



## Continuous evolution algebras

Fernando Montaner<sup>a</sup>, Irene Paniello<sup>b</sup>

<sup>a</sup>Departamento de Matemáticas, Universidad de Zaragoza, Spain

<sup>b</sup>Departamento de Estadística, Informática y Matemáticas, Universidad Pública de Navarra, Spain

**Abstract.** We formulate the notion of continuous evolution algebra in terms of differentiable matrix-valued functions, to then study those such algebras arising as solutions of ODE problems. Given their dependence on natural bases, matrix Lie groups provide a suitable framework where considering time-variant evolution algebras. We conclude by broadening our approach by considering continuous evolution algebras stemming as flow lines on matrix Lie groups.

### 1. Introduction

Evolution algebras were introduced in [1, 2] aimed to study dynamics in non-Mendelian genetic systems, and soon were connected to a broad number of other areas of research. One of their most fruitful interactions stems from homogeneous discrete-time (HDT) Markov chains. This connection, already settled in [1], has been recently revisited in [3], where the HDT case was generalized to a more general continuous-time framework.

Characterized as nonassociative algebras admitting (natural) bases, for which the only non-vanishing products arise from the squares of the natural basis elements, evolution algebras have square (and not cubic) structure matrices. A real evolution algebra is then called Markov when a natural basis exists yielding a nonnegative row stochastic (i.e. Markov) structure matrix [3, Remark 2.1]. The passage from time-invariant to time-variant Markov evolution algebras was considered in [3], and defined in terms of standard stochastic semigroups.

Given a finite dimensional (real) vector space  $\mathcal{E}$  with basis  $\mathcal{B} = \{e_1, \dots, e_n\}$ , a family  $\mathcal{E} = \{\mathcal{E}_t = (\mathcal{E}, m(t))\}_{t \geq 0}$  of evolution algebras with multiplication:

$$m(t)(e_i \otimes e_j) = e_i \cdot_t e_j = \begin{cases} \sum_{k=1}^n a_{ik}(t)e_k, & i = j = 1, \dots, n; \\ 0, & \text{otherwise;} \end{cases}$$

is a *continuous time Markov evolution algebra* (CT-Markov EA) if the structure matrices  $\{\mathbf{A}(t)\}_{t \geq 0}$  (of each  $\mathcal{E}_t$  w.r.t.  $\mathcal{B}$ ) define a *standard stochastic semigroup* on the finite index set  $\Lambda = \{1, \dots, n\}$ . Then, for each  $t, s \geq 0$ :

---

2020 *Mathematics Subject Classification.* Primary 17D92; Secondary 17A01, 92D25

*Keywords.* Continuous evolution algebra, Matrix Lie group, Flow, Vector field

Received: 27 January 2022; Revised: 06 October 2022; Accepted: 03 November 2022

Communicated by Dijana Mosić

Supported by grants MTM2017-83506-C2-1-P (AEI/FEDER, UE) and PID2021-123461NB-C21, funded by MCIN/AEI/10.13039/501100011033 and by "ERDF A way of making Europe". The first author is also supported by Diputación General de Aragón (Grupo de Investigación Álgebra y Geometría).

*Email addresses:* [fmontane@unizar.es](mailto:fmontane@unizar.es) (Fernando Montaner), [irene.paniello@unavarra.es](mailto:irene.paniello@unavarra.es) (Irene Paniello)

- (i)  $\mathbf{A}(t)$  is a Markov matrix.
- (ii)  $\mathbf{A}(0) = \mathbf{I}_n$ .
- (iii)  $\mathbf{A}(t + s) = \mathbf{A}(t)\mathbf{A}(s)$  (Chapman-Kolmogorov equation or semigroup property).
- (iv)  $\lim_{t \rightarrow 0^+} \mathbf{A}(t) = \mathbf{A}(0) = \mathbf{I}_n$ , componentwise (standard property).

Finite state standard stochastic semigroups are solutions of Backward and Forward Kolmogorov differential equations: (B)  $\mathbf{A}'(t) = \mathbf{Q}\mathbf{A}(t)$  and (F)  $\mathbf{A}'(t) = \mathbf{A}(t)\mathbf{Q}$ , with initial condition  $\mathbf{A}(0) = \mathbf{I}_n$ . The unique solution is  $\mathbf{A}(t) = e^{t\mathbf{Q}}$ , for a *rate matrix* or *Markov generator*  $\mathbf{Q}$ , a matrix with nonnegative off-diagonal entries and row sums equal to zero. It also holds  $\mathbf{A}'(0) = \mathbf{Q}$ . Moreover, since  $\det(\mathbf{A}(t)) = e^{tr(t\mathbf{Q})}$ , matrices in finite standard stochastic semigroups are nonsingular matrices in the stochastic group  $S(n, \mathbb{R})$  [4].

Aimed by Tian’s outlined notion of continuous evolution algebra [1, Subsection 6.2.4], in the second section we formulate continuous evolution algebras to be a family of finite dimensional algebras, defined on the same  $\mathbb{K}$ -vector space, endowed with a natural basis for which the corresponding structure matrices define differentiable matrix-valued functions, that is, a differentiable curve in  $M_n(\mathbb{K})$ , for a field  $\mathbb{K}$  (usually  $\mathbb{R}$  or  $\mathbb{C}$ ). It is then straightforward to consider those algebras arising as solutions of first-order matrix ODE problems  $\mathbf{A}'(t) = \mathbf{A}(t)\mathbf{X}$  (equivalently  $\mathbf{A}'(t) = \mathbf{X}\mathbf{A}(t)$ ) with initial condition  $\mathbf{A}(0) = \mathbf{I}_n$ .

In the third section matrix Lie groups provide a suitable framework where considering continuous evolution algebras. One-parameter subgroups provide immediate examples of  $G$ -continuous evolution algebras, that is, continuous evolution algebras with structure matrices into a matrix Lie group  $G$ . These algebras are then characterized by their velocity vectors (structure matrix derivative at  $t = 0$ ) and an initial condition (structure matrix  $\mathbf{A}(0) = \mathbf{I}_n$  at  $t = 0$ ). This brings into the Lie algebra  $\mathfrak{g}$  of the matrix Lie group  $G$  or equivalently the tangent space  $T_{\mathbf{I}_n}(G)$  of  $G$  at  $\mathbf{I}_n$ . Different initial conditions lead us to consider the tangent bundle  $T(G)$  of  $G$ .

Continuous time Markov evolution algebras [3] do fit into the previous scheme, as it is shown in section four, by taking into account the properties of the exponential matrix series.

In the fifth section we show how  $G$ -continuous evolution algebras, resulting in Section 3, make way towards continuous evolution algebras understood as flow lines for global flows on matrix Lie groups.

In the last section, besides briefly summarizing the main guidelines along the work, we pose for further study continuous evolution algebras arising from dynamical systems obeying more general differential equations.

## 2. Continuous evolution algebras

Let  $\mathbb{K}$  be  $\mathbb{R}$  or  $\mathbb{C}$ . Matrices will be denoted boldface.

An  $n$ -dimensional evolution  $\mathbb{K}$ -algebra  $\mathcal{E}$  is a  $\mathbb{K}$ -vector space admitting a basis  $\mathcal{B} = \{e_1, \dots, e_n\}$ , of the underlying vector space, with associated multiplication given by:

$$\begin{cases} e_i^2 = e_i e_i = \sum_{j=1}^n a_{ij} e_j, & i = 1, \dots, n; \\ e_i e_j = e_j e_i = 0, & i \neq j. \end{cases}$$

Such bases are called *natural bases* [1]. Dynamics of evolution algebras arise from their *evolution operator*  $L_e$ , defined as the one-sided multiplication by the evolution element  $e = \sum_{i=1}^n e_i$  given by  $\mathcal{B}$  [1]. Continuity and dynamics of evolution operators were considered in [5].

