Boundedness of Hardy–Cesàro operators on variable exponent Morrey–Herz spaces

Kieu Huu Dung\(^a\), Do Lu Cong Minh\(^a\), Tran Thi Nang\(^a\)

\(^a\)Faculty of Fundamental Sciences, Van Lang University, Ho Chi Minh City, Vietnam

Abstract. In this paper, we give the necessary and sufficient conditions for the boundedness of Hardy–Cesàro operators on some weighted function spaces such as the weighted central Morrey, weighted local central Morrey, weighted non-local central Morrey, weighted Herz and weighted Morrey–Herz type spaces with variable exponent.

1. Introduction

Let \( f \) be a non-negative measurable function on \( \mathbb{R}^+ \), and one-dimensional Hardy operator be given by

\[
H(f)(x) = \frac{1}{x} \int_0^x f(t) dt, \quad x > 0.
\]

G. H. Hardy [16] obtained a famous integral inequality as follows.

\[
\|H\|_{L^q(\mathbb{R}^+)} \leq \frac{q}{q-1},
\]

where \( q \in (1, \infty) \) and \( \frac{q}{q-1} \) is the sharpest constant.

In 1984, C. Carton–Lebrun and M. Fosset [5] defined the weighted Hardy–Littlewood average operator \( U_\psi \):

\[
U_\psi(f) = \int_0^1 \psi(t)f(tx) dt, \quad x \in \mathbb{R}^n,
\]

where \( \psi : [0, 1] \to [0, \infty) \) is a measurable function and \( f \) is a measurable function on \( \mathbb{R}^n \). It is clear to see that if we choose \( n = 1 \) and \( \psi \equiv 1 \), the Hardy–Littlewood average operator \( U_\psi \) reduces to the classical Hardy operator \( H \). Next, J. Xiao [30] proved that \( U_\psi \) is bounded on \( L^p(\mathbb{R}^n) \) if and only if

\[
\mathcal{A}_{n,p,\psi} := \int_0^1 t^{-n/p} \psi(t) dt < \infty.
\]
Let $\Omega$ be a measurable function on $\mathbb{R}^n$. The Hardy–Cesàro operator is defined by

$$U_{\psi,s,d}(f)(x) = \int_{[0,1]^d} \psi(t)f(s(t)x)dt,$$

for a measurable complex-valued function $f$ on $\mathbb{R}^n$.

In case $d = 1$, the Hardy–Cesàro operator $U_{\psi,s,1}$ was researched by N. M. Chuong and H. D. Hung [10]. The authors gave the sufficient and necessary conditions for the boundedness of $U_{\psi,s,1}$ on the weighted Lebesgue and weighted BMO spaces. Note that, when we take $d = 1$ and $s(t) = t$, the Hardy–Cesàro operator $U_{\psi,s}$ becomes to the Hardy–Littlewood average operator $U_{\psi}$.

It is well-known that the Hardy–Cesàro operators and their commutators have attracted much more attention on real Euclidean spaces (see [8, 9, 14, 17, 25]).

The theory of function spaces with variable exponents is developed in the field of electronic fluid mechanics, elasticity, fluid dynamics, recovery of graphics, differential equations, harmonic analysis and partial differential equations (see [6], [12], [13], [18]). In particular, the maximal operators, the Calderón-Zygmund singular operators, the Kantorovich operators, the Hardy-type operators and their commutators have been extensively studied on the Lebesgue, Herz, Morrey, and Morrey-Herz spaces with variable exponent (see, e.g., [2], [3], [4], [7], [11], [15], [19], [21], [22], [23], [24], [26], [27], [28], [29] and others).

Motivated by above mentioned results, the goal of this paper is to establish the necessary and sufficient conditions for the boundedness of $U_{\psi,s,d}$ on the weighted central Morrey, weighted local central Morrey, weighted non-local central Morrey, weighted Herz and weighted Morrey-Herz spaces with variable exponent. In each case, the estimates for operator norms are also discussed.

Our paper is organized as follows. In Section 2, we give the necessary preliminaries on weighted Lebesgue spaces, central Morre spaces, Herz spaces and Morrey-Herz spaces with variable exponent. Our main theorems are given and proved in Section 3.

