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Approximation properties of semi-exponential Szász-Mirakyan-Kantorovich operators

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Abstract. In the present paper, we deal with the approximation properties of semi-exponential Szász-Mirakyan-Kantorovich operators. Here, we establish the relation between semi-exponential Szász-Mirakyan-operators and its Kantorovich variant. Further, we propose the modification of the Kantorovich variant so as to preserve the test functions e^{Ax} and e^{2Ax} and we derive the Voronovskaya-type result.

1. Introduction

Ismail and May [20] considered exponential type operators, which preserve linear functions. In ([15], [16]), the authors have studied the approximation properties of such operators by considering different basis functions. This paper is about the study of semi-exponential Szász-Mirakyan-Kantorovich operators, which is extension of exponential type operators. The motivation comes from ([19], [24]), wherein the semi-exponential operators are introduced and their different types of examples are discussed. In the year 1954, Butzer [11] considered the integral modification of Szász-Mirakyan operators, namely Szász-Mirakyan-Kantorovich operators. The approximation properties of such operators are studied in various research papers such as ([10], [13], [14], [17], [25]). Very recently, Herzog [19] proposed the semi-exponential Szász-Mirakyan operators defined by

$$(S_n^\beta f)(x) = \sum_{k=0}^{\infty} s_{nk}^\beta(x) f\left(\frac{k}{n}\right), \ \ \beta > 0 \tag{1}$$

where

$$s_{nk}^\beta(x) = e^{-(n+\beta)x}\frac{(n+\beta)^k x^k}{k!}.$$

These operators satisfy the differential equation

$$(D+\beta)s_{nk}^\beta(x) = \frac{(k-nx)}{x}s_{nk}^\beta(x),$$

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where D represents differential operator i.e. $D \equiv \frac{d}{dx}$. As a particular case, if $\beta = 0$, the operators (1) reduce to exponential-type operators viz.

$$(S_n f)(x) := (S_n^0 f)(x) = \sum_{k=0}^{\infty} s_{nk}(x) f\left(\frac{k}{n}\right), \tag{2}$$

where

$$s_{nk}(x) = e^{-nx}\frac{n^k x^k}{k!}.$$

Motivated by the recent studies, we now consider the Kantorovich variant of (1) as below:

$$(K_n^\beta f)(x) = n\sum_{k=0}^{\infty} s_{nk}^\beta(x) \int_{k/n}^{(k+1)/n} f(t)dt. \tag{3}$$

The paper is organised in the following manner:
In section 2, we estimate the moments and central moments for the operators (3) and find the bounds for their basis function. Section 3, provides a relation between the operators (1) and (3), the rate of convergence for the function of bounded variation and some direct estimates. Also, this section includes the Voronovskaya type asymptotic result for the operators (3). In section 4, we modify the operators (3) so as to preserve the test functions e^{Ax} and e^{2Ax} for $A > 0$. Further, we obtain the moments, central moments and the asymptotic formula for our modified operators.

2. Auxiliary Results

In the sequel, we use the following results. Throughout the paper we denote $e_i(t) = t^i, i = 0, 1, 2, ...$

Lemma 2.1. *For m-th order moment given by* $(S_n^\beta e_m)(x) = \sum_{k=0}^{\infty} s_{nk}^\beta(x)\left(\frac{k}{n}\right)^m$, $m \in N \cup \{0\}$, *we have the following:*

$$\begin{aligned}
(S_n^\beta e_0)(x) &= 1, \ (S_n^\beta e_1)(x) = \frac{x(\beta+n)}{n},\\
(S_n^\beta e_2)(x) &= \frac{x(\beta+n)}{n^2}\Big\{x(\beta+n)+1\Big\},\\
(S_n^\beta e_3)(x) &= \frac{x(\beta+n)}{n^3}\Big\{x(\beta+n)(\beta x+nx+3)+1\Big\},\\
(S_n^\beta e_4)(x) &= \frac{x(\beta+n)}{n^4}\Big\{x(\beta+n)\Big(\beta^2x^2+2\beta nx^2+6\beta x+n^2x^2+6nx+7\Big)+1\Big\},\\
(S_n^\beta e_5)(x) &= \frac{x(\beta+n)}{n^5}\Big\{x(\beta+n)(\beta^3x^3+3\beta^2nx^3+10\beta^2x^2+3\beta n^2x^3+20\beta nx^2\\
&\quad +25\beta x+n^3x^3+10n^2x^2+25nx+15)+1\Big\},\\
(S_n^\beta e_6)(x) &= \frac{x(\beta+n)}{n^6}\Big\{x(\beta+n)(\beta^4x^4+4\beta^3nx^4+15\beta^3x^3+6\beta^2n^2x^4+45\beta^2nx^3\\
&\quad +65\beta^2x^2+4\beta n^3x^4+45\beta n^2x^3+130\beta nx^2+90\beta x+n^4x^4+15n^3x^3\\
&\quad +65n^2x^2+90nx+31)+1\Big\}.
\end{aligned}$$

Proof. The moment generating function of the operators defined by (1) is given as

$$
\begin{aligned}
(S_n^{\beta} e^{At})(x) &= \sum_{k=0}^{\infty} e^{-(n+\beta)x} \frac{(n+\beta)^k x^k}{k!} e^{\frac{Ak}{n}} \\
&= e^{-(n+\beta)x} e^{(n+\beta)xe^{\frac{A}{n}}} \\
&= exp\Big\{(n+\beta)x(e^{\frac{A}{n}}-1)\Big\}
\end{aligned}
$$

and

$$(S_n^{\beta} e_r)(x) = \left[\frac{\partial^r}{\partial A^r}\Big(exp\Big\{(n+\beta)x(e^{\frac{A}{n}}-1)\Big\}\Big)\right]_{A=0}.$$