If  $\mathcal{E}$  is a Markov (real) evolution algebra, then its evolution operator  $L_e$  satisfies Chapman-Kolmogorov like equations [1, 4.1.3] and its (Markov) structure matrix  $\mathbf{A} = (a_{ij})_{i,j=1}^n$  becomes the transition probability matrix of the underlying HDT Markov chain [1, 4.1.2].

We denote  $\mathfrak{E} = \{\mathcal{E}_t = (\mathcal{E}, m(t)) \mid t \in \mathbb{R}\}$  the family of  $n$ -dimensional evolution  $\mathbb{K}$ -algebras, defined on the  $\mathbb{K}$ -vector space  $\mathcal{E}$ , with multiplications, for a fixed natural basis  $\mathcal{B}$ :

$$\begin{cases} e_i \cdot_t e_i = \sum_{j=1}^n a_{ij}(t) e_j, & i = 1, \dots, n; \\ e_i \cdot_t e_j = e_j \cdot_t e_i = 0, & i \neq j, \end{cases}$$

and by  $\mathbf{A}(t) = (a_{ij}(t))_{i,j=1}^n \in M_n(\mathbb{K})$  the structure matrix of  $\mathcal{E}_t$ , for all  $t \in \mathbb{R}$ . Once the natural basis  $\mathcal{B}$  is fixed, we identify  $\mathfrak{E}$  to the set of  $\mathbb{R}$ -indexed structure matrices  $\{\mathbf{A}(t) \mid t \in \mathbb{R}\}$ .

**Definition 2.1.** [1, 6.2.4] A continuous evolution algebra is a family  $\mathfrak{E} = \{\mathcal{E}_t = (\mathcal{E}, m(t)) \mid t \in \mathbb{R}\}$  of evolution algebras such that the functions  $a_{ij}(t)$  are differentiable, for all  $i, j = 1, \dots, n$ .

The set  $M_n(\mathbb{K})$ , of  $n \times n$  matrices with entries in  $\mathbb{K}$ , with the operator norm is a Banach algebra. In fact, any two norms in  $M_n(\mathbb{K})$  give rise to the same topology. The general linear group  $GL_n(\mathbb{K})$ , the group of all invertible matrices in  $M_n(\mathbb{K})$ , and all its subgroups carry the subspace topology inherited from  $M_n(\mathbb{K})$ .

**Definition 2.2.** [6, Definition 2.11] A differentiable curve in  $M_n(\mathbb{K})$  is a curve  $\gamma : (a, b) \rightarrow M_n(\mathbb{K})$  whose derivative

$$\gamma'(t) = \lim_{h \rightarrow 0} \frac{\gamma(t+h) - \gamma(t)}{h}$$

exists, in  $M_n(\mathbb{K})$  for each  $t \in (a, b) \subseteq \mathbb{R}$  (that is, componentwise, the maps  $\gamma_{ij}(t)$  are differentiable, for all  $i, j \in 1, \dots, n$ ).

Fixed a natural basis  $\mathcal{B}$ , we will consider continuous evolution algebras to be defined by differentiable curves in  $M_n(\mathbb{K})$ .

**Proposition 2.3.**  $\mathfrak{E} = \{\mathcal{E}_t = (\mathcal{E}, m(t)) \mid t \in \mathbb{R}\}$  is a continuous evolution algebra if and only if the map  $\gamma : \mathbb{R} \rightarrow M_n(\mathbb{K})$ , given by  $\gamma(t) = \mathbf{A}(t)$  defines a differentiable curve.

*Proof.* It suffices to recall that continuous evolution algebras are defined by componentwise differentiable structure matrices  $\mathbf{A}(t) = (a_{ij}(t))_{i,j=1}^n \in M_n(\mathbb{K})$ , defining differentiable matrix-valued functions.  $\square$

**Corollary 2.4.** Any (time-invariant) evolution algebra  $\mathcal{E}$ , with structure matrix  $\mathbf{A}$ , defines a continuous evolution algebra  $\mathfrak{E} = \{\mathcal{E}_t = (\mathcal{E}, m(t)) \mid t \in \mathbb{R}\}$  such that  $\mathcal{E}_t = \mathcal{E}$  (equivalently  $\mathbf{A}(t) = \mathbf{A}$ ) for all  $t \in \mathbb{R}$ .

*Proof.* Consider the differentiable curve  $\gamma(t) = \mathbf{A}$  for all  $t \in \mathbb{R}$ .  $\square$

We will say that a continuous evolution algebra  $\mathfrak{E} = \{\mathcal{E}_t = (\mathcal{E}, m(t)) \mid t \in \mathbb{R}\}$  satisfies a property  $\mathcal{P}$  if each  $\mathcal{E}_t$  satisfies  $\mathcal{P}$  for all  $t \in \mathbb{R}$ , and, in particular that it is perfect if  $\mathcal{E}_t^2 = \mathcal{E}_t$ , or, equivalently,  $\mathbf{A}(t)$  is nonsingular, for all  $t \in \mathbb{R}$ .

**Example 2.5.** Let  $\mathbf{A} \in M_n(\mathbb{K})$ . A quite simple example of (non time-invariant) differentiable curve is  $\gamma : \mathbb{R} \rightarrow M_n(\mathbb{K})$ , defined as  $\gamma(t) = \mathbf{I}_n + t\mathbf{A}$ , with  $\gamma(0) = \mathbf{I}_n$  and  $\gamma'(0) = \mathbf{A}$ . It defines a continuous evolution algebra  $\mathfrak{E}$  with multiplication  $e_i \cdot_t e_j = \sum_{k=1}^n (\delta_{ik} + a_{ik}t)e_k$ , if  $i = j \in \{1, \dots, n\}$  (here  $\delta_{ik}$  denotes the Kronecker's delta), and  $e_i \cdot_t e_j = 0$  otherwise. Here, non-singularity of  $\gamma(t)$ , i.e. of  $\mathbf{A}(t)$ , is only ensured when  $|t| < \frac{1}{|\lambda|}$  |  $\lambda$  a nonzero eigenvalue of  $\mathbf{A}$  [6, p. 76]. Thus, not all algebras  $\mathcal{E}_t$  are perfect.

The matrix series  $\exp(\mathbf{X}) = e^{\mathbf{X}} = \sum_{m=0}^{\infty} \frac{1}{m!} \mathbf{X}^m$  converges absolutely for any  $\mathbf{X} \in M_n(\mathbb{K})$ . It defines the matrix exponential function  $\exp : M_n(\mathbb{K}) \rightarrow GL_n(\mathbb{K})$  [7, Theorem 2.11], which is infinitely differentiable (smooth), and injective on an open neighbourhood of  $\mathbf{O}_n$ , the  $n \times n$  matrix with all zero entries. Also it holds  $\exp(\mathbf{X})^{-1} = \exp(-\mathbf{X})$  [7, Theorem 2.3(3)]. Exponentials of matrices are easily computed for diagonalizable or nilpotent matrices. SN decompositions of complex matrices [7, Appendix B. Theorem B.6] and Jordan canonical forms [7, Appendix B.4] are useful tools for matrix exponentiation [6, Chapter 2, Sections 2.1, 2.2, 2.3]. A different procedure, based on solving an  $n$ th-order scalar differential equation derived from the matrix characteristic polynomial, can be found in [8, Appendix 5.14.1].