2. Preliminaries

Let us give some basic facts and notations which will be used throughout this paper. The letter $C$ denotes a positive constant which is independent of the main parameters, but may be different from line to line. For any $a \in \mathbb{R}^n$ and $r > 0$, let $B(a, r)$ denote the ball centered at $a$ with radius $r$. With a given measurable set $\Omega$, let $\chi_{\Omega}$ be a characteristic function, $\chi_{k} = \chi_{C_{k} \setminus C_{k-1}}$ and $C_{k} = B_{k} \setminus B_{k-1}$ and $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, for all $k \in \mathbb{Z}$. Next, we write $a \leq b$ to mean that there is a positive constant $C$, independent of the main parameters, such that $a \leq Cb$. The symbol $f \asymp g$ means that $f$ is equivalent to $g$ (i.e. $C^{-1} f \leq g \leq Cf$). In this paper, as usual, we will let $\omega(\cdot)$ represent a non-negative weighted function on $\mathbb{R}^n$.

Now, we give some notations and definitions of Lebesgue, Herz and Morrey-Herz spaces with constant parameters (see [20]).

**Definition 2.1.** Let $1 \leq p < \infty$, we define the weighted Lebesgue space $L^p(\omega)$ of a measurable function $f$ by

$$\|f\|_{L^p(\omega)} = \left( \int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx \right)^{\frac{1}{p}} < \infty.$$
Definition 2.2. Let \( \alpha \in \mathbb{R}, 0 < q < \infty \), and \( 0 < p < \infty \). The weighted homogeneous Herz-type space \( K^{\alpha,p}_q(\omega) \) is defined by
\[
K^{\alpha,p}_q(\omega) = \left\{ f \in L^q_{\text{loc}}(\mathbb{R}^n \setminus \{0\}, \omega) : \| f \|_{K^{\alpha,p}_q(\omega)} < \infty \right\},
\]
where \( \| f \|_{K^{\alpha,p}_q(\omega)} = \left( \sum_{k=\omega}^{\infty} 2^{k\alpha q}\| f \chi_k \|_{L^p(\omega)}^p \right)^{\frac{1}{p}} \).

Definition 2.3. Let \( \alpha \in \mathbb{R}, 0 < p < \infty, 0 < q < \infty, \lambda \geq 0 \). The homogeneous weighted Morrey-Herz type space \( M K^{\alpha,p}_q(\omega) \) is defined by
\[
M K^{\alpha,p}_q(\omega) = \left\{ f \in L^q_{\text{loc}}(\mathbb{R}^n \setminus \{0\}, \omega) : \| f \|_{M K^{\alpha,p}_q(\omega)} < \infty \right\},
\]
where \( \| f \|_{M K^{\alpha,p}_q(\omega)} = \sup_{k, k \in \mathbb{Z}} 2^{-k\lambda} \left( \sum_{k=\omega}^{\infty} 2^{k\alpha q}\| f \chi_k \|_{L^p(\omega)}^p \right)^{\frac{1}{p}} \).

Remark 1. It is helpful to note that \( K^0_p(\mathbb{R}^n) = L^p(\mathbb{R}^n) \) for \( 0 < p < \infty \); \( K^{\alpha,p}_0(\mathbb{R}^n) = L^p(|x|^\alpha dx) \) for all \( 0 < p < \infty \) and \( \alpha \in \mathbb{R} \). Since \( M K^{\alpha,0}_p(\mathbb{R}^n) = K^{0,p}_\alpha(\mathbb{R}^n) \), it follows that the Herz space is a special case of Morrey-Herz space.

Let us present the definition of the Lebesgue space with variable exponent. The reader may find in the works [6], [12] and [13].

