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The (Ψ, Φ)-orthogonal interpolative contractions and an application to fractional differential equations

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Abstract. In this manuscript, we introduce the (Ψ, Φ)-orthogonal interpolative contraction as a generalization of an orthogonal interpolative contraction. We prove several fixed point theorems stating conditions under which (Ψ, Φ)-orthogonal interpolative contraction admits a fixed point. Our fixed point results are improvements of several known results in literature. As an application, we resolve a fractional differential equation.

1. Introduction

The interpolative contraction principles consist of product of distances having exponents satisfying some conditions. The term "interpoltive contraction" was introduced by the renowned mathematician Erdal Karapinar in his paper [33] published in 2018. The interpoltive contraction is defined as follows:

A self-mapping S defined on a metric space $(\mathcal{A}, d)$ is said to be an interpolative contraction, if there exist $\nu \in (0, 1]$ and $K \in [0, 1)$ such that

$$d(Sx, Sy) \leq K\left(d(x, y)\right)^{\nu}, \forall x, y \in \mathcal{A}.$$

Note that for $\nu = 1$, S is a Banach contraction. If the mapping S defined on a metric space $(\mathcal{A}, d)$ satisfies the following inequalities:

$$\begin{aligned}
d(Sx, Sy) &\leq K\left(d(x, Sx)\right)^{\nu}\left(d(y, Sy)\right)^{1-\nu}, \\
d(Sx, Sy) &\leq K\left(d(x, Sy)\right)^{\nu}\left(d(x, Sy)\right)^{1-\nu}, \\
d(Sx, Sy) &\leq K\left(d(x, y)\right)^{\eta}\left(d(x, Sx)\right)^{\nu}\left(d(y, Sy)\right)^{1-\nu-\eta}, \ \nu + \eta < 1 \\
d(Sx, Sy) &\leq K\left(d(x, y)\right)^{\nu}\left(d(x, Sx)\right)^{\eta}\left(d(y, Sy)\right)^{\gamma} \\
&\quad \left(\frac{1}{2}(d(x, Sy) + d(y, Sx))\right)^{1-\eta-\nu-\gamma}, \ \nu + \eta + \gamma < 1.
\end{aligned}$$

for all $x, y \in \mathcal{A}$, then S is called interpolative Kannan type contraction, interpolative Chatterjea type contraction, interpolative Ćirić-Reich-Rus type contraction and interpolative Hardy Rogers type contraction

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respectively. Recently, many classical and advanced contractions have been revisited via interpolation (see [8, 9, 19, 20, 26–32, 34–36, 42]).
Boyd-Wong [14] contraction principle generalizes the contraction principle introduced by Rakotch [45]. Moreover, the Boyd-Wong idea has been generalized by Matkowski [39], Samet *et al.* [48], Karapinar *et al.* [25], Pasicki [43]. The F-contraction [52] is another remarkable generalization of the Banach contraction principle (BCP) and during the last decade many research papers have been published addressing fixed points (common fixed points) of F-contractions (see [12, 13, 47, 49, 51] and references there in). Proinov [44](2020), presented some fixed point theorems that extended the earlier results in [14, 25, 39, 43, 48, 52]
Gordji *et al.* [22] (2017) introduced the notion of orthogonal set (a non-empty set whose elements obey a special relation called *orthogonal relation*) to present a new generalization of the Banach fixed point theorem (BFPT). Gordji *et al.* [22] (2017) explained the notion of orthogonal set by presenting many examples (see [22, Example 2.2-Example 2.11]). The metric defined on the orthogonal set is called *orthogonal metric space*. The orthogonal metric space contains partially ordered metric space and graphical metric space. Baghani *et al.*[12] extended the work in [22] to F-contractions, moreover, the investigation done in [12] was generalized by Nazam *et al.* [41](2021).
In this paper, motivated by the contraction principles presented in [22, 44], we introduce (Ψ, Φ)-orthogonal interpolative contractions which generalize interpolative contractions and unify several interpolative contractions in the orthogonal metric spaces. We show that every interpolative contraction is an orthogonal interpolative contraction but not conversely. We investigate different conditions on the functions Ψ, Φ to show the existence of fixed-points of (Ψ, Φ)-orthogonal interpolative Kannan type contractions, (Ψ, Φ)-orthogonal interpolative Chatterjea type contractions, (Ψ, Φ)-orthogonal interpolative Ćirić-Reich-Rus type contractions and (Ψ, Φ)-orthogonal interpolative Hardy-Rogers type contractions. We also present an application to resolve a fractional differential type equation and some examples in support of the obtained results.

2. Preliminaries

In this section, we define orthogonal set, $\perp$-regular space and O-sequence (a sequence whose terms are pair wise orthogonal). The binary relation $\perp$ (orthogonal relation) is a generalization of the partial order, α-admissible function and directed graph. It also contains the notion of orthogonality in the inner product spaces. The following definition is one of the key notions of this paper.

Definition 2.1. *[22] Let $\perp$ be a binary relation defined on a non-empty set $\mathcal{A}$ (i.e., $\perp \subset \mathcal{A} \times \mathcal{A}$). If $\perp$ satisfies the property (P), then we call it orthogonal relation and the pair $(\mathcal{A}, \perp)$ is called orthogonal set.*

(P): $\exists\ x_0 \in \mathcal{A}$: either $(\forall y, x_0 \perp y)$ or $(\forall y, y \perp x_0)$.

To illustrate the orthogonal set, we have the following examples.

Example 2.2. *[22] Let $\mathcal{A}$ be a inner product space with the inner product $\langle \cdot, \cdot \rangle$. Define $x \perp y$ if $\langle x, y \rangle = 0$. It is easy to see that $0 \perp y$ for all $y \in \mathcal{A}$. Hence $(\mathcal{A}, \perp)$ is an O-set.*

Example 2.3. *[22] In graph theory, a wheel graph W_n is a graph with n vertices for each $n \geq 4$, formed by connecting a single vertex to all vertices of an $(n-1)$-cycle. Let $\mathcal{A}$ be the set of all vertices of W_n for each $n \geq 4$. Define $x \perp y$ if there is a connection from x to y. Then $(\mathcal{A}, \perp)$ is an orthogonal set.*

Example 2.4. *Let $\mathcal{A}$ be the set of integers. Consider, $a \perp \theta$ if and only if $a \equiv 1 \pmod{\theta}$. Then $(\mathcal{A}, \perp)$ is an O-set. Indeed, $1 \perp \theta$ for each θ.*

Example 2.5. *Let $\mathcal{A}$ be the set of all persons in the word. Define $x \perp y$ if x can give blood to y. According to the blood transfusion protocol, if x_0 is a person such that his (her) blood type is O−, then we have $x_0 \perp y$ for all $y \in \mathcal{A}$. This means that $(\mathcal{A}, \perp)$ is an O-set (orthogonal set). In this O-set, x_0 is not unique. Note that, x_0 may be a person with blood type AB+. In this case, we have $y \perp x_0$ for all $y \in \mathcal{A}$.*

Definition 2.6. *[22]*

(a) A sequence $\{\hbar_n : n \in \mathbb{N}\}$ *is said to be an O-sequence if either* $x_n \perp x_{n+1}$ *or* $x_{n+1} \perp x_n$ *for all* n.

(b) The O-set $(\mathcal{A}, \perp)$ *endowed with a metric* d *is called an O-metric space (in short, OMS) denoted by* $(\mathcal{A}, \perp, d)$.

(c) The O-sequence $\{x_n\} \subset \mathcal{A}$ *is said to be O-Cauchy if* $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. *If each O-Cauchy sequence converges in* $\mathcal{A}$, *then* $(\mathcal{A}, \perp, d)$ *is called O-complete.*

(d) Let $(\mathcal{A}, \perp, d)$ *be an orthogonal metric space. A mapping* $f : \mathcal{A} \to \mathcal{A}$ *is said to be an orthogonal contraction if there exists* $k \in [0, 1)$ *such that*

$$d(fx, fy) \leq kd(x, y) \ \forall x, y \in \mathcal{A} \text{ with } x \perp y.$$

In the following, we give some comparisons between fundamental notions.

1. The continuity implies orthogonal continuity but converse is not true. If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = [x], \ \forall x \in \mathbb{R}$ and the relation $\perp \subseteq \mathbb{R} \times \mathbb{R}$ is defined by

$$x \perp y \text{ if } x, y \in \left(i + \frac{1}{3}, i + \frac{2}{3}\right), i \in \mathbb{Z} \text{ or } x = 0.$$

Then f is $\perp$-continuous while f is discontinuous on $\mathbb{R}$.

2. The completeness of the metric space implies O-completeness but the converse is not true. We know that $\mathcal{A} = [0, 1)$ with Euclidean metric d is not complete metric space. If we define the relation $\perp \subseteq \mathcal{A} \times \mathcal{A}$ by

$$x \perp y \iff x \leq y \leq \frac{1}{2} \text{ or } x = 0.$$

Then $(\mathcal{A}, \perp, d)$ is an O-complete.

3. The Banach contraction implies orthogonal contraction but converse is not true. Let $\mathcal{A} = [0, 10)$ with Euclidean metric d so that $(\mathcal{A}, d)$ is a metric space. If we define the relation $\perp \subseteq \mathcal{A} \times \mathcal{A}$ by

$$x \perp y \text{ if } xy \leq x \vee y.$$

Then $(\mathcal{A}, \perp, d)$ is an O-metric space. Define $f : \mathcal{A} \to \mathcal{A}$ by $f(x) = \frac{x}{2}$ (if $x \leq 2$) and $f(x) = 0$ (if $x > 2$). Since $d(f(3), f(2)) > kd(3, 2)$, so, f is not a contraction while it is an orthogonal contraction.