**Example 2.6.** (i) Let  $\mathbf{X} \in M_n(\mathbb{K})$ . Then  $\gamma(t) = \exp(t\mathbf{X})$  defines a differentiable curve in  $GL_n(\mathbb{K})$ , with  $\gamma(0) = \mathbf{I}_n$ . Moreover,  $\gamma'(t) = \gamma(t)\mathbf{X} = \mathbf{X}\gamma(t)$  with  $\gamma'(0) = \left(\frac{d}{dt}\gamma(t)\right)_{t=0} = \mathbf{X}$ , and  $\gamma(s+t) = \gamma(s)\gamma(t)$  for all  $s, t \in \mathbb{R}$  [7, Proposition 2.3(4), Proposition 2.4].

(ii) Consider the flip flop process given by a Poisson process of intensity  $\lambda > 0$ . Its standard stochastic semigroup is

$$\mathbf{A}(t) = \frac{1}{2} \begin{pmatrix} 1 + e^{-2\lambda t} & 1 - e^{-2\lambda t} \\ 1 - e^{-2\lambda t} & 1 + e^{-2\lambda t} \end{pmatrix}, \quad \text{for all } t \geq 0,$$

with infinitesimal generator (or rate matrix)  $\mathbf{Q} = \begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix}$  so that  $\mathbf{A}(t) = \exp(t\mathbf{Q})$  [9, Example 2.2, Exercise 8.2.4]. Considered, by default, only for nonnegative time values ( $t \geq 0$ ), (finite-state) standard stochastic semigroups define continuous time Markov processes. Clearly nonnegativity of the matrices  $\mathbf{A}(t)$  is only ensured when  $t \geq 0$ , but the map  $\gamma(t) = \mathbf{A}(t)$  still defines a differentiable curve in  $GL_2(\mathbb{R})$  such that  $\gamma(s + t) = \gamma(s)\gamma(t)$  for all  $s, t \in \mathbb{R}$  [9, Chapter 8, Section 2.2]. Besides, as noted in the introduction, Forward and Backward Kolmogorov differential equations  $\mathbf{A}'(t) = \mathbf{A}(t)\mathbf{Q} = \mathbf{Q}\mathbf{A}(t)$ , with initial condition  $\mathbf{A}(0) = \mathbf{I}_n$ , are satisfied.

**Lemma 2.7.** *Examples 2.6(i) and 2.6(ii) above define continuous evolution algebras.*

Continuous evolution algebras arising from the differentiable curves in Example 2.6 are perfect. The next result is straightforward.

**Lemma 2.8.** *Perfect continuous evolution algebras correspond to differentiable curves in  $GL_n(\mathbb{K})$ .*

Given a matrix  $\mathbf{X} \in M_n(\mathbb{K})$ , the differentiable curve  $\gamma(t) = \exp(t\mathbf{X})$ , defining the continuous evolution algebra  $\mathfrak{E}$  with structure matrices  $\mathbf{A}(t) = \gamma(t) = \exp(t\mathbf{X})$  (see, for instance, Example 2.6) is the (unique) solution of the matrix ordinary differential equation (ODE):

$$\begin{cases} \mathbf{A}'(t) = \mathbf{A}(t)\mathbf{X}, \\ \mathbf{A}(0) = \mathbf{I}_n. \end{cases}$$

Approaching continuous evolution algebras as solutions of differential equations agrees to their original appearance in [1] as tools to study the dynamics of particular genetic systems, and will be considered in the following sections.

### 3. Continuous evolution algebras on matrix Lie groups

In this section we consider continuous evolution algebras arising as solutions of first-order (matrix) ordinary differential equations. The additional assumption on being solutions of ODEs leads us to consider their structure matrices within matrix Lie groups.

**Definition 3.1.** [7, Definition 1.4] *A matrix Lie group  $G$  is a subgroup  $G \leq GL_n(\mathbb{K})$ , which is closed in  $GL_n(\mathbb{K})$ , that is, for any sequence  $\{\mathbf{A}_m\}_{m \geq 1}$  of matrices in  $G$ , converging, componentwise, as  $m \rightarrow \infty$ , to a matrix  $\mathbf{A}$ , then either  $\mathbf{A} \in G$  or  $\mathbf{A}$  is not invertible.*

Besides the general linear groups  $GL_n(\mathbb{K})$ , examples of matrix Lie groups are the special linear groups  $SL_n(\mathbb{K}) = \{\mathbf{A} \in M_n(\mathbb{K}) \mid \det(\mathbf{A}) = 1\}$ . If  $\mathbb{K} = \mathbb{R}$ , the orthogonal group  $O(n) = \{\mathbf{A} \in M_n(\mathbb{R}) \mid \mathbf{A}^T \mathbf{A} = \mathbf{I}_n\}$  and the special orthogonal group  $SO(n) = O(n) \cap SL_n(\mathbb{R})$ , and for  $\mathbb{K} = \mathbb{C}$ , the unitary group  $U(n) = \{\mathbf{A} \in M_n(\mathbb{C}) \mid \mathbf{A}^* \mathbf{A} = \mathbf{I}_n\}$  and the special unitary group  $SU(n) = U(n) \cap SL_n(\mathbb{C})$  are also matrix Lie groups. The  $n$ -dimensional affine group

$$Aff_n(\mathbb{K}) = \left\{ \begin{pmatrix} \mathbf{A} & \mathbf{t} \\ 0 & 1 \end{pmatrix} \mid \mathbf{A} \in GL_n(\mathbb{K}), \mathbf{t} \in \mathbb{K}^n \right\} \leq GL_{n+1}(\mathbb{K});$$

consisting of the set of transformations in  $\mathbb{K}^n$  of the form  $\mathbf{x} \mapsto \mathbf{A}\mathbf{x} + \mathbf{t}$ , with nonsingular  $\mathbf{A}$ , is a closed subgroup of  $GL_{n+1}(\mathbb{K})$  and a matrix Lie group.

**Remark 3.2.** (i) The matrix Lie groups  $O(n)$ ,  $SO(n)$ ,  $U(n)$  and  $SU(n)$  are compact, but  $GL_n(\mathbb{K})$  and  $SL_n(\mathbb{K})$  are not [7, Definition 1.6, Examples 1.3.1 and 1.3.2].

(ii)  $GL_n(\mathbb{C})$  is connected, as well as  $SO(n)$ ,  $U(n)$ ,  $SU(n)$  and  $SL_n(\mathbb{K})$ .  $GL_n(\mathbb{R})$  and  $O(n)$  have two connected components given by the sign of the determinant. The set of nonsingular real matrices with positive determinant  $GL_n(\mathbb{R})^+$  is the connected component of  $GL_n(\mathbb{R})$  containing the identity  $\mathbf{I}_n$  [7, Definition 1.7, Proposition 1.8 - Proposition 1.12]. Note  $O(n)^+ = O(n) \cap GL_n(\mathbb{R})^+ = SO(n)$ .

(iii) Matrix Lie groups are Lie groups, smooth manifolds [7, Appendix C.2.6]. Conversely, many, but not all, Lie groups have a matrix form [7, Appendix C.3]. See also [6, Section 7.7].