Thus the result follows by above identity. □

Lemma 2.2. *The moment generating function of the operators defined by (3) is given as*

$$(K_n^{\beta} e^{At})(x) = \frac{n(e^{\frac{A}{n}}-1)}{A} exp\Big\{(n+\beta)x(e^{\frac{A}{n}}-1)\Big\}.$$

Proof. By definition (3), we have

$$
\begin{aligned}
(K_n^{\beta} e^{At})(x) &= n\sum_{k=0}^{\infty} s_{nk}^{\beta}(x) \int_{k/n}^{(k+1)/n} e^{At} dt \\
&= \frac{n}{A} \sum_{k=0}^{\infty} s_{nk}^{\beta}(x) (e^{\frac{A(k+1)}{n}} - e^{\frac{Ak}{n}}) \\
&= \frac{n(e^{\frac{A}{n}}-1)}{A} \sum_{k=0}^{\infty} \frac{e^{-(n+\beta)x}[(n+\beta)xe^{\frac{A}{n}}]^k}{k!} \\
&= \frac{n(e^{\frac{A}{n}}-1)}{A} exp\Big\{(n+\beta)x(e^{\frac{A}{n}}-1)\Big\}.
\end{aligned}
$$

Hence the result follows. □

Remark 2.3. *Expanding the right hand side of the Lemma 2.2 using Mathematica, we have*

$$
\begin{aligned}
(K_n^{\beta} e^{At})(x) &= 1 + \frac{A(2\beta x + 2nx + 1)}{2n} + \frac{A^2\left(3\beta^2x^2 + 6\beta nx^2 + 6\beta x + 3n^2x^2 + 6nx + 1\right)}{6n^2} \\
&+ \frac{A^3}{24n^3}\Big\{4\beta^3x^3 + 12\beta^2nx^3 + 18\beta^2x^2 + 12\beta n^2x^3 + 36\beta nx^2 + 14\beta x + 4n^3x^3 + 18n^2x^2 \\
&+ 14nx + 1\Big\} + \frac{A^4}{120n^4}\Big\{5\beta^4x^4 + 20\beta^3nx^4 + 40\beta^3x^3 + 30\beta^2n^2x^4 + 120\beta^2nx^3 + 75\beta^2x^2 \\
&+ 20\beta n^3x^4 + 120\beta n^2x^3 + 150\beta nx^2 + 30\beta x + 5n^4x^4 + 40n^3x^3 + 75n^2x^2 + 30nx + 1\Big\} \\
&+ \frac{A^5}{720n^5}\Big\{6\beta^5x^5 + 30\beta^4nx^5 + 75\beta^4x^4 + 60\beta^3n^2x^5 + 300\beta^3nx^4 + 260\beta^3x^3 + 60\beta^2n^3x^5 \\
&+ 450\beta^2n^2x^4 + 780\beta^2nx^3 + 270\beta^2x^2 + 30\beta n^4x^5 + 300\beta n^3x^4 + 780\beta n^2x^3 + 540\beta nx^2 \\
&+ 62\beta x + 6n^5x^5 + 75n^4x^4 + 260n^3x^3 + 270n^2x^2 + 62nx + 1\Big\} + \frac{A^6}{5040n^6}\Big\{7\beta^6x^6 + 42\beta^5nx^6 \\
&+ 126\beta^5x^5 + 105\beta^4n^2x^6 + 630\beta^4nx^5 + 700\beta^4x^4 + 140\beta^3n^3x^6 + 1260\beta^3n^2x^5 + 2800\beta^3nx^4 \\
&+ 1400\beta^3x^3 + 105\beta^2n^4x^6 + 1260\beta^2n^3x^5 + 4200\beta^2n^2x^4 + 4200\beta^2nx^3 + 903\beta^2x^2 + 42\beta n^5x^6 \\
&+ 630\beta n^4x^5 + 2800\beta n^3x^4 + 4200\beta n^2x^3 + 1806\beta nx^2 + 126\beta x + 7n^6x^6 + 126n^5x^5 + 700n^4x^4 \\
&+ 1400n^3x^3 + 903n^2x^2 + 126nx + 1\Big\} + O\left(A^7\right).
\end{aligned}
$$

Lemma 2.4. *The following result holds:*

$$
\begin{aligned}
(K_n^\beta e_0)(x) &= 1,\\
(K_n^\beta e_1)(x) &= \frac{2\beta x + 2nx + 1}{2n},\\
(K_n^\beta e_2)(x) &= \frac{3\beta^2x^2 + 6\beta x(nx+1) + 3n^2x^2 + 6nx + 1}{3n^2},\\
(K_n^\beta e_3)(x) &= \frac{1}{4n^3}\Big\{4\beta^3x^3 + 6\beta^2x^2(2nx+3) + 2\beta x\left(6n^2x^2 + 18nx + 7\right) + 4n^3x^3 + 18n^2x^2 + 14nx + 1\Big\},\\
(K_n^\beta e_4)(x) &= \frac{1}{5n^4}\Big\{5\beta^4x^4 + 20\beta^3x^3(nx+2) + 15\beta^2x^2\left(2n^2x^2 + 8nx + 5\right)\\
&\quad +10\beta x\left(2n^3x^3 + 12n^2x^2 + 15nx + 3\right) + 5n^4x^4 + 40n^3x^3 + 75n^2x^2 + 30nx + 1\Big\},\\
(K_n^\beta e_5)(x) &= \frac{1}{6n^5}\Big\{6\beta^5x^5 + 15\beta^4x^4(2nx+5) + 20\beta^3x^3\left(3n^2x^2 + 15nx + 13\right)\\
&\quad +30\beta^2x^2\left(2n^3x^3 + 15n^2x^2 + 26nx + 9\right) + 2\beta x\left(15n^4x^4 + 150n^3x^3 + 390n^2x^2 + 270nx + 31\right)\\
&\quad +6n^5x^5 + 75n^4x^4 + 260n^3x^3 + 270n^2x^2 + 62nx + 1\Big\},\\
(K_n^\beta e_6)(x) &= \frac{1}{7n^6}\Big\{7\beta^6x^6 + 42\beta^5x^5(nx+3) + 35\beta^4x^4\left(3n^2x^2 + 18nx + 20\right)\\
&\quad +140\beta^3x^3\left(n^3x^3 + 9n^2x^2 + 20nx + 10\right) + 21\beta^2x^2\left(5n^4x^4 + 60n^3x^3 + 200n^2x^2 + 200nx + 43\right)\\
&\quad +14\beta x\left(3n^5x^5 + 45n^4x^4 + 200n^3x^3 + 300n^2x^2 + 129nx + 9\right) + 7n^6x^6 + 126n^5x^5 + 700n^4x^4\\
&\quad +1400n^3x^3 + 903n^2x^2 + 126nx + 1\Big\}.
\end{aligned}
$$