Definition 2.4. Let \( \mathcal{P}_b(\mathbb{R}^n) \) be the set of all measurable functions \( p(\cdot): \mathbb{R} \to [1, \infty) \) such that
\[
1 < p_- \leq p(x) \leq p_+ < \infty, \text{ for all } x \in \mathbb{R}^n,
\]
where \( p_- = \text{ess inf}_{x \in \mathbb{R}^n} p(x) \) and \( p_+ = \text{ess sup}_{x \in \mathbb{R}^n} p(x) \). For \( p(\cdot) \in \mathcal{P}_b(\mathbb{R}^n) \), the variable exponent Lebesgue space \( L^{p(\cdot)}(\mathbb{R}^n) \) is the set of all complex-valued measurable functions \( f \) defined on \( \mathbb{R}^n \) such that there exists constant \( \eta > 0 \) satisfying
\[
F_p(f / \eta) = \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\eta} \right)^{p(x)} dx < \infty.
\]

The variable exponent Lebesgue space \( L^{p(\cdot)}(\mathbb{R}^n) \) then becomes a normed space equipped with a norm given by
\[
\| f \|_{L^{p(\cdot)}} = \inf \left\{ \eta > 0 : F_p \left( \frac{f}{\eta} \right) \leq 1 \right\}.
\]

For \( p \in \mathcal{P}_b(\mathbb{R}^n) \), it is useful to remark that we have the following inequalities which are usually used in the sequel.

\[
\begin{align*}
[i] & \text{ If } F_p(f) \leq C, \text{ then } \| f \|_{L^{p(\cdot)}} \leq \max \{ C^{\frac{1}{p_+}}, C^{\frac{1}{p_-}} \}, \text{ for all } f \in L^{p(\cdot)}(\mathbb{R}^n), \\
[ii] & \text{ If } F_p(f) \geq C, \text{ then } \| f \|_{L^{p(\cdot)}} \geq \min \{ C^{\frac{1}{p_+}}, C^{\frac{1}{p_-}} \}, \text{ for all } f \in L^{p(\cdot)}(\mathbb{R}^n). \tag{1}
\end{align*}
\]

The space \( \mathcal{P}_{\infty}(\mathbb{R}^n) \) is defined by the set of all measurable functions \( p(\cdot) \in \mathcal{P}_b(\mathbb{R}^n) \) and there exists a constant \( p_\infty \) such that
\[
p_\infty = \lim_{|x| \to \infty} p(x).
\]

For \( p(\cdot) \in \mathcal{P}_b(\mathbb{R}^n) \), the weighted variable exponent Lebesgue space \( L^{p(\cdot)}_{\omega,\text{loc}}(\mathbb{R}^n) \) is the set of all complex-valued measurable functions \( f \) such that \( f \omega \) belongs to the \( L^{p(\cdot)}(\mathbb{R}^n) \) space, and the norm of \( f \) is given by \( \| f \|_{L^{p(\cdot)}_{\omega,\text{loc}}(\mathbb{R}^n \setminus \{0\})} = \| f \omega \|_{L^{p(\cdot)}(\mathbb{R}^n \setminus \{0\})} \). The set \( L^{p(\cdot)}_{\omega,\text{loc}}(\mathbb{R}^n \setminus \{0\}) \) consists of all measurable functions \( f \) on \( \mathbb{R}^n \setminus \{0\} \) satisfying
exponent weighted Morrey-Herz spaces

Let $C_0^{log}(\mathbb{R}^n)$ denote the set of all log-Hölder continuous functions $\alpha(\cdot)$ satisfying at the origin

$$|\alpha(x) - \alpha(0)| \leq \frac{C_0^\alpha}{\log(e + |x|)}, \text{ for all } x \in \mathbb{R}^n.$$ 

Denote by $C_{\infty}^{log}(\mathbb{R}^n)$ the set of all log-Hölder continuous functions $\alpha(\cdot)$ satisfying at infinity

$$|\alpha(x) - \alpha_\infty| \leq \frac{C_\infty^\alpha}{\log(e + |x|)}, \text{ for all } x \in \mathbb{R}^n.$$ 

Next, we would like to give the definition of variable exponent weighted Herz spaces $K_{p,q;\alpha}(\mathbb{R}^n)$, variable exponent weighted Morrey-Herz spaces $M_{K_{p,q;\alpha}}(\mathbb{R}^n)$ (see [21], [27] for more details) and variable exponent weighted central Morrey spaces $M_{\lambda}^{\alpha, \Lambda}(\mathbb{R}^n)$.