**Definition 3.3.** [7, Definition 2.12] Let  $G$  be a matrix Lie group. A one-parameter subgroup of  $G$  is a map  $\gamma : (\mathbb{R}, +) \rightarrow G$  such that:

- (i)  $\gamma$  is continuous for all  $t \in \mathbb{R}$ .
- (ii)  $\gamma(0) = \mathbf{I}_n$ .
- (iii)  $\gamma(s + t) = \gamma(s)\gamma(t)$  for all  $s, t \in \mathbb{R}$ .

Some references consider the slightly different notion of one-parameter semigroup  $\gamma(t)$  defined on intervals  $(-\epsilon, \epsilon) \subseteq \mathbb{R}$ , for some  $\epsilon > 0$ . Then there exists a unique extension  $\tilde{\gamma} : \mathbb{R} \rightarrow G$  to a one-parameter subgroup on  $\mathbb{R}$  such that  $\tilde{\gamma}(t) = \gamma(t)$  for all  $t \in (-\epsilon, \epsilon)$  [6, Proposition 2.16]. We thus consider the case  $\epsilon = \infty$ .

The next result follows from the characterization of differentiable one-parameter subgroups as smooth curves given by the exponential function [10, Theorem 3.2.6].

**Theorem 3.4.** Any one-parameter subgroup  $\gamma : (\mathbb{R}, +) \rightarrow G$  in a matrix Lie group  $G$  defines a perfect continuous evolution algebra  $\mathfrak{E}$ , whose structure matrices  $\mathbf{A}(t)$  are the unique solution of a matrix ODE:

$$\begin{cases} \mathbf{A}'(t) = \mathbf{A}(t)\mathbf{X}, \\ \mathbf{A}(0) = \mathbf{I}_n, \end{cases}$$

for some  $\mathbf{X} \in M_n(\mathbb{K})$ . Conversely, for any such perfect continuous evolution algebra  $\gamma(t) = \mathbf{A}(t) (= \exp(t\mathbf{X}))$  defines a one-parameter subgroup in  $GL_n(\mathbb{K})$ .

*Proof.* Let  $G \leq GL_n(\mathbb{K})$  be a matrix Lie group, and  $\gamma : (\mathbb{R}, +) \rightarrow G$  be a one-parameter subgroup in  $G$ . By [10, Theorem 3.2.6]  $\gamma(t) = \exp(t\mathbf{X})$ , for some  $\mathbf{X} \in M_n(\mathbb{K})$ . By Proposition 2.3 and Lemma 2.8,  $\mathfrak{E}$  with structure matrices  $\mathbf{A}(t) = \gamma(t)$  is a perfect continuous evolution algebra. The fact that matrices  $\mathbf{A}(t)$  are the unique solution of the matrix ODE is clear. For the converse statement see again [10, Theorem 3.2.6].  $\square$

We will say that a continuous evolution algebra is  $G$ -continuous if all its structure matrices  $\mathbf{A}(t) \in G$ , for all  $t \in \mathbb{R}$ , and one-parameter  $G$ -continuous if they are as in Theorem 3.4. Clearly one can always consider  $G \leq GL_n(\mathbb{C})$ .  $G$ -continuous evolution algebras are perfect.

**Example 3.5.** The one-parameter subgroup  $\gamma : (\mathbb{R}, +) \rightarrow GL_2(\mathbb{R})$  given by  $\gamma(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$  defines a 2-dimensional  $SO(2)$ -continuous evolution algebra ( $\text{im } \gamma = SO(2)$ ). For a fixed natural basis  $\mathcal{B} = \{e_1, e_2\}$ , the multiplication in  $\mathfrak{E}$  is given by:

$$\begin{cases} e_1 \cdot_t e_1 = e_1 \cos t + e_2 \sin t, \\ e_2 \cdot_t e_2 = -e_1 \sin t + e_2 \cos t. \end{cases}$$

Here  $\gamma(t) = \exp(t\mathbf{X})$ , where  $\mathbf{X} = \gamma'(0)$  is the skew-symmetric matrix  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ .

The (real) Lie algebra of a matrix Lie group  $G$  is  $\mathfrak{g} = \{\mathbf{X} \in M_n(\mathbb{K}) \mid \exp(\mathbb{R}\mathbf{X}) \subseteq G\}$  [7, Subsection 2.15]. This is the tangent space to  $G$  at  $\mathbf{I}_n$ :

$$T_{\mathbf{I}_n}G = \{\gamma'(0) \in M_n(\mathbb{K}) \mid \gamma \text{ is a differentiable curve in } G \text{ with } \gamma(0) = \mathbf{I}_n\}.$$

Derivatives at origin of differentiable curves on matrix Lie groups with  $\gamma(0) = \mathbf{I}_n$  (i.e. elements of the Lie algebra  $\mathfrak{g}$ ) are called velocity vectors.

**Theorem 3.6.** *Let  $G$  be a matrix Lie group. Each  $\mathbf{X} \in \mathfrak{g}$  defines a (one-parameter)  $G$ -continuous evolution algebra with velocity  $\mathbf{X}$  at origin.*

*Proof.* Let  $\mathfrak{g}$  be the Lie algebra of  $G$ , and let us denote by  $\text{Hom}(\mathbb{R}, G)$  the set of all continuous group homomorphisms  $(\mathbb{R}, +) \rightarrow G$ . By [10, Lemma 4.1.5], there exists a bijection  $\mathfrak{g} \rightarrow \text{Hom}(\mathbb{R}, G)$ , given by  $\mathbf{X} \mapsto \gamma_{\mathbf{X}}$  with  $\gamma_{\mathbf{X}}(t) = \exp(t\mathbf{X})$ . The result now follows from Theorem 3.4.  $\square$

**Corollary 3.7.** *Let  $\mathbf{X} \in \mathfrak{g}$ . The  $G$ -continuous evolution algebra  $\mathfrak{E}$  with structure matrices  $\mathbf{A}(t) = \exp(t\mathbf{X})$  is the unique solution of the ODE problem:*

$$\begin{cases} \mathbf{A}'(t) = \mathbf{A}(t)\mathbf{X}, \\ \mathbf{A}(0) = \mathbf{I}_n. \end{cases}$$

*Proof.* See Theorem 3.4 and note that now  $\mathbf{X} \in \mathfrak{g}$  ensures  $\mathfrak{E}$  is  $G$ -continuous.  $\square$

Let  $\mathfrak{E} = \{\mathcal{E}_t = (\mathcal{E}, m(t)) \mid t \in \mathbb{R}\}$  be as in Corollary 3.7. Then its structure matrices are not only nonsingular matrices in  $G \leq GL_n(\mathbb{K})$ , but all them fall inside the same connected component of  $G$ , that containing  $\mathbf{A}(0) = \mathbf{I}_n$ . Thus, if  $\mathbb{K} = \mathbb{R}$ , we have  $\mathbf{A}(t) \in G \cap GL_n(\mathbb{R})^+$  (matrices with positive determinant). See, for instance, Example 3.5, where  $\text{im } \gamma \subseteq O(2) \cap GL_2(\mathbb{R})^+ = SO(2)$ . A similar situation occurs when considering the Lorentz group  $Lor(1, 1)$  inside the generalized orthogonal group  $O(1, 1)$ . Only one of the four connected components of  $O(1, 1)$  is reached.