Proof. We know that the $m-th$ order moment is the coefficient of $\frac{A^m}{m!}$ in the expansion of moment generating function, hence from Remark 2.3 we obtain the required result. □

Lemma 2.5. *If we denote $\mu^\beta_{n,m}(x) = (K_n^\beta(e_1 - xe_0)^m)(x)$, then we have*

$$
\begin{aligned}
\mu^\beta_{n,0}(x) &= 1,\\
\mu^\beta_{n,1}(x) &= \frac{2\beta x + 1}{2n},\\
\mu^\beta_{n,2}(x) &= \frac{3\beta^2x^2 + 6\beta x + 3nx + 1}{3n^2},\\
\mu^\beta_{n,3}(x) &= \frac{4\beta^3x^3 + 18\beta^2x^2 + 2\beta x(6nx+7) + 10nx + 1}{4n^3},\\
\mu^\beta_{n,4}(x) &= \frac{5\beta^4x^4 + 40\beta^3x^3 + 15\beta^2x^2(2nx+5) + 10\beta x(8nx+3) + 15n^2x^2 + 25nx + 1}{5n^4}.
\end{aligned}
$$

Proof. The proof follows easily from Lemma 2.4. □

Let $k_n^\beta(x,t) = n\sum_{k=0}^{\infty} s^\beta_{nk}(x)\phi_n(t)$, where $\phi_n(t)$ represents the characteristic function of the interval $[\frac{k}{n}, \frac{k+1}{n}]$. Hence,

$$(S_n^\beta f)(x) = \int_0^\infty k_n^\beta(x,t)f(t)dt.$$

Lemma 2.6. *For $x \in (0,\infty)$, we have*

$$\int_0^y k_n^\beta(x,t)dt \le \frac{1 + 6\beta x + 3nx + 3\beta^2x^2}{3n^2(x-y)^2},\ \ 0 < y < x$$

and

$$\int_z^\infty k_n^\beta(x,t)dt \leq \frac{1+6\beta x+3nx+3\beta^2x^2}{3n^2(z-x)^2}, \quad x<z<\infty.$$

Proof. On applying Lemma 2.5, the proof follows immediately along the lines of [23]. □

Lemma 2.7. *For $x \in (0,\infty)$,*

$$s_{nk}^\beta(x) \leq \frac{1}{\sqrt{2e(n+\beta)x}}.$$

Proof. Using the bounds as given in [27], we have the above inequality. □

Lemma 2.8. *For $x>0$, we have*

$$\left|\sum_{k=0}^\infty s_{nk}^\beta(x) - \frac{1}{2}\right| \leq 0.8\frac{\sqrt{1+3x}}{\sqrt{(n+\beta)x}}.$$

Proof. The proof follows along the lines of [26]. □

3. Approximation for $(K_n^\beta f)$

Theorem 3.1. *The following relation exists between semi-exponential Szász-Mirakyan-operators and its Kantorovich variant:*

$$\left(1+\frac{\beta}{n}\right)(K_n^\beta f)(x) = (D\circ S_n^\beta \circ F)(x),$$

where S_n^β are convex of order 1 and F denotes the integral $\int_0^x f(t)dt$.

Proof. Consider

$$\begin{aligned} Ds_{nk}^\beta(x) &= \frac{kx^{k-1}(\beta+n)^k e^{x(-\beta-n)}}{k!} + \frac{(-\beta-n)x^k(\beta+n)^k e^{x(-\beta-n)}}{k!} \\ &= (n+\beta)s_{nk-1}^\beta(x) - ns_{nk}^\beta(x) - \beta s_{nk}^\beta(x). \end{aligned}$$

Thus,

$$(D+\beta)s_{nk}^\beta(x) = (n+\beta)s_{nk-1}^\beta(x) - ns_{nk}^\beta(x). \tag{4}$$

Now,

$$(D\circ S_n^\beta\circ F)(x) = \sum_{k=0}^\infty (D+\beta)s_{nk}^\beta(x)F\left(\frac{k}{n}\right) - \beta\sum_{k=0}^\infty s_{nk}^\beta(x)F\left(\frac{k}{n}\right).$$