We will use the following lemma to support the proofs.

Lemma 2.7. *[44] Let* (X, d) *be a metric space and* $\{x_n\} \subset X$ *be a sequence verifying* $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. *If the sequence* $\{x_n\}$ *is not Cauchy, then there are* $\{x_{n_k}\}$, $\{x_{m_k}\}$ *and* $\xi > 0$ *such that*

$$\lim_{k\to\infty} d(x_{n_k+1}, x_{m_k+1}) = \xi + . \tag{1}$$

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = d(x_{n_k+1}, x_{m_k}) = d(x_{n_k}, x_{m_k+1}) = \xi. \tag{2}$$

Definition 2.8. *[15] A mapping* $T : \mathcal{A} \to \mathcal{A}$ *is said to be asymptotically regular at a point* v *of* $\mathcal{A}$ *if*

$$\lim_{n\to\infty} d(T^n v, T^{n+1} v) = 0.$$

If T *is asymptotically regular at each point in* $\mathcal{A}$, *then it is named as an asymptotically regular mapping.*

3. Orthogonal interpolative contractions and related fixed-point results

The orthogonal interpolative contractions are more general than interpolative contractions. The following example shows that the orthogonal interpolative Kannan contraction (OIKC) implies interpolative Kannan contraction (IKC) but not conversely.

Example 3.1. *Let* $\mathcal{A} = \{1, 2, 3, 4, 5, 6, 7, 8, \cdots\}$ *with Euclidean metric* d *so that* $(\mathcal{A}, d)$ *is a metric space. If we define the relation* $\perp \subseteq \mathcal{A} \times \mathcal{A}$ *by*

$$x \perp y \text{ if } xy \leq x \vee y \text{ for all } x \neq y.$$

Then $(\mathcal{A}, \perp, d)$ *is an O-metric space. Define* $S : \mathcal{A} \to \mathcal{A}$ *by* $S(x) = \frac{x}{2}$ *(if* $x \leq 2$*) and* $S(x) = 0$ *(if* $x > 2$*). Since, there exist* $\nu = \frac{3}{4}$ *and* $k = 0.99$ *such that* $d(S(3), S(2)) > k[d(3, S(3))]^{\nu} \cdot [d(2, S(2)]^{1-\nu}$*, so, S is not an IKC while the following calculations show that S is an OIKC.*
For if $x = 2$ *and* $y = 1$*, then* $x \perp y$ *and*

$$d(S(2), S(1)) \leq k[d(2, S(2))]^{\nu} \cdot [d(1, S(1)]^{1-\nu} \text{ for some } k \in [0, 1) \text{ and } \nu \in (0, 1).$$

For if $x = 3$ *and* $y = 1$*, then* $x \perp y$ *and*

$$d(S(3), S(1)) \leq k[d(3, S(3))]^{\nu} \cdot [d(1, S(1)]^{1-\nu} \text{for some } k \in [0, 1) \text{ and } \nu \in (0, 1).$$

Similarly, for all other cases, S is an OIKC. Thus, joining [33, Example 2.3] and Example 3.1, we have

Kannan contraction $\rightarrow$ *IKC* $\rightarrow$ *OIKC.*

Kannan contraction $\not\leftarrow$ *IKC* $\not\leftarrow$ *OIKC.*

In the following we define $\perp$-regular and $\perp$-preserving mapping and illustrate them with examples. Let $\Lambda = \{(x, y) \in \mathcal{A} \times \mathcal{A} : x \perp y\}$.

Definition 3.2. *Let* $(\mathcal{A}, \perp, d)$ *be an OMS and* $\perp \subset \mathcal{A} \times \mathcal{A}$ *be a binary relation. The space* $(\mathcal{A}, \perp, d)$ *is called* $\perp$*-regular if for each sequence* $\{x_n\} \subset \mathcal{A}$ *so that* $x_n \perp x_{n+1}$ *for each* $n \geq 0$ *and* $x_n \to x$ *as* $n \to \infty$*, we have either* $x_n \perp x$*, or* $x \perp x_n$ *for all* $n \geq 0$.

Definition 3.3. *Let* $S : \mathcal{A} \to \mathcal{A}$ *and* $\perp \subset \mathcal{A} \times \mathcal{A}$ *be an orthogonal relation. The mapping S is called* $\perp$*-preserving if, whenever,* $x \perp y$*, we have* $Sx \perp Sy$ *for all* $x, y \in \mathcal{A}$.

Example 3.4. *Let* $\mathcal{A} = [0, 1)$ *and define the relation* $\perp \subset \mathcal{A} \times \mathcal{A}$ *by*

$$x \perp y \text{ if } xy \leq x \vee y.$$

Then $\mathcal{A}$ *is an O-set. Define* $S : \mathcal{A} \to \mathcal{A}$ *by*

$$S(x) = \begin{cases} \frac{x}{5} & \text{if } x \in \mathbb{Q} \cap \mathcal{A}, \\ 0 & \text{if } x \in \mathbb{Q}^c \cap \mathcal{A}. \end{cases}$$

Then S is a $\perp$*-preserving mapping. Indeed, for* $x = \frac{1}{3}$, $y = \frac{1}{2}$*, we have* $x \perp y$ *and since* $S\left(\frac{1}{3}\right) S\left(\frac{1}{2}\right) = \frac{1}{150} < S\left(\frac{1}{3}\right) \vee S\left(\frac{1}{2}\right)$*, so,* $S\left(\frac{1}{3}\right) \perp S\left(\frac{1}{2}\right)$*. Similarly for all the other cases, it is evident that S is a* $\perp$*-preserving mapping.*

3.1. (Ψ, Φ)-orthogonal interpolative Kannan type contraction

Let $(\mathcal{A}, \perp, d)$ be an OMS and $\Phi, \Psi : (0, \infty) \to (-\infty, \infty)$ be two functions. A mapping $S : \mathcal{A} \to \mathcal{A}$ is said to be a (Ψ, Φ)-orthogonal interpolative Kannan type contraction if there exists $\nu \in (0, 1)$ such that

$$\Psi(d(Sx, Sy)) \leq \Phi\left((d(x, Sx))^{\nu}(d(y, Sy))^{1-\nu}\right), \tag{3}$$

for all $(x, y) \in \Lambda$, and $\min\{d(Sx, Sy), d(y, Sy), d(x, Sx)\} > 0$.

The following example explains (3).

Example 3.5. *Let $\mathcal{A} = [1, 7)$ and define the relation $\perp$ on $\mathcal{A}$ by*

$$x \perp y \text{ if } xy \in \{x, y\}.$$

Then $\perp$ is an orthogonal relation and so $(\mathcal{A}, \perp)$ is an O-set. Let d be the Euclidean metric on $\mathcal{A}$, then, $(\mathcal{A}, d)$ is an incomplete metric space. Define $\Psi, \Phi : \mathbb{R}^+ \to \mathbb{R}$ by

$$\Psi(x) = \begin{cases} x + 1 & \text{if } x \in \{3.5, 5\}, \\ \frac{x}{2} & \text{if } x \in \mathbb{R}^+ - \{3.5, 5\}. \end{cases}$$

$$\Phi(x) = \begin{cases} x^2 + 1 & \text{if } x \in \{3.5, 5\}, \\ x + 10 & \text{if } x \in \mathbb{R}^+ - \{3.5, 5\}. \end{cases}$$

Let $S : \mathcal{A} \to \mathcal{A}$ be defined by

$$S(x) = \begin{cases} 6 & \text{if } 1 \leq x < 2, \\ 2.5 & \text{if } 2 \leq x < 3, \\ 1.5 & \text{if } 3 \leq x < 7. \end{cases}$$

Our calculations show that $d(Sx, Sy) = 3.5$, $d(x, Sx) = 5$ and $d(y, Sy) = 0.5$ if $x = 1$, $y = 2$ $(1 \perp 2)$. This information shows that

$$d(Sx, Sy) > \lambda[d(x, Sx)]^{\nu}[d(y, Sy)]^{1-\nu} \text{ for some } \lambda = \frac{1}{2}, \nu = 0.9.$$

Thus, S is not an orthogonal interpolative Kannan type contraction. However, S is a (Ψ, Φ)-orthogonal interpolative Kannan type contraction. Indeed,

$$\Psi(d(Sx, Sy)) \leq \Phi\left([d(x, Sx)]^{\nu}[d(y, Sy)]^{1-\nu}\right).$$

We obtain the same conclusions for $x = 1$, $y = 3$ $(1 \perp 3)$; $x = 1$, $y = 4$ $(1 \perp 4)$; $x = 1$, $y = 5$ $(1 \perp 5)$ and $x = 1$, $y = 6$ $(1 \perp 6)$.

Remark 3.6. *Example 3.1 and 3.5 show that interpolative contraction implies orthogonal interpolative contraction and orthogonal interpolative contraction implies (Ψ, Φ)-orthogonal interpolative contraction but converse is not true.*

Remark 3.7. *For the particular definitions of the mappings Ψ, Φ, we have the following observations which show the generality of (Ψ, Φ)-orthogonal interpolative Kannan type contraction.*

1. *Defining $\Phi(x) = \Psi(x) - \tau$ for all $x \in (0, \infty)$ in (3) we have orthogonal interpolative Kannan type F-contractions.*
2. *Defining $\Phi(x) = \Psi(x) - \tau(x)$ for all $x \in (0, \infty)$ in (3), we have orthogonal interpolative Kannan type (τ, F_T)-contraction.*

3. *Defining* $\Psi(x) = x$ *and* $\Phi(y) = \lambda y$ *for all* $x, y \in (0, \infty)$ *in (3), then we obtain the contraction introduced in [33].*

For the orthogonal relation $\perp$, self-mapping S and functions $\Psi, \Phi : (0, \infty) \to (-\infty, \infty)$, we state the following conditions:

(i) for each $\hbar_0 \in \mathcal{A}$, there is $\hbar_1 = S(\hbar_0)$ such that $\hbar_1 \perp \hbar_0$ or $\hbar_0 \perp \hbar_1$.