**Definition 2.5.** Let $0 < p < \infty, q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$ and $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. The variable exponent weighted Herz space $K_{p,q;\alpha}(\mathbb{R}^n)$ is defined by

$$K_{p,q;\alpha}(\mathbb{R}^n) = \left\{ f \in L^{q(\cdot)}_{\alpha, loc}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{K_{p,q;\alpha}} < \infty \right\},$$

with

$$\|f\|_{K_{p,q;\alpha}} = \left( \sum_{k=-\infty}^{\infty} \|2^{k\alpha} f \chi_k\|_{L^{p(\cdot)}_{\alpha, loc}}^p \right)^{1/p}.$$ 

**Definition 2.6.** Assume that $0 \leq \lambda < \infty, 0 < p < \infty, q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$ and $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. The variable exponent weighted Morrey-Herz space $M_{K_{p,q;\alpha}}(\mathbb{R}^n)$ is defined by

$$M_{K_{p,q;\alpha}}(\mathbb{R}^n) = \left\{ f \in L^{q(\cdot)}_{\alpha, loc}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M_{K_{p,q;\alpha}}} < \infty \right\},$$

where

$$\|f\|_{M_{K_{p,q;\alpha}}} = \sup_{k \in \mathbb{Z}} 2^{-k\lambda} \left( \sum_{k=-\infty}^{\infty} \|2^{k\alpha} f \chi_k\|_{L^{p(\cdot)}_{\alpha, loc}}^p \right)^{1/p}.$$ 

Note that, when $p(\cdot), q(\cdot)$ and $\alpha(\cdot)$ are constant, it is obvious to see that

$$L^{p(\cdot)}_{\alpha, loc} = L^p(\alpha), \quad K_{q,\alpha;\Lambda}^{\alpha, p} = K_{q,\alpha;\Lambda}^{-\alpha, p} \quad \text{and} \quad M^{\alpha, \Lambda}_{p,q;\alpha} = M^{\alpha, \Lambda}_{p,q;\alpha} = M^{\alpha, \Lambda}_{p,q;\alpha}.$$ 

**Theorem 2.7 (Proposition 3.8 in [1]).** If $p \in (0, \infty)$, $q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$ and $\alpha \in L^\infty(\mathbb{R}^n) \cap C_0^{log}(\mathbb{R}^n) \cap C_0^{log}(\mathbb{R}^n)$, then we have

$$\|f\|_{K_{p,q;\alpha}} \approx \left( \sum_{k=-\infty}^{\infty} 2^{k\alpha(0)p} \|f \chi_k\|_{L^{p(\cdot)}_{\alpha, loc}}^p \right)^{1/p} + \left( \sum_{k=0}^{\infty} 2^{k\alpha\lambda} \|f \chi_k\|_{L^{p(\cdot)}_{\alpha, loc}}^p \right)^{1/p}.$$ 

**Theorem 2.8 (Proposition 2.5 in [21]).** If $\lambda \in (0, \infty), p \in (0, \infty)$, $q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$ and $\alpha \in L^\infty(\mathbb{R}^n) \cap C_0^{log}(\mathbb{R}^n) \cap C_0^{log}(\mathbb{R}^n)$, then we obtain

$$\|f\|_{M_{K_{p,q;\alpha}}} \approx \max \left\{ \sup_{k_0 \leq 0} M_{1,k_0}, \sup_{k_0 > 0} M_{2,k_0} + M_{3,k_0} \right\},$$
Here

\[ M_{1,\alpha} = 2^{-k/\lambda} \left( \sum_{k=0}^{\infty} 2^{k(0)\alpha} \|f \chi_k\|_{L^p_{\xi}} \right)^{1/p}, \quad M_{2,\alpha} = 2^{-k/\lambda} \left( \sum_{k=0}^{\infty} 2^{k(0)\alpha} \|f \chi_k\|_{L^p_{\xi}} \right)^{1/p}, \]

\[ M_{3,\alpha} = 2^{-k/\lambda} \left( \sum_{k=0}^{\infty} 2^{k(0)\alpha} \|f \chi_k\|_{L^p_{\xi}} \right)^{1/p}. \]

From the definition of weighted Morrey-Heisenberg spaces with variable exponent and Proposition 2.5 in [21], we have the following result.