Using the relation (4), we obtain

$$
\begin{aligned}
(D \circ S_n^{\beta} \circ F)(x) &= \sum_{k=0}^{\infty} \left\{(n+\beta) s_{nk-1}^{\beta}(x) - n s_{nk}^{\beta}(x)\right\} F\left(\frac{k}{n}\right) - \beta \sum_{k=0}^{\infty} s_{nk}^{\beta}(x) F\left(\frac{k}{n}\right) \\
&= n \sum_{k=0}^{\infty} \left\{s_{nk-1}^{\beta}(x) - s_{nk}^{\beta}(x)\right\} F\left(\frac{k}{n}\right) + \beta \sum_{k=0}^{\infty} s_{nk-1}^{\beta}(x) F\left(\frac{k}{n}\right) - \beta \sum_{k=0}^{\infty} s_{nk}^{\beta}(x) F\left(\frac{k}{n}\right) \\
&= n \sum_{k=0}^{\infty} s_{nk}^{\beta}(x) \left\{F\left(\frac{k+1}{n}\right) - F\left(\frac{k}{n}\right)\right\} + \beta \left\{\sum_{k=0}^{\infty} s_{nk-1}^{\beta}(x) - \sum_{k=0}^{\infty} s_{nk}^{\beta}(x)\right\} F\left(\frac{k}{n}\right) \\
&= n \sum_{k=0}^{\infty} s_{nk}^{\beta}(x) \left\{F\left(\frac{k+1}{n}\right) - F\left(\frac{k}{n}\right)\right\} + \beta \sum_{k=0}^{\infty} s_{nk}^{\beta}(x) \left\{F\left(\frac{k+1}{n}\right) - F\left(\frac{k}{n}\right)\right\} \\
&= n \sum_{k=0}^{\infty} s_{nk}^{\beta}(x) \int_{k/n}^{(k+1)/n} f(t)dt + \frac{\beta}{n} \left(n \sum_{k=0}^{\infty} s_{nk}^{\beta}(x) \int_{k/n}^{(k+1)/n} f(t)dt\right) \\
&= (K_n^{\beta} f)(x) + \frac{\beta}{n} (K_n^{\beta} f)(x),
\end{aligned}
$$

which proves the required relation. □

Remark 3.2. *Here we show the validity of our Theorem 3.1, by considering few monomials. Applying Lemma 2.1 for $f(t) = e_0$, we have*

$$
\begin{aligned}
\left(\frac{n}{n+\beta}\right)\left(D \circ S_n^{\beta} \circ \int_0^x f(t)dt\right) &= \left(\frac{n}{n+\beta}\right)\left(D \circ S_n^{\beta} \circ e_1\right) = \left(\frac{n}{n+\beta}\right) D\left\{\frac{e_1(\beta+n)}{n}\right\} \\
&= 1 = (K_n^{\beta} e_0)(x).
\end{aligned}
$$

Next if we take $f(t) = e_1$,

$$
\begin{aligned}
\left(\frac{n}{n+\beta}\right)\left(D \circ S_n^{\beta} \circ \int_0^x f(t)dt\right) &= \left(\frac{n}{n+\beta}\right)\left(D \circ S_n^{\beta} \circ \frac{e_2}{2}\right) \\
&= \frac{n}{2(n+\beta)} D\left\{\frac{(\beta+n)}{n^2}(e_2(\beta+n) + e_1)\right\} \\
&= \frac{1}{2n}(2e_1(\beta+n) + e_0) \\
&= (K_n^{\beta} e_1)(x).
\end{aligned}
$$

For $f(t) = e_2$,

$$
\begin{aligned}
\left(\frac{n}{n+\beta}\right)\left(D \circ S_n^{\beta} \circ \int_0^x f(t)dt\right) &= \left(\frac{n}{n+\beta}\right)\left(D \circ S_n^{\beta} \circ \frac{e_3}{3}\right) \\
&= \frac{n}{3(n+\beta)} D\left\{\frac{(\beta+n)}{n^3}\left((\beta+n)^2 e_3 + 3(\beta+n)e_2 + e_1\right)\right\} \\
&= \frac{1}{3n^2}\left(3(\beta+n)^2 e_2 + 6(\beta+n)e_1 + e_0\right) \\
&= (K_n^{\beta} e_2)(x).
\end{aligned}
$$

This verifies the connection obtained in Theorem 3.1.

Theorem 3.3. *Assume that f is a function of bounded variation on every finite interval on $(0,\infty)$. Then for sufficiently large n, there exists a constant M, such that*

$$\left|(K_n^{\beta}f)(x) - \left\{\frac{f(x^+)+f(x^-)}{2}\right\}\right| \leq \left(0.8\frac{\sqrt{1+3x}+1}{\sqrt{(n+\beta)x}}\right)\left|f(x^+)-f(x^-)\right| + \left\{\frac{2\lambda(2\beta+n+\beta^2x)+x}{n^2x}\right\}\sum_{k=1}^{n} V_{x-\frac{x}{\sqrt{k}}}^{x+\frac{x}{\sqrt{k}}}(g_x) + \frac{M}{n^r},$$

where

$$g_x(t) = \begin{cases} f(t)-f(x^-) & 0 \leq t < x \\ 0 & t = x \\ f(t)-f(x^+) & x < t < \infty \end{cases}$$

and $V_a^b(g_x)$ denotes the total variation of g_x on $[a,b]$.

Proof. Using Lemmas 2.6, 2.7 and 2.8, we obtain the desired inequality following [17] and [18]. □

Let us define $C^*[0,\infty)$ to be the collection of all such functions g such that $|g(x)| \leq N(1+x^2)$, where the constant term N depends on g and is independent of x. The space $C^*[0,\infty)$ is equipped with the norm

$$\|g\|^* = \sup_{x\in[0,\infty)} g(x)(1+x^2)^{-1}.$$

Now, for $g \in C[0,\infty) \cap C^*[0,\infty)$, let the weighted modulus of continuity [21] be given as

$$\Omega(g,\zeta) = \sup_{|h|<\zeta, x\in[0,\infty)} |g(x+h)-g(x)|\left((1+h^2)(1+x^2)\right)^{-1}.$$

By the property of $\Omega(g,\zeta)$, the following inequality holds:

$$|g(t)-g(x)| \leq 2[1+(t-x)^2][1+|t-x|\zeta^{-1}](1+x^2)(1+\zeta^2)\Omega(g,\zeta), \tag{5}$$

where $x,t \in [0,\infty)$.
We now prove the Voronovskaya type asymptotic result, which has been extensively studied by a number of researchers in [1], [2], [4] etc..