**Example 3.8.** *Let  $G = Lor(1, 1) = \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \mid t \in \mathbb{R} \right\}$ , and consider the one-parameter subgroup  $\gamma(t) = \exp(t\mathbf{X})$ , with velocity vector at origin  $\mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . It defines a 2-dimensional real  $G$ -continuous evolution algebra with structure matrices  $\mathbf{A}(t) = \exp(t\mathbf{X}) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$ , for all  $t \in \mathbb{R}$ . Thus,  $\mathcal{E}_t$  has multiplication:*

$$\begin{cases} e_1 \cdot e_1 = e_1 \cosh t + e_2 \sinh t, \\ e_2 \cdot e_2 = e_1 \sinh t + e_2 \cosh t. \end{cases}$$

The Lorentz group  $Lor(1, 1)$  is connected, with Lie algebra  $\mathfrak{lor}(1, 1) = \{t\mathbf{X} \mid t \in \mathbb{R}\}$ , and surjective exponential map  $\exp : \mathfrak{lor}(1, 1) \rightarrow Lor(1, 1)$  [6, Remark 6.5, Theorem 6.7, Theorem 6.11]. However considered within  $O(1, 1)$  (see Example 3.11),  $\mathbf{A}(t) \in Lor(1, 1) \subsetneq O(1, 1)^+ = SO(1, 1) = O(1, 1) \cap GL_2(\mathbb{R})^+$ .

**Example 3.9.** *The classification of (time-invariant) perfect three dimensional (real or complex) evolution algebras follows from [11, Theorem 3.5(iv)]. We point out here that in [11] multiplication constants are arranged into the structure matrices by columns, whereas here, following [1, Definition 3, p. 20], we are considering them row-rearranged. Families of non-isomorphic 3-dimensional perfect real evolution algebras are detailed in [12]. Let  $\mathbb{K} = \mathbb{R}$ , and consider the second family (second row) in [12, Table 18], whose structure matrices have negative determinant. Although any of this algebras trivially defines a continuous evolution algebra (see Corollary 2.4), such real evolution algebras cannot appear within (for any  $t \in \mathbb{R}$ ) algebras as in Corollary 3.7. The same applies to 2-dimensional real evolution algebras of type  $E_7(a_4)$  [13]. For any differentiable map  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ , the matrix-valued function  $\mathbf{A}(t) = \begin{pmatrix} 0 & 1 \\ 1 & \alpha(t) \end{pmatrix}$  defines a perfect  $GL_2(\mathbb{R})$ -continuous evolution algebra, whose structure matrices have negative determinant.*

To overcome drawbacks as in Examples 3.8 and 3.9, let us next consider matrix ODEs with initial condition different from  $\mathbf{A}(0) = \mathbf{I}_n$ . Given arbitrary matrices  $\mathbf{X}$  and  $\mathbf{A}_0$  in  $M_n(\mathbb{K})$ , the matrix ODE:

$$\begin{cases} \mathbf{A}'(t) = \mathbf{A}(t)\mathbf{X}, \\ \mathbf{A}(0) = \mathbf{A}_0, \end{cases}$$

defines a differentiable curve  $\gamma(t) = \mathbf{A}_0 \exp(t\mathbf{X})$  in  $M_n(\mathbb{K})$  and, therefore a continuous evolution algebra  $\mathfrak{E}$ . Only if  $\mathbf{A}_0 \in GL_n(\mathbb{K})$ , then  $\mathbf{A}(t) = \mathbf{A}_0 \exp(t\mathbf{X}) \in GL_n(\mathbb{K})$  for all  $t \in \mathbb{R}$ , being then  $\mathfrak{E}$  a perfect continuous evolution algebra.

Recall  $GL_n(\mathbb{R})^+$  equals the set  $\det^{-1}(\mathbb{R}^+)$  of nonsingular real matrices with positive determinant, where  $\det : GL_n(\mathbb{R}) \rightarrow \mathbb{R}$  denotes the determinant function on  $GL_n(\mathbb{R})$ . Analogously we denote  $GL_n(\mathbb{R})^- = \det^{-1}(\mathbb{R}^-)$  the set of nonsingular real matrices with negative determinant.

**Proposition 3.10.** *Let  $\mathfrak{E} = \{\mathcal{E}_t = (\mathcal{E}, m(t))\}_{t \geq 0}$  be a continuous evolution algebra arising as the solution of the ODE*

$$\begin{cases} \mathbf{A}'(t) = \mathbf{A}(t)\mathbf{X}, \\ \mathbf{A}(0) = \mathbf{A}_0. \end{cases}$$

Then:

- (i)  $\mathfrak{E}$  is perfect if and only if  $\mathbf{A}_0 \in GL_n(\mathbb{K})$ . If moreover  $\mathbb{K} = \mathbb{R}$ , then for all  $t \in \mathbb{R}$ , it holds  $\mathbf{A}(t) \in GL_n(\mathbb{R})^\sigma$ , where  $\sigma = \pm$  is the sign of  $\det(\mathbf{A}_0)$ .
- (ii) Otherwise, if  $\mathbf{A}_0 \notin GL_n(\mathbb{K})$ ,  $\mathcal{E}_t$  is not perfect, for any  $t \in \mathbb{R}$ .

*Proof.* It follows from  $\mathbf{A}(t) = \mathbf{A}_0 \exp(t\mathbf{X})$ , and the continuity of the determinant map in  $GL_n(\mathbb{R})$ .  $\square$

**Example 3.11.** *Elements of the generalized orthogonal group  $O(1, 1)$  are of the form [6, Example 6.4]:*

$$\mathbf{A}(t) = \exp(t\mathbf{X}), \quad \text{with } \mathbf{X} \text{ as in Example 3.8,}$$

$$\mathbf{A}_2(t) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{A}(t) = \mathbf{A}_2 \mathbf{A}(t),$$

$$\mathbf{A}_3(t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{A}(t) = \mathbf{A}_3 \mathbf{A}(t),$$

$$\mathbf{A}_4(t) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{A}(t) = \mathbf{A}_4 \mathbf{A}(t),$$

For  $i = 1, 2, 3, 4$  ( $\mathbf{A}_1 = \mathbf{I}_2$ ),  $\mathbf{A}_i(t) = \mathbf{A}_i \exp(t\mathbf{X})$  defines a  $O(1, 1)$ -continuous evolution algebra, as in Proposition 3.10, contained in each one of the four different connected components of  $O(1, 1)$ .

**Example 3.12.** *Consider the matrices  $\mathbf{A}_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $\mathbf{X} = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$ . Then  $\mathbf{A}_1(t) = \exp(t\mathbf{X}) = \begin{pmatrix} 1 & t\alpha \\ 0 & 1 \end{pmatrix}$  defines a  $SL_2(\mathbb{R})$ -continuous evolution algebra, whereas  $\mathbf{A}_2(t) = \mathbf{A}_0 \exp(t\mathbf{X})$  gives rise to a continuous evolution algebra, with  $\mathcal{E}_t$  of type  $E_7(t\alpha)$  (see Example 3.9).*

The next Corollary is clear.

**Corollary 3.13.** *If  $\mathbf{A}_0 \in G$  and  $\mathbf{X} \in \mathfrak{g}$ , the continuous evolution algebra  $\mathfrak{E}$ , with  $\mathbf{A}(t) = \mathbf{A}_0 \exp(t\mathbf{X})$  is  $G$ -continuous and lies in the connected component of  $\mathbf{A}_0$ .*

One of the main consequences of this section is formulated in the following Remark 3.14 for further reference.