(ii) Ψ is non-decreasing and $\Phi(x) < \Psi(x)$ for all $x > 0$.

(iii) $\limsup_{x \to \delta+} \Phi(x) < \Psi(\delta+)$ for all $\delta > 0$.

(iv) $\limsup_{a \to 0} \Phi(a) \leq \lim \inf_{a \to \xi+} \Psi(a)$.

(v) $\Psi(a^\nu b^\eta) \leq \Psi(a)$ and $\Phi(x) < \Psi(x)$ for all $x > 0$.

(vi) $\inf_{a > \xi} \Psi(a) > -\infty$.

(vii) if $\{\Psi(\hbar_n)\}$ and $\{\Phi(\hbar_n)\}$ are converging to same limit and $\{\Psi(\hbar_n)\}$ is strictly decreasing, then $\lim_{n \to \infty} \hbar_n = 0$.

(viii) $\limsup_{a \to 0} \Phi(a) < \liminf_{a \to \xi} \Psi(a)$ for all $\xi > 0$.

The following is one of the main theorems that states the conditions for the existence of the fixed points of a self-mapping S satisfying (3).

Theorem 3.8. *Let* $\perp$ *be a transitive orthogonal relation, then, every* $\perp$*-preserving self-mapping defined on a* $\perp$*-regular O-complete metric space* $(\mathcal{A}, \perp, d)$ *(in short, OCMS) verifying* (3) *and* (i)-(iv), *admits a fixed point in* $\mathcal{A}$.

Proof. By (i), as $\hbar_0 \in \mathcal{A}$ is such that $\hbar_0 \perp \hbar_1$ or $\hbar_1 \perp \hbar_0$ for each $\hbar_1 \in \mathcal{A}$, then by using the $\perp$-preserving nature of S, we construct an orthogonal sequence $\{\hbar_n\}$ such that $\hbar_n = S(\hbar_{n-1}) = S^n(\hbar_0)$ and $\hbar_{n-1} \perp \hbar_n$ for each $n \in \mathbb{N}$. Note that, if $\hbar_n = S(\hbar_n)$, then $\hbar_n$ is a fixed point of S for all $n \geq 0$. We assume that $\hbar_n \neq \hbar_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Let $d_n = d(\hbar_n, \hbar_{n+1})$ for all $n \geq 0$. By the first part of (ii) and (3), we have

$$\begin{aligned} \Psi(d_n) &\leq \Psi(d(S(\hbar_{n-1}), S(\hbar_n))) \leq \Phi\left((d(\hbar_{n-1}, S\hbar_{n-1}))^\nu (d(\hbar_n, S\hbar_n))^{1-\nu}\right) \\ &= \Phi\left((d_{n-1})^\nu (d_n)^{1-\nu}\right). \end{aligned}$$

In view of second part of (ii), we have

$$\Psi(d_n) \leq \Phi\left((d_{n-1})^\nu (d_n)^{1-\nu}\right) < \Psi\left((d_{n-1})^\nu (d_n)^{1-\nu}\right). \tag{4}$$

Since Ψ is non-decreasing, one gets $d_n < d_{n-1}$ for each $n \geq 1$. This shows that the sequence $\{d_n\}$ is decreasing, so there is $L \geq 0$ so that $\lim_{n \to \infty} d_n = L+$. If $L > 0$, by (4), one gets

$$\Psi(L+) = \lim_{n \to \infty} \Psi(d_n) \leq \lim_{n \to \infty} \sup \Phi((d_{n-1})^\nu (d_n)^{1-\nu}) \leq \lim_{a \to L+} \sup \Phi(a).$$

This contradicts (iii), so $L = 0$, that is, S is asymptotically regular.

We claim that $\{\hbar_n\}$ is Cauchy. Suppose on the contrary that $\{\hbar_n\}$ is not a Cauchy sequence. Then, by Lemma 2.7, there are subsequences $\{\hbar_{n_k}\}$, $\{\hbar_{m_k}\}$ of $\{\hbar_n\}$ and $\xi > 0$ such that (1) and (2) hold. By (1), we infer that $d(\hbar_{n_k+1}, \hbar_{m_k+1}) > \xi$. Since $\hbar_n \perp \hbar_{n+1}$ for all $n \geq 0$, by transitivity of $\perp$, we have $\hbar_{n_k} \perp \hbar_{m_k}$ for all $k \geq 1$. Letting $\ell = \hbar_{n_k}$ and $\jmath = \hbar_{m_k}$ in (3), we have for each $k \geq 1$,

$$\begin{aligned} \Psi(d(\hbar_{n_k+1}, \hbar_{m_k+1})) &\leq \Psi(d(S\hbar_{n_k}, S\hbar_{m_k})) \\ &\leq \Phi\left((d(\hbar_{n_k}, S\hbar_{n_k}))^\nu (d(\hbar_{m_k}, S\hbar_{m_k}))^{1-\nu}\right) \\ &= \Phi\left((d(\hbar_{n_k}, \hbar_{n_k+1}))^\nu (d(\hbar_{m_k}, \hbar_{m_k+1}))^{1-\nu}\right). \end{aligned}$$

If $\hbar_k = d(\hbar_{n_k+1}, \hbar_{m_k+1})$, $b_k = d(\hbar_{n_k}, \hbar_{n_k+1})$ and $c_k = d(\hbar_{m_k}, \hbar_{m_k+1})$, we have

$$\Psi(\hbar_k) \leq \Phi\left((b_k)^{\nu}(c_k)^{1-\nu}\right), \text{ for all } k \geq 1. \tag{5}$$

By (1), we have $\lim_{k\to\infty} \hbar_k = \xi+$ and (5) implies

$$\liminf_{a\to\xi+} \Psi(a) \leq \liminf_{k\to\infty} \Psi(\hbar_k) \leq \limsup_{k\to\infty} \Phi\left((b_k)^{\nu}(c_k)^{1-\nu}\right) \leq \limsup_{a\to 0} \Phi(a).$$

It is a contradiction to (iv), so $\{\hbar_n\}$ is Cauchy sequence in the OCMS $(\mathcal{A}, \perp, d)$, hence there is $a^* \in \mathcal{A}$ so that $\hbar_n \to a^*$ as $n \to \infty$, and the $\perp$-regularity of $(\mathcal{A}, \perp, d)$ yields that $\hbar_n \perp a^*$ or $a^* \perp \hbar_n$. We claim that $d(a^*, S(a^*)) = 0$. Assume on contrary that $d(\hbar_{n+1}, S(a^*)) > 0$ for infinitely many values of n. By (3),

$$\begin{aligned} \Psi(d(\hbar_{n+1}, S(a^*))) &\leq \Psi(d(S(\hbar_n), S(a^*))) \leq \Phi\left((d(\hbar_n, S\hbar_n))^{\nu}(d(a^*, Sa^*))^{1-\nu}\right) \\ &< \Psi\left((d(\hbar_n, S\hbar_n))^{\nu}(d(a^*, Sa^*))^{1-\nu}\right) = \Psi\left((d_n)^{\nu}(d(a^*, Sa^*))^{1-\nu}\right). \end{aligned}$$

By the first part of (ii), we get $d(\hbar_{n+1}, S(a^*)) < (d(\hbar_n, \hbar_{n+1}))^{\nu}(d(a^*, Sa^*))^{1-\nu}$. Letting $n \to \infty$, we obtain $d(a^*, S(a^*)) < (d(a^*, Sa^*))^{1-\nu}$, which is a false statement. Thus, $d(a^*, S(a^*)) = 0$, and hence, $a^* = S(a^*)$. □

The next result is the second main theorem stating some conditions for the existence of fixed points of S verifying (3).

Theorem 3.9. *Let $\perp$ be a transitive orthogonal relation, then, every $\perp$-preserving self-mapping defined on a $\perp$-regular OCMS $(\mathcal{A}, \perp, d)$ verifying* (3) *and* (i),(iv)-(viii)*, admits a fixed point in $\mathcal{A}$.*

Proof. Let $\hbar_0 \in \mathcal{A}$ be such that $\hbar_0 \perp \hbar_1$ or $\hbar_1 \perp \hbar_0$ for each $\hbar_1 \in \mathcal{A}$, then by using the $\perp$-preserving nature of S, we construct an orthogonal sequence $\{\hbar_n\}$ such that $\hbar_n = S(\hbar_{n-1}) = S^n(\hbar_0)$ and $\hbar_{n-1} \perp \hbar_n$ for each $n \in \mathbb{N}$. Note that, if $\hbar_n = S(\hbar_n)$ then $\hbar_n$ is a fixed point of S for all $n \geq 0$. We assume that $\hbar_n \neq \hbar_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Let $d_n = d(\hbar_n, \hbar_{n+1})$ for all $n \geq 0$. By (v) and (3), we have

$$\begin{aligned} \Psi(d(\hbar_n, \hbar_{n+1})) &\leq \Psi(d(S(\hbar_{n-1}), S(\hbar_n))) = \Phi((d(\hbar_{n-1}, S\hbar_{n-1}))^{\nu}(d(\hbar_n, S\hbar_n))^{1-\nu}) \\ &\leq \Phi((d_{n-1})^{\nu}(d_n)^{1-\nu}) < \Psi\left((d_{n-1})^{\nu}(d_n)^{1-\nu}\right) \\ &\leq \Psi(d_{n-1}). \end{aligned} \tag{6}$$

The inequality (6) shows that $\{\Psi(d(\hbar_{n-1}, \hbar_n))\}$ is strictly decreasing. If it is not bounded below, in view of (vi), we get $\inf_{d(\hbar_{n-1},\hbar_n)>\xi} \Psi(d(\hbar_{n-1}, \hbar_n)) > -\infty$. This implies that

$$\lim_{d_{n-1}\to\xi+} \inf \Psi(d_{n-1}) > -\infty.$$

Thus, $\lim_{n\to\infty} d(\hbar_{n-1}, \hbar_n) = 0$, otherwise we have

$$\lim_{d_{n-1}\to\xi+} \inf \Psi(d_{n-1}) = -\infty, \text{ (a contradiction to (vi))}.$$

If it is bounded below, then $\{\Psi(d(\hbar_{n-1}, \hbar_n))\}$ is a convergent sequence and by (6), $\{\Phi(d(\hbar_{n-1}, \hbar_n))\}$ also converges and both have same limit. Thus, by (vii), one gets $\lim_{n\to\infty} d(\hbar_{n-1}, \hbar_n) = 0$. Hence, S is asymptotically regular.

