. Stability of the disease-free equilibrium

Theorem 4.1. *If $\mathcal{R}_0 < 1$, then the disease-free equilibrium E_0 of system* (1) *is globally attractive.*

Proof. By the positivity of the solutions, it follows from the first equation of (1) that, for each $\epsilon > 0$, there exists a $t_S > 0$ such that $S(t,x) \leq S^0(x) + \epsilon$ for all $t > t_S$. From this and using that $g(y) \leq g'(0)y$, we get

$$\begin{aligned} B(t,x) \leq& \left[S^0(x)+\epsilon\right]g'(0)\int_0^t \beta(a,x)\pi(a,x)\int_\Omega \Gamma_2(a,x,y)B(t-a,y)\,\mathrm{d}y\,\mathrm{d}a \\ &+ \left[S^0(x)+\epsilon\right]\left[\beta_v(x)v(t,x)+F(t,x)\right], \qquad \forall t > t_S,\ x \in \Omega. \end{aligned}$$

Let $B^\infty := \limsup_{t\to+\infty} B(t,\cdot)$. Taking the lim sup on both sides of the above inequality, we obtain

$$B^\infty(x) \leq \mathcal{R}[B^\infty](x), \qquad \forall x \in \Omega.$$

Therefore, if $\mathcal{R}_0 < 1$, then $B^\infty(x) \to 0$ as $t \to \infty$ for every fixed $x \in \Omega$. Hence, $\lim_{t\to\infty} B(t,\cdot) = 0$. Substituting $B(t,x) = 0$ into the first equation of (5), we get

$$\begin{aligned} &\frac{\partial S(t,x)}{\partial t} = \nabla\cdot[D_S(x)\nabla S(t,x)] + \Lambda(x) - \mu(x)S(t,x), && x \in \Omega; \\ &[D_S(x)\nabla S(t,x)]\cdot\mathbf{n} = 0, && x \in \partial\Omega. \end{aligned} \tag{10}$$

By [8, Lemma 2.2], we know that (10) has a unique positive steady state $S^0(x)$, which is globally attractive. Therefore, the DFE E^0 is globally attractive for system (1) when $\mathcal{R}_0 < 1$. □

4.2. Uniform persistence

In the following, we will prove the uniform persistence of system (1).

Define

$$D_0 = \left\{(\phi_S,\phi_i,\phi_v) \in X_+ \times Y_+ \times X_+ : \phi_S(x)\left[\int_0^\infty \beta(a,x)\, g\big(\phi_i(a,x)\big)\,\mathrm{d}a + \beta_v(x)\phi_v(x)\right] > 0 \text{ for some } x \in \Omega\right\}.$$

The (strong) uniform persistence in D_0 means that there exists a positive constant $\epsilon > 0$ such that

$$\liminf_{t\to+\infty} \|B(t,\cdot)\|_X > \epsilon$$

for any initial condition $\phi \in D_0$. To stablish the uniform persistence of system (1), we first need to prove the following lemma.

Lemma 4.2. *Suppose that $\mathcal{R}_0 > 1$. Then there exists a positive constant $\epsilon > 0$ such that*

$$\limsup_{t\to+\infty} \|B(t,\cdot)\|_X > \epsilon \tag{11}$$

for any initial condition $\phi \in D_0$.

Proof. Since $\mathcal{R}_0 > 1$, we can choose a constant $\epsilon > 0$ such that, for a sufficiently large T_1,

$$\frac{\Lambda(x)-\epsilon}{\mu(x)}\left[1-e^{-\mu(x)T_1}\right]\left[\int_0^\infty \beta(a,x)\, g\Big(\pi(a,x)\Big)\,\mathrm{d}a + \frac{\beta_v(x)}{c(x)}\int_0^\infty p(b,x)\pi(b,x)\,\mathrm{d}b\right] > 1, \qquad \forall x\in\Omega. \tag{12}$$