Theorem 3.4. *For $f', f'' \in C[0,\infty) \cap C^*[0,\infty)$, we have*

$$\left|(K_n^{\beta}f)(x) - f(x) - f'(x)\left[\frac{2\beta x+1}{2n}\right] - \frac{f''(x)}{2}\left[\frac{3\beta^2x^2+6\beta x+3nx+1}{3n^2}\right]\right| \leq 8.O(n^{-1})(1+x^2)\Omega(f'', n^{\frac{-1}{2}}).$$

Proof. By Taylor's expansion,

$$f(u) = f(x) + (u-x)f'(x) + (u-x)^2\frac{f''(x)}{2} + \xi(u,x)(u-x)^2,$$

where $\xi(u,x) = \frac{1}{2}(f''(\theta)-f''(x))$ is a continuous function vanishing at 0 and $\theta \in (x,u)$.
Now, applying the operator K_n^{β} to the above inequality, we obtain

$$\left|(K_n^{\beta}f)(x) - f(x) - f'(x)\left[\frac{2\beta x+1}{2n}\right] - \frac{f''(x)}{2}\left[\frac{3\beta^2x^2+6\beta x+3nx+1}{3n^2}\right]\right| \leq \left(K_n^{\beta}(|\xi(u,x)|(u-x)^2)\right)(x).$$

Also by Lemma 2.5, we conclude that

$$\begin{aligned}\left(K_n^\beta(|\xi(u,x)|(u-x)^2)\right)(x) &= 8\left[\mu_{n,2}^\beta(x)+\zeta^{-4}\mu_{n,6}^\beta(x)\right](1+x^2)\Omega(f'',\zeta)\\ &= 8\left[O(n^{-1})+\zeta^{-4}O(n^{-3})\right](1+x^2)\Omega(f'',\zeta).\end{aligned}$$

If we choose $\zeta = n^{\frac{-1}{2}}$, then

$$\left(K_n^\beta(|\xi(u,x)|(u-x)^2)\right)(x) \leq 8.O(n^{-1})(1+x^2)\Omega(f'',n^{\frac{-1}{2}}),$$

which lead us to the desired result. □

Corollary 3.5. *For $f', f'' \in C[0,\infty) \cap C^*[0,\infty)$, we have*

$$\lim_{n\to\infty} n\left|(K_n^\beta f)(x) - f(x)\right| = f'(x)\left[\frac{2\beta x+1}{2}\right] - \frac{x}{2}f''(x).$$

Lemma 3.6. *For a function f which is bounded on $[0,\infty)$, let $\|f\| = \sup_{x\in[0,\infty)} |f(x)|$. Then*

$$\left|(K_n^\beta f)(x)\right| \leq \|f\|.$$

Let $C_B[0,\infty)$ denote the space of all uniformly continuous and bounded functions on $[0,\infty)$ and

$$C_B^{**}[0,\infty) = \left\{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\right\}.$$

For $\alpha > 0$, the $K-$ functional is given as

$$K_2(f,\alpha) = inf\left\{\|f-g\| + \alpha\|g''\|\right\},$$

where $g \in C_B^{**}[0,\infty)$. Then $K_2(f,\alpha) \leq C\omega_2(f,\alpha^{\frac{1}{2}})$, where ω_2 denotes the modulus of continuity of second order and C is a positive absolute constant. For more details, one may refer [12, pp. 177, Theorem 2.4].

Theorem 3.7. *Fot $f \in C_B[0,\infty)$,*

$$\left|(K_n^\beta f)(x) - f(x)\right| \leq C\omega_2(f,\sqrt{\alpha_n^\beta(x)}) + \omega\left(f, \frac{2\beta x+1}{2n}\right),$$

where ω is first order modulus of continuity and

$$\alpha_n^\beta(x) = \frac{24\beta^2x^2 + 36\beta x + 12n + 7}{12n^2}.$$

Proof. Let us define the operators $\hat{K}_n^\beta : C_B[0,\infty) \to C_B[0,\infty)$ by

$$(\hat{K}_n^\beta f)(x) = (K_n^\beta f)(x) - f\left(\frac{2\beta x + 2nx + 1}{2n}\right) + f(x).$$

In view of Lemma 2.4, it is clear that these operators preserve linear functions. By Taylor's expansion, we may write

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-v)g''(v)dv,$$

where $g \in C_B^{**}[0,\infty)$ and $x, t \in [0,\infty)$.
Thus,

$$\begin{aligned}\left|(\hat{K}_n^{\beta} g)(x) - g(x)\right| &\leq \left(\hat{K}_n^{\beta}\left|\int_x^t (t-v)g''(v)dv\right|\right)(x)\\ &\leq \left(K_n^{\beta}\left|\int_x^t (t-v)g''(v)dv\right|\right)(x) + \left|\int_x^{\frac{2\beta x+2nx+1}{2n}}\left(\frac{2\beta x+2nx+1}{2n} - v\right)g''(v)dv\right|\\ &\leq \mu_{n,2}^{\beta}(x)\|g''\| + \left|\int_x^{\frac{2\beta x+2nx+1}{2n}}\left(\frac{2\beta x+2nx+1}{2n} - v\right)dv\right|\|g''\|.\end{aligned}$$