Now, we claim that $\{\hbar_n\}$ is a Cauchy sequence. If $\{\hbar_n\}$ is not a Cauchy sequence, so by Lemma 2.7, there exist $\{\hbar_{n_k}\}$, $\{\hbar_{m_k}\}$ and $\xi > 0$ such that (1) and (2) hold. By (1), we infer that $d(\hbar_{n_k+1}, \hbar_{m_k+1}) > \xi$. Since $\hbar_n \perp \hbar_{n+1}$ for all $n \geq 0$ so by transitivity of $\perp$, we have $\hbar_{n_k} \perp \hbar_{m_k}$. Letting $x = \hbar_{n_k}$ and $y = \hbar_{m_k}$ in (3), we have, for all $k \geq 1$,

$$\begin{aligned} \Psi(d(\hbar_{n_k+1}, \hbar_{m_k+1})) &\leq \Psi(d(S\hbar_{n_k}, S\hbar_{m_k})) \\ &\leq \Phi\left((d(\hbar_{n_k}, S\hbar_{n_k}))^{\nu}(d(\hbar_{m_k}, S\hbar_{m_k}))^{1-\nu}\right) \\ &= \Phi\left((d(\hbar_{n_k}, \hbar_{n_k+1}))^{\nu}(d(\hbar_{m_k}, \hbar_{m_k+1}))^{1-\nu}\right). \end{aligned}$$

If $\hbar_k = d(\hbar_{n_k+1}, \hbar_{m_k+1})$, $b_k = d(\hbar_{n_k}, \hbar_{n_k+1})$ and $c_k = d(\hbar_{m_k}, \hbar_{m_k+1})$, we have

$$\Psi(\hbar_k) \leq \Phi\left((b_k)^{\nu}(c_k)^{1-\nu}\right), \text{ for all } k \geq 1. \tag{7}$$

By (1), we have $\lim_{k\to\infty} \hbar_k = \xi+$ and (7) implies

$$\liminf_{a\to\xi+} \Psi(a) \leq \liminf_{k\to\infty} \Psi(\hbar_k) \leq \limsup_{k\to\infty} \Phi\left((b_k)^{\nu}(c_k)^{1-\nu}\right) \leq \limsup_{a\to 0} \Phi(a).$$

It contradicts (vi), so $\{\hbar_n\}$ is a Cauchy sequence in the OCMS $\mathcal{A}$. Hence, there is $a^* \in \mathcal{A}$ in order that $\hbar_n \to a^*$ as $n \to \infty$.

To show that $Sa^* = a^*$, we have two cases:

Case 1. If $d(\hbar_{n+1}, Sa^*) = 0$ for some $n \geq 0$, then we have the following information:

$$d(a^*, Sa^*) \leq d(a^*, \hbar_{n+1}) + d(\hbar_{n+1}, Sa^*) = d(a^*, \hbar_{n+1}).$$

Letting $n \to \infty$ on both sides, we have $d(a^*, Sa^*) \leq 0$. This implies $d(a^*, S(a^*)) = 0$. Hence, $a^* = Sa^*$.

Case 2. If for all $n \geq 0$, $d(\hbar_{n+1}, Sa^*) > 0$, then by $\perp$-regularity of $\mathcal{A}$, we find $\hbar_n \perp a^*$ or $a^* \perp \hbar_n$. By (3), we have

$$\Psi(d(\hbar_{n+1}, Sa^*)) \leq \Psi(d(S\hbar_n, Sa^*)) \leq \Phi\left((d(\hbar_n, S\hbar_n))^{\nu}(d(a^*, Sa^*)^{1-\nu}\right) \text{ for all } n \geq 0.$$

By taking $\ell_n = d(\hbar_{n+1}, Sa^*)$, we have

$$\Psi(\ell_n) \leq \Phi\left((d_n)^{\nu}(d(a^*, Sa^*)^{1-\nu}\right) \text{ for all } n \geq 0. \tag{8}$$

Take $\xi = d(a^*, Sa^*)$. Note that $\ell_n \to \xi$ and $d_n \to 0$ as $n \to \infty$. Applying limits on (8), we have

$$\liminf_{a\to\xi} \Psi(a) \leq \liminf_{n\to\infty} \Psi(\hbar_n) \leq \limsup_{n\to\infty} \Phi((b_n)^{\nu}\xi^{1-\nu}) \leq \liminf_{a\to 0} \Phi(a).$$

This contradicts (viii) if $\xi > 0$. Thus, we have $d(a^*, Sa^*) = 0$, that is, a^* is a fixed point of S. □

3.2. (Ψ, Φ)-*orthogonal modified interpolative Chatterjea contraction*

Let $(\mathcal{A}, \perp, d)$ be an OMS and $\Psi, \Phi : (0, \infty) \to (-\infty, \infty)$ be two functions. The mapping $S : \mathcal{A} \to \mathcal{A}$ is said to be a (Ψ, Φ)-orthogonal modified interpolative Chatterjea contraction if there exists $\nu \in (0, 1]$ such that

$$\Psi(d(Sx, Sy)) \leq \Phi\left(\left(\left(\frac{d(x, Sy)}{2}\right)^{\frac{1}{\nu}} + \left(\frac{d(y, Sx)}{2}\right)^{\frac{1}{\nu}}\right)^{\nu}\right), \tag{9}$$

for all $(x, y) \in \Lambda$, and $\min\{d(Sx, Sy), d(x, Sy)\} > 0$.

The following example explains (9).

Example 3.10. *Let* $\mathcal{A} = [1, 7)$ *and define the relation* $\perp$ *on* $\mathcal{A}$ *by*

$$x \perp y \text{ if } xy \in \{x, y\}.$$

Then $\perp$ *is an orthogonal relation and so* $(\mathcal{A}, \perp)$ *is an O-set. Let d be the Euclidean metric on* $\mathcal{A}$, *then,* $(\mathcal{A}, d)$ *is an incomplete metric space. Define* $\Psi, \Phi : \mathbb{R}^+ \to \mathbb{R}$ *by*

$$\Psi(x) = \begin{cases} x + 1 & \text{if } x \in \{3.5, 4.5\}, \\ \frac{x}{2} & \text{if } x \in \mathbb{R}^+ - \{3.5, 4.5\}. \end{cases}$$

$$\Phi(x) = \begin{cases} x^2 + 1 & \text{if } x \in \{3.5, 4.5\}, \\ x + 10 & \text{if } x \in \mathbb{R}^+ - \{3.5, 4.5\}. \end{cases}$$

Let $S : \mathcal{A} \to \mathcal{A}$ be defined by

$$S(x) = \begin{cases} 6 & \text{if } 1 \leq x < 2, \\ 2.5 & \text{if } 2 \leq x < 3, \\ 1.5 & \text{if } 3 \leq x < 7. \end{cases}$$

Our calculations show that $d(Sx, Sy) = 3.5$, $d(x, Sy) = 1.5$ and $d(y, Sx) = 4$ if $x = 1$, $y = 2$ $(1 \perp 2)$. This information shows that

$$d(Sx, Sy) > \lambda\left(\left(\frac{d(x, Sy)}{2}\right)^{\frac{1}{\nu}} + \left(\frac{d(y, Sx)}{2}\right)^{\frac{1}{\nu}}\right)^{\nu} \text{ for some } \lambda = \frac{1}{2}, \nu = 0.9.$$

Thus, S is not an orthogonal modified interpolative Chatterjea contraction. However, S is a (Ψ, Φ)-orthogonal modified interpolative Chatterjea contraction. Indeed,

$$\Psi(d(Sx, Sy)) \leq \Phi\left(\left(\left(\frac{d(x, Sy)}{2}\right)^{\frac{1}{\nu}} + \left(\frac{d(y, Sx)}{2}\right)^{\frac{1}{\nu}}\right)^{\nu}\right).$$

We obtain the same conclusion for $x = 1$, $y = 3$ $(1 \perp 3)$; $x = 1$, $y = 4$ $(1 \perp 4)$; $x = 1$, $y = 5$ $(1 \perp 5)$; $x = 1$, $y = 6$ $(1 \perp 6)$.