Suppose by contradiction that (11) does not hold. Then, we can take $T_1 > 0$ such that $B(t,x) \le \epsilon$ for all $t \ge T_1$ and $x \in \Omega$. By the first equation of system (5), we have

$$\frac{\partial S(t,x)}{\partial t} \ge \nabla\cdot[D_S(x)\nabla S(t,x)] + \Lambda(x) - \epsilon - \mu(x)S(t,x) \tag{13}$$

for all $t \ge T_1$. Integrating equation (13) yields

$$\begin{aligned} S(t,x) &\ge \int_0^t e^{-\mu(x)a}\int_\Omega \Gamma_1(a,x,y)\Big[\Lambda(x)-\epsilon\Big]\,\mathrm{d}y\,\mathrm{d}a \\ &= \frac{\Lambda(x)-\epsilon}{\mu(x)}\left[1-e^{-\mu(x)t}\right] \ge \frac{\Lambda(x)-\epsilon}{\mu(x)}\left[1-e^{-\mu(x)T_1}\right]. \end{aligned}$$

By (4) and the fourth equation of (5), we have

$$\begin{aligned} v(t,x) &\ge \int_0^t e^{-c(x)(t-a)}\int_\Omega \Gamma_3(t-a,x,y)\int_0^\infty p(b,x)i(a,b,y)\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}a \\ &\ge \int_0^t e^{-c(x)a}\int_\Omega \Gamma_3(a,x,y)\int_0^{t-a} p(b,x)\pi(b,x)\int_\Omega \Gamma_2(b,y,z)B(t-a-b,z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}a. \end{aligned}$$

Then, by the second equation of (5), we obtain

$$\begin{aligned} B(t,x) \ge \frac{\Lambda(x)-\epsilon}{\mu(x)}\left[1-e^{-\mu(x)T_1}\right]\Bigg[&\int_0^t \beta(a,x)\, g\left(\pi(a,x)\int_\Omega \Gamma_2(a,x,y)B(t-a,y)\,\mathrm{d}y\right)\mathrm{d}a \\ &+ \beta_v(x)\int_0^t e^{-c(x)a}\int_\Omega \Gamma_3(a,x,y)\int_0^{t-a} p(b,x)\pi(b,x)\int_\Omega \Gamma_2(b,y,z)B(t-a-b,z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}a\Bigg]. \end{aligned} \tag{14}$$

Taking the Laplace transform on both sides of (14), we have

$$\begin{aligned} \widehat{B(\lambda,x)} \ge{}& \frac{\Lambda(x)-\epsilon}{\mu(x)}\left[1-e^{-\mu(x)T_1}\right]\Bigg[\int_0^\infty e^{-\lambda t}\int_0^t \beta(a,x)\, g\left(\pi(a,x)\int_\Omega \Gamma_2(a,x,y)B(t-a,y)\,\mathrm{d}y\right)\mathrm{d}a\,\mathrm{d}t \\ &+ \beta_v(x)\int_0^\infty e^{-\lambda t}\int_0^t e^{-c(x)a}\int_\Omega \Gamma_3(a,x,y)\int_0^{t-a} p(b,x)\pi(b,x) \\ &\int_\Omega \Gamma_2(b,y,z)B(t-a-b,z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}a\,\mathrm{d}t\Bigg] \\ ={}& \frac{\Lambda(x)-\epsilon}{\mu(x)}\left[1-e^{-\mu(x)T_1}\right]\Bigg[\int_0^\infty \beta(a,x)\, g\left(\pi(a,x)\int_\Omega \Gamma_2(a,x,y)\int_a^\infty e^{-\lambda t}B(t-a,y)\,\mathrm{d}t\,\mathrm{d}y\right)\mathrm{d}a \\ &+ \beta_v(x)\int_0^\infty e^{-c(x)a}\int_\Omega \Gamma_3(a,x,y)\int_a^\infty e^{-\lambda t}\int_0^{t-a} p(b,x)\pi(b,x) \\ &\int_\Omega \Gamma_2(b,y,z)B(t-a-b,z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}a\Bigg]. \end{aligned}$$