Now, on applying Lemma 2.5, we are led to

$$\begin{aligned}\left|(\hat{K}_n^{\beta} g)(x) - g(x)\right| &\leq \left\{\mu_{n,2}^{\beta}(x) + \left(\frac{2\beta x+1}{2n}\right)^2\right\}\|g''\|\\ &= \left\{\frac{3\beta^2x^2+6\beta x+3nx+1}{3n^2} + \left(\frac{2\beta x+1}{2n}\right)^2\right\}\|g''\|\\ &:= \alpha_n^{\beta}(x)\|g''\|.\end{aligned}$$

By the definition of the operators $\hat{K}_n^{\beta}$ and Lemma 3.6, we obtain

$$\|(\hat{K}_n^{\beta} f)(x)\| \leq \|(K_n^{\beta} f)(x)\| + 2\|f\| \leq 3\|f\|.$$

Finally, we may conclude that

$$\begin{aligned}\left|(K_n^{\beta} f)(x) - f(x)\right| &\leq \left|(\hat{K}_n^{\beta} g)(x) - g(x)\right| + \left|(\hat{K}_n^{\beta}(f-g))(x) - (f-g)(x)\right| + \left|f\left(\frac{2\beta x+2nx+1}{2n}\right) - f(x)\right|\\ &\leq \alpha_n^{\beta}(x)\|g''\| + 4\|f-g\| + \left|f\left(\frac{2\beta x+2nx+1}{2n}\right) - f(x)\right|\\ &\leq C\left\{\alpha_n^{\beta}(x)\|g''\| + \|f-g\|\right\} + \omega\left(f, \frac{2\beta x+1}{2n}\right).\end{aligned}$$

Considering infimum over $g \in C_B^{**}[0,\infty)$ and using the property of K- functional, we obtain the required assertion. □

4. Preservation of $e^{Aqx}, q = 1, 2$

Many researchers have explored the preservation of exponential functions in the recent past. ([3], [5], [6], [7], [8] and [22]) can be used for more in-depth research in the related area. For showing the preservation of exponential function, we consider the following form of semi-exponential Szász-Mirakyan-Kantorovich operators:

$$(\widetilde{K}_n^{\beta} f)(x) = n\sum_{k=0}^{\infty} s_{nk}^{\beta}(a_n^{\beta}(x))\int_{k/n}^{(k+1)/n} f(t)dt. \tag{6}$$

For preservation of e^{Ax}, we must have

$$\begin{aligned}(\widetilde{K}_n^{\beta} e^{At})(x) = e^{Ax} &= n\sum_{k=0}^{\infty} s_{nk}^{\beta}(a_n^{\beta}(x))\int_{k/n}^{(k+1)/n} f(t)dt\\ &= \frac{n(e^{\frac{A}{n}}-1)}{A} exp\left\{(n+\beta)a_n^{\beta}(x)(e^{\frac{A}{n}}-1)\right\},\end{aligned}$$

which implies

$$a_n^{\beta}(x) = \frac{\ln(Ae^{Ax}) - \ln n(e^{A/n} - 1)}{(n+\beta)(e^{A/n} - 1)}.$$

Clearly, $\lim_{n\to\infty} a_n^{\beta}(x) = x$.

Next, we define the operators, which preserve e^{Ax} and e^{2Ax}, in the following way:

$$(\widetilde{\widetilde{K}}_n^{\beta} f)(x) = ne^{Ax} \sum_{k=0}^{\infty} s_{nk}^{\beta}(a_n^{\beta}(x)) \int_{k/n}^{(k+1)/n} e^{-At} f(t)dt. \tag{7}$$

Lemma 4.1. *For $x \in (0,\infty)$ and $n \in N$, the following relations hold:*

$$\begin{aligned}
(\widetilde{\widetilde{K}}_n^{\beta} e_0)(x) &= \frac{ne^{Ax}(1-e^{\frac{-A}{n}})}{A} exp\Big\{(n+\beta)\left(\frac{\ln(Ae^{Ax}) - \ln n(e^{A/n}-1)}{(n+\beta)\,(e^{A/n}-1)}\right)\left(e^{\frac{-A}{n}} - 1\right)\Big\},\\
(\widetilde{\widetilde{K}}_n^{\beta} e^{3At})(x) &= \frac{ne^{Ax}(e^{\frac{2A}{n}}-1)}{2A} exp\Big\{(n+\beta)\left(\frac{\ln(Ae^{Ax}) - \ln n(e^{A/n}-1)}{(n+\beta)\,(e^{A/n}-1)}\right)\left(e^{\frac{2A}{n}} - 1\right)\Big\},\\
(\widetilde{\widetilde{K}}_n^{\beta} e^{4At})(x) &= \frac{ne^{Ax}(e^{\frac{3A}{n}}-1)}{3A} exp\Big\{(n+\beta)\left(\frac{\ln(Ae^{Ax}) - \ln n(e^{A/n}-1)}{(n+\beta)\,(e^{A/n}-1)}\right)\left(e^{\frac{3A}{n}} - 1\right)\Big\}.
\end{aligned}$$