Remark 3.11. *For the particular definitions of the mappings Ψ, Φ, we have the following observations which show the generality of (Ψ, Φ)-orthogonal modified interpolative Chatterjea contraction.*

1. *Defining $\Phi(x) = \Psi(x) - \tau$ for all $x \in (0, \infty)$ in (9), we have the orthogonal interpolative modified Chatterjea F-contractions.*
2. *Defining $\Phi(x) = \Psi(x) - \tau(x)$ for all $x \in (0, \infty)$ in (9), we have the orthogonal interpolative modified Chatterjea (τ, F_T)-contraction.*
3. *Defining $\Psi(x) = x$ and $\Phi(y) = \lambda y$ for all $x, y \in (0, \infty)$ in (9), then we obtain the contraction introduced in [20].*
4. *Defining $\Phi(x) = \beta(x)x$ and $\Psi(x) = x$ for all $x > 0$ and $\beta : (0, \infty) \to (0, 1)$ verifying $\limsup_{x \to \xi+} \beta(x) < 1$ for each $\xi > 0$, we obtain orthogonal interpolative modified Chatterjea type Geraghty contractions.*
5. *For $\nu = 1$, we have*

   $$\Psi(d(Sx, Sy)) \leq \Phi\left(\frac{1}{2}(d(x, Sy) + d(y, Sx))\right),$$

   a classical (Ψ, Φ)-orthogonal Chatterjea contraction.

For the (Ψ, Φ)- orthogonal interpolative Chatterjea type contractions, we have the following theorem.

Theorem 3.12. *Let $\perp$ be a transitive orthogonal relation, then, every $\perp$-preserving self-mapping defined on a $\perp$-regular O-complete metric space $(\mathcal{A}, \perp, d)$ (in short, OCMS) verifying* (9) *and* (i)-(iv), *admits a fixed point in $\mathcal{A}$.*

Proof. Proceeding as in proof of Theorem 3.8, we have

$$\begin{aligned} \Psi(d_n) &\leq \Psi(d(S(\hbar_{n-1}), S(\hbar_n))) \\ &\leq \Phi\left(\left(\left(\frac{d(\hbar_{n-1}, S\hbar_n)}{2}\right)^{\frac{1}{\nu}} + \left(\frac{d(\hbar_n, S\hbar_{n-1})}{2}\right)^{\frac{1}{\nu}}\right)^{\nu}\right) \\ &= \Phi\left(\frac{d(\hbar_{n-1}, \hbar_{n+1})}{2}\right) \leq \Phi\left((d_{n-1} + d_n)/2\right). \end{aligned} \tag{10}$$

Suppose that $d_{n-1} < d_n$ for some $n \geq 1$, then by (10) and the second part of (ii), we have

$$\Psi(d_n) \leq \Phi(d_n) < \Psi(d_n).$$

This is a contradiction to the definition of Ψ. Consequently, we have

$$\Psi(d_n) \leq \Phi\left((d_{n-1})\right) < \Psi\left((d_{n-1})\right) \ \forall\, n \geq 1.$$

Continuing as in the proof of Theorem 3.8, we have $\hbar_n \to a^*$ as $n \to \infty$, and the $\perp$-regularity of the space $(\mathcal{A}, \perp, d)$ implies $\hbar_n \perp a^*$ or $a^* \perp \hbar_n$. We claim that $d(a^*, S(a^*)) = 0$. Assume on contrary that $d(\hbar_{n+1}, S(a^*)) > 0$ for infinitely many values of n. By (9),

$$\begin{aligned}
\Psi(d(\hbar_{n+1}, S(a^*))) &\leq \Psi(d(S(\hbar_n), S(a^*))) \\
&\leq \Phi\left(\left(\left(\frac{d(\hbar_n, Sa^*)}{2}\right)^{\frac{1}{\nu}} + \left(\frac{d(a^*, S\hbar_n)}{2}\right)^{\frac{1}{\nu}}\right)^{\nu}\right) \\
&= \Phi\left(\left(\left(\frac{d(\hbar_n, Sa^*)}{2}\right)^{\frac{1}{\nu}} + \left(\frac{d(a^*, \hbar_{n+1})}{2}\right)^{\frac{1}{\nu}}\right)^{\nu}\right) \\
&< \Psi\left(\left(\left(\frac{d(\hbar_n, Sa^*)}{2}\right)^{\frac{1}{\nu}} + \left(\frac{d(a^*, \hbar_{n+1})}{2}\right)^{\frac{1}{\nu}}\right)^{\nu}\right).
\end{aligned}$$

Due to (ii), we get

$$d(\hbar_{n+1}, S(a^*)) < \left(\left(\frac{d(\hbar_n, Sa^*)}{2}\right)^{\frac{1}{\nu}} + \left(\frac{d(a^*, \hbar_{n+1})}{2}\right)^{\frac{1}{\nu}}\right)^{\nu}.$$

Applying limit $n \to \infty$, we obtain $d(a^*, S(a^*)) < \frac{d(a^*, Sa^*)}{2}$, that is an absurdity. Thus, $d(a^*, S(a^*)) = 0$. We get, $a^* = S(a^*)$. □

Theorem 3.13. *Let $\perp$ be a transitive orthogonal relation, then, every $\perp$-preserving self-mapping defined on a $\perp$-regular OCMS $(\mathcal{A}, \perp, d)$ verifying* (9) *with $\Phi(\jmath) < \Psi(\jmath)$ for each $\jmath > 0$ and* (i),(iv),(vi)-(viii), *admits a fixed point in $\mathcal{A}$.*

Proof. Proceeding as in the proof of Theorem 3.9 and then following the arguments used in the proof of Theorem 3.12, we have the required result. □

3.3. (Ψ, Φ)-*orthogonal interpolative Ćirić-Reich-Rus type contraction*

Let $(\mathcal{A}, \perp, d)$ be an OMS and $\Psi, \Phi : (0, \infty) \to (-\infty, \infty)$ be two functions. The mapping $S : \mathcal{A} \to \mathcal{A}$ is said to be (Ψ, Φ)-orthogonal interpolative Ćirić-Reich-Rus type contraction if there exist $\nu, \eta \in [0, 1)$ with $\nu + \eta < 1$ such that

$$\Psi(d(Sx, Sy)) \leq \Phi\left(d(x, y)^{\nu} d(x, Sx)^{\eta} d(y, Sy)^{1-\eta-\nu}\right), \tag{11}$$

for all $(x, y) \in \Lambda$, and $\min\{d(Sx, Sy), d(x, y), d(y, Sy), d(x, Sx)\} > 0$.

Remark 3.14. *For the particular definitions of the mappings Ψ, Φ, we have the following observations which show the generality of* (Ψ, Φ)*-orthogonal interpolative Ćirić-Reich-Rus type contraction.*

1. *Defining $\Phi(x) = \Psi(x) - \tau$ for all $x \in (0, \infty)$ in (11), we have the orthogonal interpolative Ćirić-Reich-Rus type F-contractions.*
2. *Defining $\Phi(x) = \Psi(x) - \tau(x)$ for all $x \in (0, \infty)$ in (11), we have the orthogonal interpolative Ćirić-Reich-Rus type (τ, F_T)-contraction.*

3. *Defining* $\Psi(x) = x$ *and* $\Phi(y) = \lambda y$ *for all* $x, y \in (0, \infty)$ *in (11), then we obtain the contraction introduced in [3].*

4. *Defining* $\Phi(x) = \beta(x)x$ *and* $\Psi(x) = x$ *for all* $x > 0$ *and* $\beta : (0, \infty) \to (0, 1)$ *verifying* $\limsup_{x\to\xi+} \beta(x) < 1$ *for each* $\xi > 0$*, we obtain orthogonal interpolative Ćirić-Reich-Rus type Geraghty's contractions.*

5. *Letting* $\nu = 0$ *in (11), we obtain* (Ψ, Φ)*-orthogonal interpolative Kannan type contraction.*

The following two theorems state the conditions for the existence of fixed-point of (Ψ, Φ)-orthogonal interpolative Ćirić-Reich-Rus type contraction.

Theorem 3.15. *Let* $\perp$ *be a transitive orthogonal relation, then, every* $\perp$*-preserving self-mapping defined on a* $\perp$*-regular O-complete metric space* $(\mathcal{A}, \perp, d)$ *(in short, OCMS) verifying* (11) *and* (i)-(iv)*, admits a fixed point in* $\mathcal{A}$.

Proof. Proceeding as in the proof of Theorem 3.8, we have

$$\begin{aligned}
\Psi(d_n) &\leq \Psi(d(S(\hbar_{n-1}), S(\hbar_n))) \\
&\leq \Phi\left((d(\hbar_{n-1}, \hbar_n))^{\nu}\,(d(\hbar_{n-1}, S\hbar_{n-1}))^{\eta}\,(d(\hbar_n, S\hbar_n))^{1-\eta-\nu}\right) \\
&= \Phi\left((d_{n-1})^{\nu}\,(d_{n-1})^{\eta}\,(d_n)^{1-\eta-\nu}\right) \\
&= \Phi\left((d_{n-1})^{\nu+\eta}\,(d_n)^{1-\eta-\nu}\right) \\
&< \Psi\left((d_{n-1})^{\nu+\eta}\,(d_n)^{1-\eta-\nu}\right). \qquad (12)
\end{aligned}$$

By (12) and the first part of (ii), we have

$$(d_n)^{\nu+\eta} < (d_{n-1})^{\nu+\eta}, \text{for all n} \geq 1.$$

Continuing as in the proof of Theorem 3.8, we have $\hbar_n \to a^*$ as $n \to \infty$, and $\perp$-regularity of $(\mathcal{A}, \perp, d)$ implies $\hbar_n \perp a^*$ or $a^* \perp \hbar_n$. We claim that $d(a^*, S(a^*)) = 0$. Assume on contrary that $d(\hbar_{n+1}, S(a^*)) > 0$ for infinitely many values of n. By (11),

$$\begin{aligned}
\Psi(d(\hbar_{n+1}, S(a^*))) &\leq \Psi(d(S(\hbar_n), S(a^*))) \\
&\leq \Phi\left((d(\hbar_n, a^*))^{\nu}\,(d(\hbar_n, S\hbar_n))^{\eta}\,(d(a^*, Sa^*))^{1-\eta-\nu}\right) \\
&= \Phi\left((d(\hbar_n, a^*))^{\nu}\,(d_n)^{\eta}\,(d(a^*, Sa^*))^{1-\eta-\nu}\right) \\
&< \Psi\left((d(\hbar_n, a^*))^{\nu}\,(d_n)^{\eta}\,(d(a^*, Sa^*))^{1-\eta-\nu}\right).
\end{aligned}$$

Using (ii), we get

$$d(\hbar_{n+1}, S(a^*)) < (d(\hbar_n, a^*))^{\nu}\,(d_n)^{\eta}\,(d(a^*, Sa^*))^{1-\eta-\nu}.$$

Letting $n \to \infty$, we find $d(a^*, S(a^*)) < 0$. A contradiction, thus, $d(a^*, S(a^*)) = 0$. So, $a^* = S(a^*)$. □

The following example explains Theorem 3.15.