After some algebraic manipulations, we get

$$\begin{aligned}
\widehat{B(\lambda,x)} \geq\ & \frac{\Lambda(x)-\epsilon}{\mu(x)}\left[1-e^{-\mu(x)T_1}\right] \\
& \times\Big[\int_0^\infty \beta(a,x)\, g\left(\pi(a,x)\int_\Omega \Gamma_2(a,x,y)e^{-\lambda a}\int_0^\infty e^{-\lambda t}B(t,y)\,\mathrm{d}t\,\mathrm{d}y\right)\mathrm{d}a \\
& + \beta_v(x)\int_0^\infty e^{-c(x)a}\int_\Omega \Gamma_3(a,x,y)\int_0^\infty e^{-\lambda(t+a)}\int_0^t p(b,x)\pi(b,x) \\
& \int_\Omega \Gamma_2(b,y,z)B(t-b,z)\,\mathrm{d}z\,\mathrm{d}b\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}a\Big] \\
\geq\ & \frac{\Lambda(x)-\epsilon}{\mu(x)}\left[1-e^{-\mu(x)T_1}\right] \\
& \times\Big[\int_0^\infty \beta(a,x)\, g\left(\pi(a,x)\int_\Omega \Gamma_2(a,x,y)e^{-\lambda a}\int_0^\infty e^{-\lambda t}B(t,y)\,\mathrm{d}t\,\mathrm{d}y\right)\mathrm{d}a \\
& + \beta_v(x)\int_0^\infty e^{-c(x)a}\int_\Omega \Gamma_3(a,x,y)e^{-\lambda a}\int_0^\infty p(b,x)\pi(b,x)e^{-\lambda b}\int_0^\infty e^{-\lambda t} \\
& \int_\Omega \Gamma_2(b,y,z)B(t,z)\,\mathrm{d}z\,\mathrm{d}t\,\mathrm{d}b\,\mathrm{d}y\,\mathrm{d}a\Big].
\end{aligned} \tag{15}$$

Similarly to [16, Section 5], we can define $\hat{\pi}(a,x)=\pi(a,x)e^{-D_i(a,x)\zeta_i a}$, where ζ_i is an eigenvalue of $-\Delta$ on Ω with the homogeneous Neumann boundary conditions. Let $\widehat{P(\lambda,x)}$ and $\widehat{K(\lambda,x)}$ be the Laplace transforms of $p(\cdot,x)\hat{\pi}(\cdot,x)$ and $\beta(\cdot,x)g\big(\hat{\pi}(\cdot,x)\big)$, respectively. Then, letting $\tilde{B}(\lambda,\hat{x})=\min_{x\in\Omega}\widehat{B(\lambda,x)}$ and

$$\tilde{R} := \frac{\Lambda(x)-\epsilon}{\mu(x)}\left[1-e^{-\mu(x)T_1}\right]\left[\widehat{K(\lambda,x)}+\frac{\beta_v(x)}{c(x)}\widehat{P(\lambda,x)}\right],$$

we obtain from (15) that $\tilde{B}(\lambda,\hat{x}) \geq \tilde{R}\tilde{B}(\lambda,\hat{x})$, contradicting (12). This completes the proof. □

The following theorem establishes the strong $\|\cdot\|_X$ persistence of the disease for system (1). The proof of the theorem can be obtained analogously to [16], replacing only

$$\int_\Omega \Gamma_2(a,x,y)\hat{B}(t,y)\,\mathrm{d}y\ \pi(a)$$

with

$$g\left(\int_\Omega \Gamma_2(a,x,y)\hat{B}(t,y)\,\mathrm{d}y\ \pi(a,x)\right)$$

in the proof of [16, Lemma 6.3].