**Lemma 2.9.** Let \( \alpha(x) \in L^{\infty}(\mathbb{R}^n) \), \( q(x) \in \mathcal{P}(\mathbb{R}^n) \), \( p \in (0, \infty) \) and \( \lambda \in (0, \infty) \). If \( \alpha(x) \) is log-Hölder continuous both at the origin and at infinity, then

\[
\|f\chi\|_{L^p_{\xi}} \leq C 2^{j(\lambda-\alpha(0))} \|f\|_{M_{\alpha,\lambda}^{c,\mu}(\xi)}, \quad \text{for all } j \in \mathbb{Z};
\]

\[
\|f\chi\|_{L^p_{\xi}} \leq C 2^{j(\lambda-\alpha(\infty))} \|f\|_{M_{\alpha,\lambda}^{c,\mu}(\xi)}, \quad \text{for all } j \in \mathbb{N}.
\]

**Proof.** The proof of this lemma is found in [27]. □

**Definition 2.10.** Assume that \( \kappa > 0 \), \( q(x) \in \mathcal{P}(\mathbb{R}^n) \). The variable exponent weighted central Morrey space \( M_{\alpha}^{c,\mu}(\xi) \) is defined by

\[
M_{\alpha}^{c,\mu}(\xi) = \left\{ f \in L^{q}_{\alpha,\lambda}(\mathbb{R}^n) : \|f\|_{M_{\alpha}^{c,\mu}(\xi)} < \infty \right\},
\]

where \( \|f\|_{M_{\alpha}^{c,\mu}(\xi)} = \sup_{k \in \mathbb{Z}} \frac{1}{\alpha(B_k)} \|f\|_{L^q_{\alpha,\lambda}(B_k)} \).

**Definition 2.11.** Let \( \kappa > 0 \), \( q(x) \in \mathcal{P}(\mathbb{R}^n) \). The variable exponent weighted local central Morrey space \( B_{\alpha,\lambda}^{c,\mu}(\xi) \) is defined by

\[
B_{\alpha,\lambda}^{c,\mu}(\xi) = \left\{ f \in L^{q}_{\alpha,\lambda}(\mathbb{R}^n) : \|f\|_{B_{\alpha,\lambda}^{c,\mu}(\xi)} < \infty \right\},
\]

where \( \|f\|_{B_{\alpha,\lambda}^{c,\mu}(\xi)} = \sup_{0 < r < 1} \frac{1}{\alpha(B(0,r))} \|f\|_{L^q_{\alpha,\lambda}(B(0,r))} \).

**Definition 2.12.** Let \( \kappa > 0 \), \( q(x) \in \mathcal{P}(\mathbb{R}^n) \). The variable exponent weighted non-local central Morrey space \( B_{\alpha,\lambda}^{c,\mu}(\xi) \) is defined by

\[
B_{\alpha,\lambda}^{c,\mu}(\xi) = \left\{ f \in L^{q}_{\alpha,\lambda}(\mathbb{R}^n) : \|f\|_{B_{\alpha,\lambda}^{c,\mu}(\xi)} < \infty \right\},
\]

where \( \|f\|_{B_{\alpha,\lambda}^{c,\mu}(\xi)} = \sup_{r \geq 1} \frac{1}{\alpha(B(0,r))} \|f\|_{L^q_{\alpha,\lambda}(B(0,r))} \).