**Remark 3.14.** *Corollary 3.13 tells us that any element of the tangent bundle  $T(G)$  of the matrix Lie group  $G$  defines a  $G$ -continuous evolution algebra. Indeed, recall  $T(G)$  is the disjoint union of the tangent spaces  $T_{\mathbf{B}}G$  at  $\mathbf{B} \in G$ , that is:  $T(G) = \bigsqcup T_{\mathbf{B}}G = \{(\mathbf{B}, \mathbf{V}) \mid \mathbf{B} \in G, \mathbf{V} \in T_{\mathbf{B}}G\}$ , where  $T_{\mathbf{B}}G = \mathbf{B}\mathfrak{g}$ . Given  $(\mathbf{B}, \mathbf{V}) \in T(G)$ , it suffices to consider  $\mathcal{E}(t)$  with structure matrices  $\mathbf{A}(t) = \mathbf{B} \exp(t\mathbf{X})$ , and  $\mathbf{X} = \mathbf{B}^{-1}\mathbf{V}$ .*

To conclude this section we provide two examples of continuous evolution algebras, both deeply relying on the properties of the underlying Lie group  $G$ .

**Example 3.15.** The Heisenberg group  $H(3)$  is the Lie group:

$$H(3) = \left\{ \mathbf{A}(\alpha, \beta, \delta) = \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \delta \\ 0 & 0 & 1 \end{pmatrix} \mid \alpha, \beta, \delta \in \mathbb{R} \right\}$$

with Lie algebra  $\mathfrak{g} = \left\{ \mathbf{X}(a, b, c) = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$  [7, p. 162]. Here the exponential map  $\exp : \mathfrak{g} \rightarrow H(3)$

is one-to-one and onto [7, Exercise II.26], and  $\exp(\mathbf{X}(a, b, c)) = \mathbf{A}(a, b + \frac{1}{2}ac, c)$  [14, Vol. 2, 10.6.1], providing an explicit description for  $H(3)$ -continuous evolution algebras in Theorem 3.6. More generally, given differentiable maps  $\alpha, \beta, \delta : \mathbb{R} \rightarrow \mathbb{R}$ , then  $\mathbf{A}(t) = \mathbf{A}(\alpha(t), \beta(t), \delta(t))$  defines a  $H(3)$ -continuous evolution algebra, with multiplication (w.r.t a natural basis  $\mathcal{B} = \{e_1, e_2, e_3\}$ ):

$$\begin{cases} e_1 \cdot_t e_1 = e_1 + \alpha(t)e_2 + \beta(t)e_3, \\ e_2 \cdot_t e_2 = e_2 + \delta(t)e_3, \\ e_3 \cdot_t e_3 = e_3 \end{cases}$$

Conversely,

$$\mathbf{A}(t) = \exp \begin{pmatrix} 0 & a(t) & b(t) \\ 0 & 0 & c(t) \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{A} \left( a(t), b(t) + \frac{1}{2}a(t)c(t), c(t) \right),$$

also defines a  $H(3)$ -continuous evolution algebra for any differentiable maps  $a, b, c : \mathbb{R} \rightarrow \mathbb{R}$ .

The exponential map is not however surjective, nor injective, on any matrix Lie group  $G$ . If  $G$  is compact and connected, then any element of  $G$  does fall within some one-parameter subgroup and can therefore be embedded (mirroring Markov chains terminology [3]) into a  $G$ -continuous evolution algebra as those in Corollary 3.7, but otherwise it would depend on  $G$ . Matrices in the previous Example 3.15 have also a form  $\mathbf{A}(t) = \mathbf{A}(\alpha(t), \beta(t), \delta(t)) = \exp(\beta(t)E_{13}) \exp(\delta(t)E_{23}) \exp(\alpha(t)E_{12})$ , as product of exponential matrices. (Here  $E_{ij}$  stands for the usual matrix units.) In fact, there is a number of matrix Lie groups whose elements admit similar parametrizations.

**Example 3.16.** Consider the special linear group  $SL_2(\mathbb{R})$ , whose Lie algebra  $\mathfrak{sl}_2(\mathbb{R})$  is the set of traceless  $2 \times 2$  real matrices. The exponential map  $\exp : \mathfrak{sl}_2(\mathbb{R}) \rightarrow SL_2(\mathbb{R})$  is not surjective. Indeed, for any  $\mathbf{X} \in \mathfrak{sl}_2(\mathbb{R})$ ,  $\text{tr}(e^{\mathbf{X}}) \geq -2$ , and it is not difficult to find  $\mathbf{A} \in SL_2(\mathbb{R})$  with  $\text{tr}(\mathbf{A}) < -2$  [15, Problem II.2.2]. Since  $\mathfrak{sl}_2(\mathbb{R})$  is semisimple [7, p. 162], using Iwasawa decomposition [16, Theorem 6.46], any  $\mathbf{A} \in SL_2(\mathbb{R})$  decomposes as a product of exponential matrices  $\mathbf{A} = \exp(\alpha E_1) \exp(\beta E_2) \exp(\delta E_3)$ , where  $\{E_1 = E_{21} - E_{12}, E_2 = E_{11} - E_{22}, E_3 = E_{12}\}$  is a basis of  $\mathfrak{sl}_2(\mathbb{R})$  [14, Vol 2, Subsection 10.6.3]. Thus, for any differentiable maps  $\alpha, \beta, \delta : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$\begin{aligned} \mathbf{A}(t) &= \mathbf{A}(\alpha(t), \beta(t), \delta(t)) = \exp(\alpha(t)E_1) \exp(\beta(t)E_2) \exp(\delta(t)E_3) = \\ &= \begin{pmatrix} \cos \alpha(t) & -\sin \alpha(t) \\ \sin \alpha(t) & \cos \alpha(t) \end{pmatrix} \begin{pmatrix} e^{\beta(t)} & 0 \\ 0 & e^{-\beta(t)} \end{pmatrix} \begin{pmatrix} 1 & \delta(t) \\ 0 & 1 \end{pmatrix} \end{aligned}$$

defines a  $SL_2(\mathbb{R})$ -continuous evolution algebra.

#### 4. Continuous time Markov evolution algebras

Continuous time Markov (CT-Markov) evolution algebras yield on standard stochastic semigroups [3, Definition 6.1(i)]. In this section we settle CT-Markov EAs as continuous evolution algebras. Along this section  $\mathbb{K} = \mathbb{R}$ .

**Theorem 4.1.** Continuous time Markov evolution algebras are perfect real continuous evolution algebras.



*Proof.* Let  $\mathfrak{C}$  be a CT-Markov evolution algebra with defining standard stochastic semigroup  $\{\mathbf{A}(t)\}_{t \geq 0}$ . Since  $\mathfrak{C}$  is finite-dimensional,  $\{\mathbf{A}(t)\}_{t \geq 0}$  satisfies both Forward and Backward Kolmogorov differential equations [9, Subsection 3.3]. Thus  $\mathbf{A}(t) = \exp(t\mathbf{Q})$ , where  $\mathbf{Q}$  denotes the process rate matrix, defines a, clearly perfect, continuous evolution algebra [9, Theorem 2.21].  $\square$

**Corollary 4.2.** *Continuous time Markov evolution algebras define one-parameter subgroups.*

*Proof.* Let  $\gamma(t) = \mathbf{A}(t) = \exp(t\mathbf{Q})$  be as in the proof of Theorem 4.1. It defines a differentiable, thus continuous, curve, with  $\gamma(0) = \mathbf{I}_n$ . Moreover, see [6, Proposition 2.1(i)],  $\exp((t + s)\mathbf{Q}) = \exp(t\mathbf{Q})\exp(s\mathbf{Q})$ , for all  $t, s \in \mathbb{R}$ , implying the so-called group homomorphism property  $\gamma(t + s) = \gamma(t)\gamma(s)$ . Hence, see Definition 3.3,  $\gamma(t)$  defines a one-parameter subgroup in  $GL_n(\mathbb{R})$ .  $\square$