Theorem 4.3. *If $\mathcal{R}_0 > 1$, then there exists a positive constant $\epsilon > 0$ such that*

$$\liminf_{t\to+\infty}\|B(t,\cdot)\|_X > \epsilon \tag{16}$$

for any initial condition $\phi \in D_0$.

Finally, we obtain the following corollary.

Corollary 4.4. *If $\mathcal{R}_0 > 1$, then system (1) is uniformly strongly persistent, i.e., there exists a positive value ϵ such that for any solution with initial conditions in D_0,*

$$\liminf_{t\to+\infty,\ x\in\Omega} S(t,x) > \epsilon, \quad \liminf_{t\to+\infty,\ x\in\Omega} i(t,a,x) > \epsilon\underline{\pi}(a), \quad \liminf_{t\to+\infty,\ x\in\Omega} v(t,x) > \epsilon \tag{17}$$

for all $x\in\Omega$, $a\in\mathbb{R}_+$, where $\underline{\pi}(a)=\min_{x\in\Omega}\pi(a,x)$.

Proof. By Theorem 4.3, we know that for any $(a, x) \in \mathbb{R}_+ \times \Omega$ there exist $\epsilon_i > 0$ and $T_1 > 0$ such that $i(t, a, x) \geq \epsilon_i \pi(a, x) \geq \epsilon_i \underline{\pi}(a)$ for all $t > T_1$. Therefore, there exist $T_2 > T_1 > 0$ and a sufficiently small $\epsilon^i > 0$ such that $i(t, a, x) > \epsilon_i \underline{\pi}(a) - \epsilon^i$ for all $t > T_2$, $x \in \Omega$.

Let $P(x) = \int_0^\infty p(a, x)\pi(a, x)\,\mathrm{d}a$ and define $\underline{P} = \min_{x\in\Omega} P(x)$, $\bar{c} = \max_{x\in\Omega} c(x)$. By the third equation of (1), we have

$$\frac{\partial v(t,x)}{\partial t} = \nabla \cdot [D_v(x)\nabla v(t,x)] + [\epsilon_i P(x) - \epsilon^i] - c(x)v(t,x), \qquad x \in \Omega.$$

Integrating the above equation yields

$$v(t,x) \geq (\epsilon_i \underline{P} - \epsilon^i) \int_0^t e^{-\bar{c}a} \int_\Omega \Gamma_3(a,x,y)\,\mathrm{d}y\,\mathrm{d}a = \frac{\epsilon_i \underline{P} - \epsilon^i}{\bar{c}}(1 - e^{-\bar{c}t}).$$

Hence, there exists a positive constant $T_3 > T_2$ such that $v(t, x) > \frac{\epsilon_i \underline{P} - \epsilon^i}{\bar{c}} - \epsilon^v =: \epsilon_v$ for all $t > T_3$, $x \in \Omega$. By the positivity of $S(t, x)$, we readily have $S(t, x) > \epsilon_s$.

Therefore, $\epsilon = \min\{\epsilon_s, \epsilon_i \epsilon_v\}$ satisfies (17). □

5. Conclusion

In this paper, we considered an age-since-infection brucellosis model which includes spacial diffusion with the Neumann (no-flux) boundary condition. We computed the location-dependent next generation operator $\mathcal{R}(x)$, given by the explicit formula (9), by means of a renewal process instead of the abstract method used in [8]. This allowed us to determine the basic reproduction number $\mathcal{R}_0$ as the spectral radius of $\mathcal{R}(x)$ and prove that the system exhibits a threshold phenomenon in terms of $\mathcal{R}_0$.

The model studied in this work can be considered as a generalization of the model proposed by Yang et al. [16], since we incorporate heterogeneous, space-dependent coefficients ($D_S(x)$, $D_i(a, x)$ and $D_v(x)$) and assume that the incidence rate by contact with infected animals depends on a nonlinear function $g(i(t, a, x))$, which is more general than the linear incidence rate considered in [16].

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