3. Main results and their proofs

Let us introduce some notations which will be used throughout this section. Assume that \( p \in (1, \infty) \), \( q \in \mathcal{P}(\mathbb{R}^n) \), \( \alpha \in L^{\infty}(\mathbb{R}^n) \cap C^0_{\log}(\mathbb{R}^n) \cap C^0_{\log}(\mathbb{R}^n) \) and \( \omega \) be a weighted function. For simplicity of notation, we put

\[
\theta_{\sup}(t) = \esssup_{x \in \mathbb{R}^n} \frac{\omega(s^{-1}(t)z)}{\omega(z)}, \quad \text{and} \quad \theta_{\inf}(t) = \essinf_{x \in \mathbb{R}^n} \frac{\omega(s^{-1}(t)z)}{\omega(z)}.
\]

\[
\mathcal{K}_{\theta_{\sup}}(t) = \theta_{\sup}(t) \max\left\{ |s(t)|^{\frac{1}{\mu}}, |s(t)|^{\frac{1}{\mu}} \right\} \quad \text{and} \quad \mathcal{K}_{\theta_{\inf}}(t) = \theta_{\inf}(t) \min\left\{ |s(t)|^{\frac{1}{\mu}}, |s(t)|^{\frac{1}{\mu}} \right\}.
\]

Now, we are ready to state our first main result in this paper.
Theorem 3.1. Let $q(s^{-1}(t)) = q(\cdot)$ for almost everywhere $t \in \text{supp}(\psi)$, $\lambda > 0$ and $\alpha(0) - \alpha_\infty \geq 0$.

(i) If

$$C_{1,\sup} = \int_{[0,1]^t} \psi(t) \mathcal{K}_{s,0,\omega}(t) \max \left\{ |s(t)|^{\lambda-\alpha(0)}, |s(t)|^{\lambda-\alpha_\infty} \right\} dt < \infty,$$

then $U_{\varphi,s,d}$ is bounded from $MK^{(\lambda),\lambda}_{p,q,\omega}$ to itself.

(ii) Suppose that $U_{\varphi,s,d}$ is bounded from $MK^{(\lambda),\lambda}_{p,q,\omega}$ to itself, and either $\lambda = \alpha(0)$ or $q_+ = q_-$. We have

$$C_{1,\inf} = \int_{[0,1]^t} \psi(t) \mathcal{K}_{s,0,\omega}(t) |s(t)|^{\lambda-\alpha(0)} dt < \infty.$$

Moreover,

$$C_{1,\inf} \leq \|U_{\varphi,s,d}\|_{MK^{(\lambda),\lambda}_{p,q,\omega}}.$$

Proof. Now, we will prove for the case (i). By the Minkowski inequality, we have

$$\|U_{\varphi,s,d}(f)\chi_k\|_{L^{\infty}_w} \leq \int_{[0,1]^t} \psi(t) \|f(s(t))\chi_k\|_{L^{\infty}_w} dt. \quad (2)$$

For $\eta > 0$ and $t \in [0,1]^t$ such that $|s(t)| \neq 0$, we estimate

$$\int_{\mathbb{R}^t} \left( \frac{|f(s(t)x)| \chi_k(\omega(x))}{\eta} \right)^{\eta} dx = \int_{s(t)C_{ik}} \left( \frac{|f(z)| \omega^{-1}(t) z}{\eta} \right)^{\eta} |s^{-1}(t)|^\eta dz = |s^{-1}(t)|^\eta \int_{s(t)C_{ik}} \left( \frac{|f(z)| \omega(z)}{\eta} \right)^{\eta} |s^{-1}(t)|^\eta dz.$$

$$\leq \int_{\mathbb{R}^t} \left( \frac{\mathcal{K}_{s,0,\omega}(t) |f(z)| \chi_{k+\ell-1}(z) + |f(z)| \chi_{k+\ell}(z) \omega(z)}{\eta} \right)^{\eta} dz,$$

where $\ell = \ell(t) \in \mathbb{Z}$ such that $2^{\ell-1} < |s(t)| \leq 2^\ell$. Thus

$$\|f(s(t))\chi_k\|_{L^{\infty}_w} \leq \mathcal{K}_{s,0,\omega}(t) \left( \|f\chi_{k+\ell-1}\|_{L^{\infty}_w} + \|f\chi_{k+\ell}\|_{L^{\infty}_w} \right).$$

From this, (2) we have

$$\|U_{\varphi,s,d}(f)\chi_k\|_{L^{\infty}_w} \leq \int_{[0,1]^t} \psi(t) \mathcal{K}_{s,0,\omega}(t) \left( \|f\chi_{k+\ell-1}\|_{L^{\infty}_w} + \|f\chi_{k+\ell}\|_{L^{\infty}_w} \right) dt. \quad (3)$$