**Remark 4.3.** (i) *Under the current approach, rate matrices, i.e. infinitesimal generators,  $\mathbf{Q}$  of CT-Markov evolution algebras become tangent or velocity vectors at the identity  $\mathbf{I}_n$  (for  $t = 0$ ).*

(ii) *Given a nonsingular Markov matrix, that is, a nonnegative matrix in the stochastic group  $S(n, \mathbb{R})$  of nonsingular real matrices with row sums equal to one, its inverse matrix does not need to be nonnegative (i.e. still a Markov matrix). Standard stochastic semigroups only ensure the nonnegativity of matrices  $\gamma(t) = \exp(t\mathbf{Q})$  for nonnegative time values  $t \geq 0$ , whereas  $\gamma(-t) = \exp(-t\mathbf{Q}) = \exp(t\mathbf{Q})^{-1} = \gamma(t)^{-1} \in S(n, \mathbb{R})$  may not be nonnegative [7, Proposition 2.3(3)]. See, for instance, Example 2.6(ii).*

(iii) *Any  $n$ -dimensional CT-Markov evolution algebra defines a one-parameter subgroup on the  $(n - 1)$ -dimensional affine group, and, therefore, an  $Aff_{n-1}(\mathbb{R})$ -continuous evolution algebra. Recall  $S(n, \mathbb{R}) \cong Aff_{n-1}(\mathbb{R})$  [4, Theorem B].*

Of special interest are continuous evolution algebras defined on generalized doubly-stochastic Lie groups [17, Proposition 3.1, Proposition 3.2].

**Example 4.4.** *Let  $\mathbf{A}(t)$  and  $\mathbf{Q}$  be as in Example 2.6(ii). Then  $\mathbf{A}(t)$  is a nonsingular 1-generalized doubly stochastic (i.e. real with row and column sums equal to 1) matrix in the Lie group  $G = \Omega^1(2, \mathbb{R}) \cap GL_2(\mathbb{R})$ , whereas the rate matrix  $\mathbf{Q}$  is in its Lie algebra  $\mathfrak{g} = \Omega^0(2, \mathbb{R})$  [17, Proposition 3.1].*

Given a CT-Markov evolution algebra, the extension to negative times, from the stochastic process to the differentiable curve, considered in Theorem 4.1, is related to the notion of reversibility of continuous-time Markov chains [9, Chapter 8, Section 5.2]. See, for instance, the Reversibility Theorem for irreducible and ergodic regular jump HMC on a finite state space [9, Chapter 8, Theorem 5.4] which provides sufficient conditions on the infinitesimal generator  $\mathbf{Q}$  (the so-called detailed-balance equations) for the reverse process to be right-continuous and statistically equivalent to the direct process (see also [18, Theorem 7.1]).

**Example 4.5.** *Reverse-time continuous Markov processes define CT-Markov evolution algebras and therefore continuous evolution algebras. This includes examples of birth and death process. Even if many reversible processes are defined on infinite state spaces, under additional assumptions, truncations of reversible processes remain reversible, providing examples of (finite dimensional) continuous evolution algebras arising from many different areas of research.*

## 5. Evolution algebras and vector fields

Evolution algebras were first considered to model dynamics in non-mendelian genetics. Since system evolution is often described by differential equations, and working with evolution algebras usually comes with dependence on in advance fixed natural bases, matrix Lie groups seemed to be a natural framework where considering continuous evolution algebras. Even if this restricts ourselves to perfect algebras.

From the (matrix) Lie group viewpoint,  $G$ -continuous evolution algebras as in Corollary 3.7 are (maximal) integral curves for left-invariant (thus complete) vector fields on  $G$ . In fact, for any  $\mathbf{X} \in \mathfrak{g}$ ,  $\gamma_{\mathbf{X}}(t) = \exp_G(t\mathbf{X})$  is the unique solution of the problem:

$$\begin{cases} \gamma'(t) = X(\gamma(t)) (= \gamma(t)\mathbf{X}), \\ \gamma(0) = \mathbf{I}_n, \end{cases}$$

where  $X : G \rightarrow T(G)$  is the left-invariant vector field on  $G$  with  $X(\mathbf{I}_n) = \mathbf{X} \in T_{\mathbf{I}_n}G$ . Recall that left-invariant vector fields are determined by their value at the group identity element, which results in identifying the set of all left-invariant vector fields with  $T_{\mathbf{I}_n}G = \mathfrak{g}$ . This leads us to approach previous results in terms of flow lines.

**Definition 5.1.** [10, Definition 8.5.9] Let  $G$  be a matrix Lie group. A global flow on  $G$  is a smooth map  $\Phi : \mathbb{R} \times G \rightarrow G$ , such that  $\Phi(0, \mathbf{A}) = \mathbf{A}$  and  $\Phi(s, \Phi(t, \mathbf{A})) = \Phi(s + t, \mathbf{A})$  for all  $s, t \in \mathbb{R}$  and  $\mathbf{A} \in G$ . Maps  $\Phi_{\mathbf{A}} : \mathbb{R} \rightarrow G$  defined by  $\Phi_{\mathbf{A}}(t) = \Phi(t, \mathbf{A})$  are called flow lines.

**Remark 5.2.** Let  $X$  be the left-invariant vector field on  $G$  with  $X(\mathbf{I}_n) = \mathbf{X} \in \mathfrak{g}$ . It defines a global flow  $\Phi^X : \mathbb{R} \times G \rightarrow G$ , with  $\Phi^X(t, \mathbf{A}) = \mathbf{A} \gamma_X(t) = \mathbf{A} \exp_G(t\mathbf{X})$ . Then for each  $\mathbf{A} \in G$ , the flow line  $(\Phi^X)_{\mathbf{A}}(t) = \Phi^X(t, \mathbf{A})$  is an integral curve for  $X$ , and a perfect  $G$ -continuous evolution algebra. See [10, Lemma 9.2.4].

Not all vector fields (resp. flows) are complete (resp. global). This holds, for instance, when  $G$  is compact [10, Corollary 8.5.8], but existence of integral curves is, in general, only ensured locally. Thinking of vector fields as first-order differential operators, this amounts to have a first-order ordinary differential equation having no solution defined for all  $t \in \mathbb{R}$ . In what follows we will restrict to complete vector fields, with associated global flows.

Let now  $\Phi : \mathbb{R} \times G \rightarrow G$  be a global flow. It carries a complete (smooth) vector field  $X^\Phi(\mathbf{A}) = \left(\frac{d}{dt}\right)_{|t=0} \Phi(t, \mathbf{A}) = \Phi'_{\mathbf{A}}(0) \in T_{\mathbf{A}}G$ , whose maximal integral curves equal the flow lines  $\Phi_{\mathbf{A}}(t) = \Phi(t, \mathbf{A})$ . Then  $\Phi_{\mathbf{A}}(0) = \Phi(0, \mathbf{A}) = \mathbf{A}$  and  $\Phi'_{\mathbf{A}}(0) = X^\Phi(\mathbf{A})$  for all  $\mathbf{A} \in G$  [10, Lemma 8.5.10].

**Theorem 5.3.** Let  $\Phi$  be a global flow on a matrix Lie group  $G$  with associated complete vector field  $X^\Phi$ . Then for any  $\mathbf{A} \in G$ ,  $\Phi_{\mathbf{A}}(t) = \Phi(t, \mathbf{A})$  defines a perfect  $G$ -continuous evolution algebra with  $\Phi_{\mathbf{A}}(0) = \mathbf{A}$  and  $\Phi'_{\mathbf{A}}(0) = X^\Phi(\mathbf{A})$ .