Proof. Using Remark 2.2 and the value of $a_n^{\beta}(x)$, we have

$$\begin{aligned}
(\widetilde{\widetilde{K}}_n^{\beta} e_0)(x) &= \frac{ne^{Ax}(1-e^{\frac{-A}{n}})}{A} exp\Big\{(n+\beta)a_n^{\beta}(x)(e^{\frac{-A}{n}} - 1)\Big\}\\
&= \frac{ne^{Ax}(1-e^{\frac{-A}{n}})}{A} exp\Big\{(n+\beta)\left(\frac{\ln(Ae^{Ax}) - \ln n(e^{A/n}-1)}{(n+\beta)\,(e^{A/n}-1)}\right)\left(e^{\frac{-A}{n}} - 1\right)\Big\},\\
(\widetilde{\widetilde{K}}_n^{\beta} e^{3At})(x) &= \frac{ne^{Ax}(e^{\frac{2A}{n}}-1)}{2A} exp\Big\{(n+\beta)a_n^{\beta}(x)(e^{\frac{2A}{n}} - 1)\Big\}\\
&= \frac{ne^{Ax}(e^{\frac{2A}{n}}-1)}{2A} exp\Big\{(n+\beta)\left(\frac{\ln(Ae^{Ax}) - \ln n(e^{A/n}-1)}{(n+\beta)\,(e^{A/n}-1)}\right)\left(e^{\frac{2A}{n}} - 1\right)\Big\},\\
(\widetilde{\widetilde{K}}_n^{\beta} e^{4At})(x) &= \frac{ne^{Ax}(e^{\frac{3A}{n}}-1)}{3A} exp\Big\{(n+\beta)a_n^{\beta}(x)(e^{\frac{3A}{n}} - 1)\Big\}\\
&= \frac{ne^{Ax}(e^{\frac{3A}{n}}-1)}{3A} exp\Big\{(n+\beta)\left(\frac{\ln(Ae^{Ax}) - \ln n(e^{A/n}-1)}{(n+\beta)\,(e^{A/n}-1)}\right)\left(e^{\frac{3A}{n}} - 1\right)\Big\}.
\end{aligned}$$

□

Remark 4.2. *Let $\psi_{A,x}^{i}(t) = (e^{At} - e^{Ax})^{i}$, $i = 0, 1, 2, \ldots$. Then*

$$\begin{aligned}
(\widetilde{\widetilde{K}}_n^{\beta} \psi_{A,x}^{1}(t))(x) &= e^{Ax}\left[1 - (\widetilde{\widetilde{K}}_n^{\beta} e_0)(x)\right]\\
&= e^{Ax}\left[1 - \frac{ne^{Ax}(1-e^{\frac{-A}{n}})}{A} exp\Big\{(n+\beta)\left(\frac{\ln(Ae^{Ax}) - \ln n(e^{A/n}-1)}{(n+\beta)\,(e^{A/n}-1)}\right)\left(e^{\frac{-A}{n}} - 1\right)\Big\}\right],\\
(\widetilde{\widetilde{K}}_n^{\beta} \psi_{A,x}^{2}(t))(x) &= (\widetilde{\widetilde{K}}_n^{\beta} e^{2At})(x) - 2e^{Ax}(\widetilde{\widetilde{K}}_n^{\beta} e^{At})(x) + e^{2Ax}(\widetilde{\widetilde{K}}_n^{\beta} e_0)(x)\\
&= e^{2Ax}\left[\frac{ne^{Ax}(1-e^{\frac{-A}{n}})}{A} exp\Big\{(n+\beta)\left(\frac{\ln(Ae^{Ax}) - \ln n(e^{A/n}-1)}{(n+\beta)\,(e^{A/n}-1)}\right)\left(e^{\frac{-A}{n}} - 1\right)\Big\} - 1\right],
\end{aligned}$$

$$
\begin{aligned}
(\widetilde{\widetilde{K}}_n^{\beta}\psi_{A,x}^4(t))(x) &= (\widetilde{\widetilde{K}}_n^{\beta}e^{4At})(x) - 4e^{Ax}(\widetilde{\widetilde{K}}_n^{\beta}e^{3At})(x) + 6e^{2Ax}(\widetilde{\widetilde{K}}_n^{\beta}e^{2At})(x) \\
&\quad -4e^{3Ax}(\widetilde{\widetilde{K}}_n^{\beta}e^{At})(x) + e^{4Ax}(\widetilde{\widetilde{K}}_n^{\beta}e_0)(x) \\
&= e^{Ax}\left\{\frac{n(e^{\frac{3A}{n}}-1)}{3A}exp\left\{(n+\beta)\left(\frac{\ln(Ae^{Ax}) - \ln n(e^{A/n}-1)}{(n+\beta)(e^{A/n}-1)}\right)\left(e^{\frac{3A}{n}}-1\right)\right\}\right. \\
&\quad -4\frac{ne^{Ax}(e^{\frac{2A}{n}}-1)}{2A}exp\left\{(n+\beta)\left(\frac{\ln(Ae^{Ax}) - \ln n(e^{A/n}-1)}{(n+\beta)(e^{A/n}-1)}\right)\left(e^{\frac{2A}{n}}-1\right)\right\} \\
&\quad +2e^{3Ax} + e^{4Ax}\frac{n(1-e^{\frac{-A}{n}})}{A}exp\left\{(n+\beta)\left(\frac{\ln(Ae^{Ax}) - \ln n(e^{A/n}-1)}{(n+\beta)(e^{A/n}-1)}\right)\left(e^{\frac{-A}{n}}-1\right)\right\}.
\end{aligned}
$$

Let $\widehat{C}[0,\infty)$ be the subspace of $C[0,\infty)$ consisting of continuous real valued functions on positive real axis such that $\lim_{x\to\infty} f(x)$ is finite.