Example 3.16. *Let* $\mathcal{A} = [1, 7)$ *and define the relation* $\perp$ *on* $\mathcal{A}$ *by*

$$x \perp y \text{ if } xy \in \{x, y\}.$$

Then $\perp$ *is an orthogonal relation and so* $(\mathcal{A}, \perp)$ *is an O-set. Let d be the Euclidean metric on* $\mathcal{A}$*, then,* $(\mathcal{A}, d)$ *is an incomplete metric space but* $(\mathcal{A}, \perp, d)$ *is an O-complete metric space. Define* $\Psi, \Phi : \mathbb{R}^+ \to \mathbb{R}$ *by*

$$\Psi(x) = \begin{cases} x & \text{if } x \in \{4\}, \\ x + 7 & \text{if } x \in \mathbb{R}^+ - \{4\}, \end{cases} \qquad \Phi(x) = \begin{cases} \frac{x}{3} & \text{if } x \in \{4\}, \\ x + 5 & \text{if } x \in \mathbb{R}^+ - \{4\}. \end{cases}$$

Let $S : \mathcal{A} \to \mathcal{A}$ be defined by

$$S(x) = \begin{cases} 5 & \text{if } 1 \leq x < 2, \\ 1 & \text{if } 2 \leq x < 7. \end{cases}$$

Our calculations show that $d(Sx, Sy) = 4$, $d(x, y) = 1$, $d(x, Sx) = 4$ and $d(y, Sy) = 1$ if $x = 1$, $y = 2$ $(1 \perp 2)$. This information shows that

$$d(Sx, Sy) > \lambda d(x, y)^{\nu} d(x, Sx)^{\eta} d(y, Sy)^{1-\eta-\nu} \text{ for some } \lambda = \frac{1}{2}, \nu = 0.5, \eta = 0.4.$$

Thus, S is not an orthogonal interpolative Ćirić-Reich-Rus type contraction. However, S is a (Ψ, Φ)-orthogonal interpolative Ćirić-Reich-Rus type contraction. Indeed,

$$\Psi(d(Sx, Sy)) \leq \Phi d(x, y)^{\nu} d(x, Sx)^{\eta} d(y, Sy)^{1-\eta-\nu}.$$

We obtain the same conclusions for $x = 1$, $y = 3$ $(1 \perp 3)$; $x = 1$, $y = 4$ $(1 \perp 4)$; $x = 1$, $y = 5$ $(1 \perp 5)$; $x = 1$, $y = 6$ $(1 \perp 6)$. The mappings Ψ, Φ satisfy other conditions of Theorem 3.15. The point $x = 2.5$ is a fixed point of the mapping S.

Theorem 3.17. *Let $\perp$ be a transitive orthogonal relation, then, every $\perp$-preserving self-mapping defined on a $\perp$-regular O-complete metric space $(\mathcal{A}, \perp, d)$ (in short, OCMS) verifying* (11) *and* (i),(iv)-(viii), *admits a fixed point in $\mathcal{A}$.*

Proof. Proceeding as in the proof of Theorem 3.9 and following the arguments used in the proof of Theorem 3.15, we have the required result. □

3.4. (Ψ, Φ)-orthogonal interpolative Hardy-Rogers type contraction

Let $(\mathcal{A}, \perp, d)$ be an OMS and $\Psi, \Phi : (0, \infty) \to (-\infty, \infty)$. The mapping $S : \mathcal{A} \to \mathcal{A}$ is said to be a (Ψ, Φ)-orthogonal interpolative Hardy-Rogers type contraction if there exist $\nu, \eta, \gamma \in [0, 1)$ with $\nu + \eta + \gamma < 1$ such that

$$\begin{aligned} &\Psi(d(Sx, Sy)) \\ \leq\ & \Phi\left(d(x, y)^{\nu} d(x, Sx)^{\eta} d(y, Sy)^{\gamma} \left(\frac{1}{2}(d(x, Sy) + d(y, Sx))\right)^{1-\eta-\nu-\gamma}\right), \end{aligned} \tag{13}$$

for all $(x, y) \in \Lambda$, and $\min\{d(Sx, Sy), d(x, y), d(y, Sy), d(x, Sx), d(x, Sy)\} > 0$.

Remark 3.18. *For the particular definitions of the mappings Ψ, Φ, we have the following observations which show the generality of (Ψ, Φ)-orthogonal interpolative Hardy-Rogers type contraction.*

1. *Defining $\Phi(x) = \Psi(x) - \tau$ for all $x \in (0, \infty)$ in (13) we have orthogonal interpolative Hardy-Rogers type F-contractions.*
2. *Defining $\Phi(x) = \Psi(x) - \tau(x)$ for all $x \in (0, \infty)$ in (13), we have orthogonal interpolative Hardy-Rogers type (τ, F_T)-contraction.*
3. *Defining $\Psi(x) = x$ and $\Phi(y) = \lambda y$ for all $x, y \in (0, \infty)$ in (13), then we obtain the contraction introduced in [29].*

The following two theorems state the conditions for the existence of fixed-point of (Ψ, Φ)-orthogonal interpolative Hardy-Rogers type contraction.

Theorem 3.19. *Let $\perp$ be a transitive orthogonal relation, then, every $\perp$-preserving self-mapping defined on a $\perp$-regular O-complete metric space $(\mathcal{A}, \perp, d)$ (in short, OCMS) verifying* (13) *and* (i)-(iv), *admits a fixed point in $\mathcal{A}$.*

Proof. Proceeding as in the proof of Theorem 3.8, we have

$$\begin{aligned}
\Psi(d_n) &\leq \Psi(d(S(\hbar_{n-1}), S(\hbar_n))) \\
&\leq \Phi\left(\begin{array}{l} d(\hbar_{n-1}, \hbar_n)^{\nu} d(\hbar_{n-1}, S(\hbar_{n-1}))^{\eta} d(\hbar_n, S(\hbar_n))^{\gamma} \\ \left(\frac{1}{2}(d(\hbar_{n-1}, S(\hbar_n)) + d(\hbar_n, S(\hbar_{n-1})))\right)^{1-\eta-\nu-\gamma} \end{array} \right) \\
&= \Phi\left((d_{n-1})^{\nu+\eta} (d_n)^{\gamma} \left(\frac{1}{2} d(\hbar_{n-1}, \hbar_{n+1})\right)^{1-\eta-\nu-\gamma} \right) \\
&< \Psi\left((d_{n-1})^{\nu+\eta} (d_n)^{\gamma} \left(\frac{1}{2} (d_{n-1} + d_n)\right)^{1-\eta-\nu-\gamma} \right). \qquad (14)
\end{aligned}$$

Suppose that $d_{n-1} < d_n$ for some $n \geq 1$. Since Ψ is non-decreasing, by (14), we have $(d_n)^{\nu+\eta} < (d_n)^{\nu+\eta}$. This is not possible. Consequently, we have $d_n < d_{n-1}$ for all $n \geq 1$. Continuing as in the proof of Theorem 3.8, we have $\hbar_n \to a^*$ as $n \to \infty$, and $\perp$-regularity of the space $(\mathcal{A}, \perp, d)$ implies $\hbar_n \perp a^*$ or $a^* \perp \hbar_n$. We claim that $d(a^*, S(a^*)) = 0$. Assume on contrary that $d(\hbar_{n+1}, S(a^*)) > 0$ for infinitely many values of n. By (13),

$$\begin{aligned}
&\Psi(d(\hbar_{n+1}, S(a^*))) \leq \Psi(d(S(\hbar_n), S(a^*))) \\
\leq\ & \Phi\left(d(\hbar_n, a^*)^{\nu} d(\hbar_n, S(\hbar_n))^{\eta} d(a^*, S(a^*))^{\gamma} \left(\frac{1}{2}(d(\hbar_n, S(a^*)) + d(a^*, S(\hbar_n)))\right)^{1-\eta-\nu-\gamma} \right) \\
=\ & \Phi\left(d(\hbar_n, a^*)^{\nu} (d_n)^{\eta} d(a^*, S(a^*))^{\gamma} \left(\frac{1}{2}(d(\hbar_n, S(a^*)) + d(a^*, \hbar_{n+1}))\right)^{1-\eta-\nu-\gamma} \right) \\
<\ & \Psi\left(d(\hbar_n, a^*)^{\nu} (d_n)^{\eta} d(a^*, S(a^*))^{\gamma} \left(\frac{1}{2}(d(\hbar_n, S(a^*)) + d(a^*, \hbar_{n+1}))\right)^{1-\eta-\nu-\gamma} \right).
\end{aligned}$$