*Proof.* It suffices to notice that flows are uniquely determined by their vector fields [10, Theorem 8.5.12].  $\square$

Theorem 5.3 above provides a source of continuous evolution algebras arising as smooth solutions for differential operators given as vector fields on the matrix Lie group  $G$ .

**Corollary 5.4.** Let  $\mathcal{E}$  be a  $n$ -dimensional  $\mathbb{K}$ -vector space with basis  $\mathcal{B}$ , and let  $\Phi$  and  $X^\Phi$  be as in Theorem 5.3. Then  $\mathcal{E}$  with structure matrices (w.r.t.  $\mathcal{B}$ )  $\mathbf{A}(t) = \Phi_{\mathbf{A}}(t)$  is a perfect  $G$  continuous evolution algebra such that:

- (i)  $\mathbf{A}(0) = \mathbf{A}$ .
- (ii)  $\mathbf{A}(s)\mathbf{A}(t) = \mathbf{A}(s + t)$  for all  $s, t \in \mathbb{R}$ .
- (iii)  $\mathbf{A}'(t) = X^\Phi(\mathbf{A}(t))$  for all  $t \in \mathbb{R}$ .

*Proof.* (i) and (ii) are clear, and (iii) follows from  $\mathbf{A}(t) = \Phi_{\mathbf{A}}(t)$  being an integral curve for  $X^\Phi$ .  $\square$

We end up pointing out the similarities between continuous evolution algebras in matrix Lie groups given by vector fields, see Corollary 5.4, and the definition of CT-Markov EAs (see [3, Definition 6.1(i)] or also Section 4). The CT-Markov case corresponds then the flow lines of left and right-invariant vector fields (see Forward and Backward Kolmogorov differential equations) passing through  $\mathbf{A}(0) = \mathbf{I}_n$ , and defined on matrix Lie groups with stochastic properties. This last additional condition, together to the finiteness of the state space provides then the nonnegativity of the matrix entries, resulting on Markov structure matrices for nonnegative time values.

## 6. Further comments

Continuous evolution algebras were firstly proposed in [1, 6.2.4] as a tool to study the evolution of dynamical systems. Here we consider them as differentiable curves on  $M_n(\mathbb{K})$  and focus on those algebras arising as solutions of matrix ODEs. To do this we take advantage of the differentiability of one-parameter subgroups on matrix Lie groups.

Remark 3.14 exhibits the connection between continuous evolution algebras considered in Section 3 and tangent bundles of matrix Lie groups. Our initial approach is then generalized by introducing vector fields and global flows on matrix Lie groups and considering them as smooth manifolds. This viewpoint is in fact closer to Lie group (or even, to smooth manifold) theory, as it considers continuous evolution algebras as flows lines or, equivalently, orbits of the action of  $(\mathbb{R}, +)$  on matrix Lie groups. Indeed, given  $\mathbf{A}(0) = \mathbf{A}$ , the set of structure matrices  $\{\mathbf{A}(t) \mid t \in \mathbb{R}\}$  of the continuous evolution algebra  $\mathfrak{E}$  in Corollary 5.4 consists of the orbit map of  $\mathbf{A}$  for the smooth action of  $(\mathbb{R}, +)$  as Lie group on  $G$  determined by the global flow  $\Phi$ .

The current approach makes possible to bring into a wide variety of problems, as for instance, considering evolution algebras to model systems whose dynamics obey matrix differential equations of the form  $\mathbf{A}'(t) = \mathbf{A}(t)\mathbf{X}(t)$ , with  $\mathbf{A}(0) = \mathbf{A}_0$ . Equations that, as far as  $\mathbf{X}(t)$  and  $\int_0^t \mathbf{X}(\tau)d\tau$  commute, for all  $t$ , admit a solution of the form

$$\mathbf{A}(t) = \mathbf{A}_0 \exp\left(\int_0^t \mathbf{X}(\tau)d\tau\right).$$

If the commutativity requirement fails, then one can consider Magnus solution  $\mathbf{A}(t) = \mathbf{A}_0 \exp(\Omega(t))$  based on the inverse of the derivative of the matrix exponential [19, Section IV.7, Theorem 7.1].

### Declaration of competing interest

None.

### References

- [1] J. P. Tian, *Evolution algebras and their applications*, Lecture Notes in Biomathematics 36, Springer, Berlin, 2008.
- [2] J. Tian, P. Vojtechosky, *Mathematical concepts of evolution algebras in non-Mendelian genetics*, *Quasigroups and Related systems* 14 (2006) 111–122.
- [3] I. Paniello, *Markov evolution algebras*, *Linear and Multilinear Algebra*, DOI 10.1080/03081087.2021.1893636.
- [4] D. G. Poole, *The stochastic group*, *The American Mathematical Monthly* 102(9) (1995) 798–801.
- [5] P. Mellon, M. V. Velasco, *Analytic aspects of evolution algebras*, *Banach J Mat Anal* 13(1) (2019) 113–132.
- [6] A. Baker, *Matrix Groups. An Introduction to Lie Group Theory*. Springer Verlag 2003.
- [7] B. C. Hall, *Lie Groups, Lie Algebras and Representations. An Elementary Introduction*, *Graduate Texts in Mathematics*, vol. 222, Springer 2003.
- [8] L. J. S. Allen, *An introduction to stochastic processes with applications to biology*, Second Edition, CRC Press. Boca Raton, FL, 2011.
- [9] P. Brémaud, *Markov Chains. Gibbs fields, Monte Carlo simulation, and queues*. *Texts in Applied Mathematics*, 31. Springer-Verlag, New York, 1999.
- [10] J. Hilgert, K. H. Neeb, *Structure and Geometry of Lie Groups*, *Springer Monographs in Mathematics*, vol. 159, Springer 2012.
- [11] Y. Cabrera, M. Siles Molina, M. V. Velasco, *Classification of three dimensional evolution algebras*, *Linear Algebra and its Applications*, 524 (2017) 68–108.
- [12] Y. Cabrera, M. Siles Molina, M. V. Velasco, *Classification of three dimensional evolution algebras*, *Linear Algebra and its Applications*, arXiv:1701.07219v1.
- [13] Sh. N. Murodov, *Classification of two-dimensional real evolution algebras and dynamics of some two dimensional chains of evolution algebra*. *Uzbek. Mat. Z.* 2 (2014) 102–111.
- [14] G. S. Chirikjian, *Stochastic Models, Information Theory, and Lie Groups*, vol. 1&2. Basel, Switzerland, Birkhauser, 2012.
- [15] J. Gallier, J. Quaintance. *Differential Geometry and Lie Groups. A Computational Perspective*. Springer 2020.
- [16] A. E. Knapp, *Lie Groups Beyond an Introduction*, *Progress in Mathematics*, vol. 140 2nd Ed. Birkhauser, Basel, 2002.
- [17] B. Mourad, *On a Lie-theoretical Approach to Generalized Doubly Stochastic Matrices and Applications*, *Linear and Multilinear Algebra* 52(2) (2004) 99–113.
- [18] E. Pardoux, *Markov processes and applications: algorithms, networks, genome and finance*. Paris: Wiley; 2008. (Wiley series in Probability and statistics).
- [19] E. Hairer, C. Lubich, G. Wanner, *Geometric Numerical Integration. Structure-Preserving Algorithms for Ordinary Differential Equations*, *Springer Series in Computational Mathematics* v. 3, Springer 2006.