Theorem 4.3. *If $f \in \widehat{C}[0,\infty)$ has second order derivative in the interval $[0,\infty)$, then*

$$\lim_{n\to\infty} n\left((\widetilde{\widetilde{K}}_n^{\beta}f)(x) - f(x)\right) = A^2xf(x) - \frac{3A}{2}xf'(x) + \frac{x}{2}f''(x).$$

Proof. From Taylor's expansion, we have

$$
\begin{aligned}
f(t) = (f\circ log_A)(e^{At}) &= (f\circ log_A)(e^{Ax}) + (f\circ log_A)'(e^{Ax})\psi_{A,x}^1(t) + \frac{(f\circ log_A)''(e^{Ax})}{2}\psi_{A,x}^2(t) \\
&\quad + g(e^{At}-e^{Ax})\psi_{A,x}^2(t),
\end{aligned}
$$

where g is a continuous function which vanishes at 0. Applying the operator $\widetilde{\widetilde{K}}_n^{\beta}$ on both the sides of above inequality, we get

$$
\begin{aligned}
(\widetilde{\widetilde{K}}_n^{\beta}f)(x) &= f(x)(\widetilde{\widetilde{K}}_n^{\beta}e_0)(x) + (f\circ log_A)'(e^{Ax})(\widetilde{\widetilde{K}}_n^{\beta}\psi_{A,x}^1(t))(x) + \frac{(f\circ log_A)''(e^{Ax})}{2}(\widetilde{\widetilde{K}}_n^{\beta}\psi_{A,x}^2(t))(x) \\
&\quad + (\widetilde{\widetilde{K}}_n^{\beta}g(e^{At}-e^{Ax})\psi_{A,x}^2(t))(x).
\end{aligned}
$$

As

$$(f\circ log_A)'(e^{Ax}) = e^{-Ax}A^{-1}f'(x)$$

and

$$(f\circ log_A)''(e^{Ax}) = e^{-2Ax}(A^{-2}f''(x) - A^{-1}f'(x)),$$

therefore

$$
\begin{aligned}
(\widetilde{\widetilde{K}}_n^{\beta}f)(x) - f(x) &= f(x)\left[(\widetilde{\widetilde{K}}_n^{\beta}e_0)(x) - 1\right] + e^{-Ax}A^{-1}f'(x))(\widetilde{\widetilde{K}}_n^{\beta}\psi_{A,x}^1(t))(x) \\
&\quad + \frac{e^{-2Ax}(A^{-2}f''(x) - A^{-1}f'(x))}{2}(\widetilde{\widetilde{K}}_n^{\beta}\psi_{A,x}^2(t))(x) + (\widetilde{\widetilde{K}}_n^{\beta}g(e^{At}-e^{Ax})\psi_{A,x}^2(t))(x) \\
&= f(x)\left[\frac{ne^{Ax}(1-e^{\frac{-A}{n}})}{A}exp\left\{(n+\beta)\left(\frac{\log(Ae^{Ax}) - \log n(e^{A/n}-1)}{(n+\beta)(e^{A/n}-1)}\right)\left(e^{\frac{-A}{n}}-1\right)\right\} - 1\right] \\
&\quad + e^{-Ax}A^{-1}f'(x))e^{Ax} \\
&\quad \left[1 - \frac{ne^{Ax}(1-e^{\frac{-A}{n}})}{A}exp\left\{(n+\beta)\left(\frac{\log(Ae^{Ax}) - \log n(e^{A/n}-1)}{(n+\beta)(e^{A/n}-1)}\right)\left(e^{\frac{-A}{n}}-1\right)\right\}\right]
\end{aligned}
$$

$$+\frac{e^{-2Ax}(A^{-2}f''(x)-A^{-1}f'(x))}{2}e^{2Ax}$$
$$\left[\frac{ne^{Ax}(1-e^{\frac{-A}{n}})}{A}exp\left\{(n+\beta)\left(\frac{\log(Ae^{Ax})-\log n(e^{A/n}-1)}{(n+\beta)\,(e^{A/n}-1)}\right)\left(e^{\frac{-A}{n}}-1\right)\right\}-1\right]$$
$$+(\widetilde{\widetilde{K}}_n^{\beta}g(e^{At}-e^{Ax})\psi_{A,x}^2(t))(x)$$
$$=\left[\frac{ne^{Ax}(1-e^{\frac{-A}{n}})}{A}exp\left\{(n+\beta)\left(\frac{\log(Ae^{Ax})-\log n(e^{A/n}-1)}{(n+\beta)\,(e^{A/n}-1)}\right)\left(e^{\frac{-A}{n}}-1\right)\right\}-1\right]$$
$$\left[f(x)-\frac{3}{2A}f'(x)+\frac{1}{2A^2}f''(x)\right]+(\widetilde{\widetilde{K}}_n^{\beta}g(e^{At}-e^{Ax})\psi_{A,x}^2(t))(x).$$

From Cauchy-Schwarz inequality, we may write

$$n\left|(\widetilde{\widetilde{K}}_n^{\beta}g(e^{At}-e^{Ax})\psi_{A,x}^2(t))(x)\right|\leq\left[(\widetilde{\widetilde{K}}_n^{\beta}g^2(e^{At}-e^{Ax})(t))(x)\right]^{\frac{1}{2}}\left[n^2(\widetilde{\widetilde{K}}_n^{\beta}\psi_{A,x}^4(t))(x)\right]^{\frac{1}{2}}.$$

Next we have,

$$\lim_{n\to\infty}(\widetilde{\widetilde{K}}_n^{\beta}g^2(e^{At}-e^{Ax})(t))(x)=0$$

and simple computations lead us to

$$\lim_{n\to\infty}n\left|(\widetilde{\widetilde{K}}_n^{\beta}g(e^{At}-e^{Ax})\psi_{A,x}^2(t))(x)\right|=0.$$

Hence,

$$\lim_{n\to\infty}n\left((\widetilde{\widetilde{K}}_n^{\beta}f)(x)-f(x)\right)=A^2xf(x)-\frac{3A}{2}xf'(x)+\frac{x}{2}f''(x)$$

which proves the result. □

Remark 4.4. *The original Szász-Mirakyan operators, defined by (2), preserve linear functions but this property is not possessed by their modified form defined by (1). The preservation of affine function of different operators have been discussed by the authors in [9]. One may modify the operators (1) so as to preserve affine function. We will discuss this elsewhere.*

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