Next, by applying Lemma 2.9 and $|s(t)| = 2^\ell$, we get

$$\|f\chi_{k+\ell-1}\|_{L^{\infty}_w} \leq 2^{\max(k(\ell+1)\Lambda,k(\ell-1)\Lambda)} \|f\|_{MK^{(\lambda),\lambda}_{p,q,\omega}},$$

$$= \max\{2^{\ell-1}(1-\alpha(0)),2^{\ell+1}(1-\alpha_\infty)\} 2^{\max(k(\ell-1)\Lambda,k(\ell+1)\Lambda)} \|f\|_{MK^{(\lambda),\lambda}_{p,q,\omega}},$$

$$\leq \max\{|s(t)|^{1-\alpha(0)},|s(t)|^{1-\alpha_\infty}\} 2^{\max(k(\ell-1)\Lambda,k(\ell+1)\Lambda)} \|f\|_{MK^{(\lambda),\lambda}_{p,q,\omega}}.$$. 
By Theorem 2.8, we infer
\[ \|f\chi_k\|_{L^p_{\psi,K}} \leq \max \left\{ |s(t)|^{\lambda - \alpha_0}, |s(t)|^{\lambda - \alpha_\infty} \right\} \cdot 2^{\max(k(\lambda - \alpha_0),k(\lambda - \alpha_\infty))} \cdot \|f\|_{M^0_{p,q}}. \]

From these above, by (3), it is clear to see that
\[ \|U_{\psi,d}(f)\chi_k\|_{L^p_{\psi,K}} \leq C_{1,\sup} \cdot 2^{\max(k(\lambda - \alpha_0),k(\lambda - \alpha_\infty))} \cdot \|f\|_{M^0_{p,q}}. \]  
(4)

By Theorem 2.8, we infer
\[ \|U_{\psi,d}(f)\|_{M^0_{p,q}(\omega)} \leq \max \left\{ \sup_{k_0 \leq 0, k_0 \in \mathbb{Z}} T_1, \sup_{k_0 > 0, k_0 \in \mathbb{Z}} (T_2 + T_3) \right\}, \]  
(5)

where
\[ T_1 = 2^{-k_0} \left( \sum_{k = -\infty}^{k_0} 2^{k(0)p} \|U_{\psi,d}(f)\chi_k\|_{L^p_{\psi,K}}^p \right)^{\frac{1}{p}}, \]
\[ T_2 = 2^{-k_0} \left( \sum_{k = -\infty}^{k_0} 2^{k(0)p} \|U_{\psi,d}(f)\chi_k\|_{L^p_{\psi,K}}^p \right)^{\frac{1}{p}}, \]
\[ T_3 = 2^{-k_0} \left( \sum_{k = 0}^{k_0} 2^{k\max(|\lambda,\lambda - \alpha_\infty| + |\alpha(0)|)} \right)^{\frac{1}{p}}. \]

By using the inequality (4) with \( k_0 \leq 0 \) and \( k_0 \in \mathbb{Z} \), we will estimate \( T_1 \) as follows.

\[ T_1 \leq C_{1,\sup} \cdot 2^{-k_0} \left( \sum_{k = -\infty}^{k_0} 2^{k(0)p} \cdot 2^{\max(k(\lambda - \alpha_0)p,k(\lambda - \alpha_\infty)p)} \cdot \|f\|_{M^0_{p,q}} \right)^{\frac{1}{p}} \]
\[ = C_{1,\sup} \cdot 2^{-k_0} \left( \sum_{k = -\infty}^{k_0} 2^{\max(k(\lambda - \alpha_0)p,k(\lambda - \alpha_\infty)p) + k(0)p} \right)^{\frac{1}{p}} \cdot \|f\|_{M^0_{p,q}} \]
\[ = C_{1,\sup} \cdot 2^{-k_0} \left( \sum_{k = -\infty}^{k_0} 2^{\max(|\lambda,\lambda - \alpha_\infty| + \alpha(0))} \right)^{\frac{1}{p}} \cdot \|f\|_{M^0_{p,q}}. \]