By (ii), we get

$$d(\hbar_{n+1}, S(a^*)) < d(\hbar_n, a^*)^{\nu} d(\hbar_n, \hbar_{n+1})^{\eta} d(a^*, S(a^*))^{\gamma} \left(\frac{1}{2}(d(\hbar_n, S(a^*)) + d(a^*, \hbar_{n+1}))\right)^{1-\eta-\nu-\gamma}.$$

Applying limit $n \to \infty$ on both sides of the last inequality, we have $d(a^*, S(a^*)) \leq 0$. This implies that $d(a^*, S(a^*)) = 0$. Hence, we get $a^* = S(a^*)$. □

Theorem 3.20. *Let $\perp$ be a transitive orthogonal relation, then, every $\perp$-preserving self-mapping defined on a $\perp$-regular O-complete metric space $(\mathcal{A}, \perp, d)$ (in short, OCMS) verifying* (13) *and* (i),(iv)-(viii), *admits a fixed point in $\mathcal{A}$.*

Proof. Proceeding as in the proof of Theorem 3.9 and following the arguments used in the proof of Theorem 3.19, we have the required result. □

Definition 3.21. *Let $(\mathcal{A}, \perp, d)$ be an OMS. A mapping $S : \mathcal{A} \to \mathcal{A}$ is said to be a (Ψ, Φ)-orthogonal interpolative Banach contraction if there exist $\nu \in (0, 1]$ such that*

$$\Psi(d(Sx, Sy)) \leq \Phi\left(d(x, y)^{\nu}\right), \qquad (15)$$

for all $(x, y) \in \Lambda$, and $\min\{d(Sx, Sy), d(x, y)\} > 0$.

The following two theorems state the conditions for the existence of fixed-point of (Ψ, Φ)-orthogonal interpolative Banach contraction.

Theorem 3.22. *Let $\perp$ be a transitive orthogonal relation, then, every $\perp$-preserving self-mapping defined on a $\perp$-regular O-complete metric space $(\mathcal{A}, \perp, d)$ (in short, OCMS) verifying* (15) *and* (i)-(iv), *admits a fixed point in $\mathcal{A}$.*

Theorem 3.23. *Let $\perp$ be a transitive orthogonal relation, then, every $\perp$-preserving self-mapping defined on a $\perp$-regular OCMS $(\mathcal{A}, \perp, d)$ verifying* (15) *with $\Phi(\jmath) < \Psi(\jmath)$ for each $\jmath > 0$ and* (i),(iv),(vi)-(viii), *admits a fixed point in $\mathcal{A}$.*

Remark 3.24. *If $\nu = 1$, the proofs of Theorem 3.22 and Theorem 3.23 are directly followed by [41]. If $0 < \nu < 1$, the proofs of Theorem 3.22 and Theorem 3.23 are similar to precedent ones.*

Remark 3.25. *For $\Phi(y) = \Psi(y) - \tau$ and orthogonal relation $\perp$ as follows:*

$$x \perp y \text{ if and only if } \alpha(x, y) \geq 1.$$

The Theorem 3.22 with $\nu = 1$ reduces to the main result presented in [18].
For $\Phi(y) = \Psi(y) = y$, $S : \mathcal{A} \to \mathcal{A}$ and orthogonal relation $\perp$ as follows:

$$x \perp y \text{ if and only if } x \leq y.$$

The Theorem 3.22 with $\nu = 1$ reduces to the main result presented in [46].

4. Consequences

It is noted that for $\Psi(y) = y$ for all $y > 0$, the Theorem 3.19 improves and generalizes the multivalued version of the interpolative Boyd-Wong fixed point theorem [35]. By defining $\Phi(y) = \Psi(y) - \tau$ in Theorem 3.22 and Theorem 3.23, we obtain the interpolative versions of the fixed point theorems established in [12, 18] and the main results of Secelean [49] and Lukacs and Kajanto [38] as follow:

Corollary 4.1. *Let $(\mathcal{A}, d)$ be a complete metric space. Let $S : \mathcal{A} \to \mathcal{A}$ be a mapping so that*

$$\Psi(d(Sx, Sy)) \leq \Psi(d(x, y))) - \tau \quad \forall\, x, y \in \mathcal{A}, \text{ provided } d(Sx, Sy) > 0,$$

where $\Psi : (0, \infty) \to \mathbb{R}$ is non-decreasing and $\tau > 0$. Then there exists a fixed point of S in $\mathcal{A}$.

By defining $\Phi(y) = \Psi(y) - \tau(y)$ in Theorem 3.22 and Theorem 3.23, we get an improvement of the fixed point theorem [51] as follow:

Corollary 4.2. *Let $(\mathcal{A}, \perp, d)$ be an $\perp$-regular OCMS. Let $S : \mathcal{A} \to \mathcal{A}$ be an $\perp$-preserving mapping so that*

$$\tau(d(x, y)) + \Psi(d(Sx, Sy)) \leq \Psi(d(x, y))) \quad \text{for all } (x, y) \in \Lambda\,, \text{ provided } d(Sx, Sy) > 0,$$

where $\Psi : (0, \infty) \to \mathbb{R}$ is non-decreasing and $\liminf_{a \to t+} \tau(a) > 0$, $\forall\, t \geq 0$,. If for each $\hbar_0 \in \mathcal{A}$, there is $\hbar_1 = S(\hbar_0)$ so that $\hbar_0 \perp \hbar_1$ or $\hbar_1 \perp \hbar_0$. Then S has a fixed point in $\mathcal{A}$.

Letting Ψ to be a lower semicontinuous function and Φ to be an upper semicontinuous function, Theorem 3.9 is an extension of the Amini-Harandi-Petrusel fixed point theorem [6]. By defining $\Phi(y) = h(\Psi(y))$ in Theorem 3.8, we get the following improvement and generalization of Moradi theorem [40] as follow:

Corollary 4.3. *Let $(\mathcal{A}, \perp, d)$ be an $\perp$-regular OCMS. Let $S : \mathcal{A} \to \mathcal{A}$ be an $\perp$-preserving mapping so that*

$$\Psi(d(Sx, Sy)) \leq h(\Psi(d(x, y))) \quad \text{for all } (x, y) \in \Lambda\,, \text{ provided } d(Sx, Sy) > 0,$$

where

(i) $h : I \to [0, \infty)$ *is an upper semi-continuous function with $h(y) < y$ for all $y \in I \subset \mathbb{R}$;*

(ii) $\Psi : (0, \infty) \to I$ *is non-decreasing.*

Assume that for each $\hbar_0 \in \mathcal{A}$ there is $\hbar_1 = S(\hbar_0)$ such that $\hbar_0 \perp \hbar_1$ or $\hbar_1 \perp \hbar_0$. Then S has a unique fixed point in $\mathcal{A}$.

Defining $h(y) = y^{\delta}; \delta \in (0,1)$ in Corollary 4.3, we have the next result.

Corollary 4.4. *Let $(\mathcal{A}, \perp, d)$ be an $\perp$-regular and OCMS. Let $S : \mathcal{A} \to \mathcal{A}$ be an $\perp$-preserving mapping so that*

$$\Psi(d(Sx, Sy)) \leq (\Psi(d(x,y)))^r \quad \text{for all } (x,y) \in \Lambda\,, \text{ provided } d(Sx,Sy) > 0,$$

where, $\Psi : (0,\infty) \to (0,1)$ is a non-decreasing function. Assume that for each $\hbar_0 \in \mathcal{A}$ there is $\hbar_1 = S(\hbar_0)$ such that $\hbar_0 \perp \hbar_1$ or $\hbar_1 \perp \hbar_0$. Then S has a fixed point in $\mathcal{A}$.

Corollary 4.4 is an improvement of Jleli-Samet fixed point theorem [24] and the results of Li and Jiang [37] and Ahmad *et al.* [4].

An improvement of Skof fixed point theorem [50] may be stated by defining $\Phi(y) = \lambda\Psi(y)$ in Theorem 3.22 and Theorem 3.23.

Corollary 4.5. *Let $(\mathcal{A}, \perp, d)$ be an $\perp$-regular OCMS. Let $S : \mathcal{A} \to \mathcal{A}$ be an $\perp$-preserving mapping so that*

$$\Psi(d(Sx, Sy)) \leq \lambda\Psi(d(x,y)) \quad \text{for all } (x,y) \in \Lambda\,, \text{ provided } d(Sx,Sy) > 0,$$

where $\Psi : (0,\infty) \to (0,\infty)$ is a non-decreasing function and $\lambda \in (0,1)$. Assume that for each $\hbar_0 \in \mathcal{A}$ there is $\hbar_1 = S(\hbar_0)$ so that $\hbar_0 \perp \hbar_1$ or $\hbar_1 \perp \hbar_0$. Then S has a unique fixed point in $\mathcal{A}$.

For a non-decreasing $\Psi : (0,\infty) \to (0,\infty)$ and $\beta : (0,\infty) \to (0,1)$ verifying $\limsup_{y\to\xi+} \beta(y) < 1$ for each $\xi > 0$, and defining $\Phi(y) = \beta(y)\Psi(y)$, $\Psi(y) = y$ for all $y > 0$ in Theorem 3.22, an improvement of the Geraghty fixed point theorem [21] is obtained.

5. An application to resolve a fractional differential equation

Lacroix (1819) proposed and studied a number of useful fractional differential properties. In 2015, Caputo and Fabrizio introduced a new fractional approach [16]. The interest for this definition was due to the necessity to describe a class of non-local systems, which cannot be well described by classical local theories or by fractional models with singular kernel [16]. The fundamental differences among the fractional derivatives are their different kernels which can be selected to meet the requirements of different applications. For example, the main differences between the Caputo fractional derivative [17], the Caputo-Fabrizio derivative [16], and the Atangana-Baleanu fractional derivative [7] are that the Caputo derivative is defined using a power law, the Caputo-Fabrizio derivative is defined using an exponential decay law, and the Atangana-Baleanu derivative is defined using a Mittag-Leffler law. Authors in [1, 2, 5, 10, 11] have recently studied a number of new Caputo-Fabrizio derivative (CFD) models. We will look at one of these models in metric spaces. For this reason, we add the following notations:

Let $I = [0, L]\,; L > 0$ and $C(I,\mathbb{R}) = \{u | u : I \to \mathbb{R}$ and u is continuous$\}$. Define the metric $d : C(I,\mathbb{R}) \times C(I,\mathbb{R}) \to [0,\infty)$ by

$$d(u,v) = \|u - v\|_\infty = \max_{l\in[0,L]} |u(l) - v(l)|\,, \text{ for all } u,v \in C(I,\mathbb{R}).$$

Then $(C(I,\mathbb{R}), d)$ is a complete metric space. Define an orthogonal relation $\perp$ on $C(I,\mathbb{R})$ by

$$u \perp v \text{ if and only if } u(l)v(l) \geq u(l) \vee v(l), \text{ for all } u,v \in C(I,\mathbb{R}).$$

Then $(C(I,\mathbb{R}), \perp, d)$ is an OCMS. Let $K_1 : I \times \mathbb{R} \to \mathbb{R}$ be a mapping such that $K_1(l,x) \geq 0$ for all $l \in I$ and $x \geq 0$. We will investigate the following CFDE:

$$^{C}D^{v} g\,(l) = K_1(l, g(l)); g \in C(I,\mathbb{R}), \tag{16}$$

with boundary conditions

$$g(0) = 0,\ Ig(1) = g^{'}(0).$$

Here, ${}^{C}D^{v}$ represents the CFD of order v given as

$${}^{C}D^{v}g(l) = \frac{1}{\Gamma(n-v)}\int_0^l (l-y)^{n-v-1}\, g(y))\,dy,$$

where

$$n-1 < v < n \text{ and } n = [v]+1,$$

and $I^{v}g$ is defined by

$$I^{v}g(l) = \frac{1}{\Gamma(v)}\int_0^l (l-y)^{v-1}\, g(y)\,dy, \text{ with } v > 0.$$

Then the equation (16) can be modified to

$$g(l) = \frac{1}{\Gamma(v)}\int_0^l (l-y)^{v-1} K_1(y, g(y))\,dy + \frac{2l}{\Gamma(v)}\int_0^L \int_0^y (y-z)^{v-1} K_1(z, g(z))\,dzdy.$$

The mapping $K_1 : I \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

(A) there exists $\tau > 0$ and $M = \min\{f(u,v)| \quad u, v \in C(I, \mathbb{R})\}$ so that,

$$\left|K_1(l,x) - K_1(l,y)\right| \le \frac{e^{-\tau}\Gamma(v+1)}{4M}\left|x-y\right|, \text{ for all } x, y \ge 0 \text{ with } xy \ge x \vee y.$$

(B) there exists $u_0 \in C(I, \mathbb{R})$ so that for any $l \in I$,

$$u_0(l) \le \frac{1}{\Gamma(v)}\int_0^l (l-y)^{v-1} K_1(y, u_0(y))\,dy + \frac{2l}{\Gamma(v)}\int_0^L \int_0^y (y-z)^{v-1} K_1(z, u_0(z))\,dzdy.$$

It is remarked that the mapping $K_1 : I \times \mathbb{R} \to \mathbb{R}$ is not necessarily Lipschitz from the given condition (A). For example, the function

$$K_1(l,x) = \begin{cases} lx & \text{if } x \le \frac{1}{2}, \\ 0 & \text{if } x > \frac{1}{2}. \end{cases}$$

verifies the condition (A) while K_1 is not continuous and monotone. Also, for $l = \frac{e^{-\tau}\Gamma(v+1)}{4M}$,

$$\left|K_1\left(l, \frac{1}{2}\right) - K_1\left(l, \frac{2}{3}\right)\right| = \frac{l}{2} > \frac{l}{6} = l\left|\frac{1}{2} - \frac{2}{3}\right|.$$

Theorem 5.1. *The equation (16) admits a solution in $C(I, \mathbb{R})$ if the conditions (A)-(B) are satisfied.*

Proof. Let $X = \{u \in C(I, \mathbb{R}) : u(l) \ge 0 \text{ for all } l \in I\}$ and define the mapping $R : X \to X$ in accordance with the above-mentioned notations by

$$R(q)(l) = \frac{1}{\Gamma(v)}\int_0^l (l-y)^{v-1} K_1(y, q(y))\,dy + \frac{2l}{\Gamma(v)}\int_0^L \int_0^y (y-z)^{v-1} K_1(z, q(z))\,dzdy.$$

We define an orthogonal relation $\perp$ in X by

$$u \perp v \text{ if and only if } u(l)v(l) \geq u(l) \vee v(l), \text{ for all } u, v \in X.$$

Obviously, R is $\perp$-preserving with respect to $\perp$. By (B), there is $u_0 \in C(I, \mathbb{R})$ such that $u_n = R^n(u_0))$ with $u_n \perp u_{n+1}$ or $u_{n+1} \perp u_n$ for all $n \geq 0$. We will check the contractive condition (15) of Theorem 3.22 in the next lines.

$$\left|R(q)(l) - R(u)(l)\right| = \left| \begin{array}{l} \frac{1}{\Gamma(v)} \int_0^l (l-y)^{v-1} K_1(y, q(y))\, dy \\ -\frac{1}{\Gamma(v)} \int_0^l (l-y)^{v-1} K_1(y, u(y))\, dy \\ +\frac{2l}{\Gamma(v)} \int_0^L \int_0^z (z-w)^{v-1} K_1(w, q(w))\, dw dz \\ -\frac{2l}{\Gamma(v)} \int_0^L \int_0^z (z-w)^{v-1} K_1(w, u(w))\, dw dz \end{array} \right|$$

implies

$$\begin{aligned} \left|R(q)(l) - R(u)(l)\right| &\leq \left| \int_0^l \left(\frac{1}{\Gamma(v)} (l-y)^{v-1} K_1(y, q(y)) - \frac{1}{\Gamma(v)} (l-y)^{v-1} K_1(y, u(y)) \right) dy \right| \\ &+ \left| \int_0^L \int_0^z \left(\frac{2}{\Gamma(v)} (z-w)^{v-1} K_1(w, q(w)) - \frac{2}{\Gamma(v)} (z-w)^{v-1} K_1(w, u(w)) \right) dw dz \right| \\ &\leq \frac{1}{\Gamma(v)} \frac{e^{-\tau}\Gamma(v+1)}{4M} \cdot \int_0^l (l-y)^{v-1} (q(y) - u(y))\, dy \\ &+ \frac{2}{\Gamma(v)} \frac{e^{-\tau}\Gamma(v+1)}{4M} \cdot \int_0^L \int_0^z (z-w)^{v-1} (u(w) - q(w))\, dw dz \\ &\leq \frac{1}{\Gamma(v)} \frac{e^{-\tau}\Gamma(v+1)}{4M} \cdot d(q,u) \cdot \int_0^l (l-y)^{v-1}\, dy \\ &+ \frac{2}{\Gamma(v)} \frac{e^{-\tau}\Gamma(v) \cdot \Gamma(v+1)}{4M\Gamma(v) \cdot \Gamma(v+1)} \cdot d(q,u) \cdot \int_0^L \int_0^z (z-w)^{v-1}\, dw dz \end{aligned}$$

$$\begin{aligned} &\leq \left(\frac{e^{-\tau}\Gamma(v) \cdot \Gamma(v+1)}{4M\Gamma(v) \cdot \Gamma(v+1)} \right) \cdot d(q,u) + 2e^{-\tau} B(v+1, 1) \frac{\Gamma(v) \cdot \Gamma(v+1)}{4M\Gamma(v) \cdot \Gamma(v+1)} d(q,u) \\ &\leq \frac{e^{-\tau}}{4M} d(q,u) + \frac{e^{-\tau}}{2M} d(q,u) < \frac{e^{-\tau}}{M} d(q,u), \end{aligned}$$

where B is the beta function. The final inequality is written as:

$$Md(R(q), R(u)) \leq d(R(q), R(u)) \leq e^{-\tau} d(q,u). \tag{17}$$

Define $\Psi(q(l)) = \ln(q(l))$ for all $q, u \in C(I, \mathbb{R}^+)$, then the inequality (17) can be written as

$$\tau + \Psi(d(R(q), R(u))) \leq \Psi(d(q,u)).$$

By Theorem 3.22, the equation (16) has a solution since the self-mapping R admits a fixed point. □

6. Conclusion and future work

The (Ψ, Φ)-orthogonal interpolative contractions generalize many well-known contractions. The presented theorems provide a general criterion for the existence of a unique fixed point of (Ψ, Φ)-orthogonal interpolative contraction. The research work done in this paper can be revisited to show the existence of PPF dependent fixed points of self-mappings (see [23] and references therein for details).

Declarations

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Not applicable.

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We would like to mention that this article does not contain any studies with animals and does not involve any studies over human being.

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