Consequently, by \( \min(\lambda,\lambda - \alpha_\infty + \alpha(0)) > 0 \) and \( \alpha(0) - \alpha_\infty \geq 0 \), \( k_0 \leq 0 \) and \( k_0 \in \mathbb{Z} \), one has
\[ T_1 \leq C_{1,\sup} \cdot 2^{k_0(\min(\lambda,\lambda - \alpha_\infty + \alpha(0)) - 1)p} \|f\|_{M^0_{p,q}} = C_{1,\sup} \cdot 2^{k_0(\min(\lambda,\lambda - \alpha_\infty + \alpha(0)) + 1)p} \|f\|_{M^0_{p,q}} \]
\[ = C_{1,\sup} \cdot \|f\|_{M^0_{p,q}}. \]  
(6)

By evaluating as \( T_1 \), we also have
\[ T_2 \leq C_{1,\sup} \cdot 2^{-k_0} \|f\|_{M^0_{p,q}}. \]  
(7)

Next step, we obtain
\[ T_3 \leq C_{1,\sup} \cdot 2^{-k_0} \left( \sum_{k = 0}^{k_0} 2^{k\max(|\lambda - \alpha_0|,\lambda)} \right)^{\frac{1}{p}} \cdot \|f\|_{M^0_{p,q}} \]
\[ = C_{1,\sup} \cdot 2^{-k_0} \left( \sum_{k = 0}^{k_0} 2^{k\max(\lambda - \alpha(0) + \alpha_0,\lambda)} \right)^{\frac{1}{p}} \cdot \|f\|_{M^0_{p,q}} = C_{1,\sup} \cdot 2^{-k_0} \left( \sum_{k = 0}^{k_0} 2^{k\lambda} \right)^{\frac{1}{p}} \cdot \|f\|_{M^0_{p,q}} \]
\[ \leq C_{1,\sup} \cdot 2^{-k_0} (2\lambda + 1) \|f\|_{M^0_{p,q}}. \]
Thus, by (5)-(7), it follows that
\[ \|U_{\psi, \sigma, \lambda}\|_{L^{q,1} \to L^{1,1}} \leq C_1 \sup_{\lambda} \|f\|_{L^{q,1} \to L^{1,1}}, \]
which gives the proof of case (i) of Theorem 3.1 is finished. Next, let us consider for case (ii). We choose
\[ f_0(x) = |x|^{-\alpha(0)-\frac{\sigma}{q_+}+\lambda} \omega(x)^{-1}. \]
It is obvious to see that
\[ \|f_0\|_{L^{q,1} \to L^{1,1}} > 0. \]
On the other hand, we calculate
\[ F_q(f_0\omega, \chi_k) = \int_{C_k} |x|^{(\lambda-\alpha(0))q} x^{-\sigma} dx = \int_{2^{k-1}}^{2^k} \int S^{-1} r^{(\lambda-\alpha(0))q} (x')^{-1} d\sigma(x') dr \leq 2^{\max(k(\lambda-\alpha(0))q, (\lambda-\alpha(0))q)}. \]
Combining this with the inequality (1), we get
\[ \|f_0\chi_k\|_{L^{q,1}} \leq \max\{2^{\max(k(\lambda-\alpha(0))q, (\lambda-\alpha(0))q)}, 2^{\max(k(\lambda-\alpha(0))q, (\lambda-\alpha(0))q)}\} \]
\[ = 2^{\max(k(\lambda-\alpha(0))q, (\lambda-\alpha(0))q)}. \]
(8)
Besides, by Theorem 2.8, we estimate
\[ \|f_0\|_{L^{q,1} \to L^{1,1}} \leq \max\left\{ \sup_{k_0 \in \mathbb{N}} G_1, \sup_{k_0 \in \mathbb{N}} (G_2 + G_3) \right\}. \]
(9)
Here
\[ G_1 = 2^{-k_0\lambda} \left( \sum_{k=0}^{k_0} 2^{k_0(0)p} \|f_0\chi_k\|_{L^{p,1}^q} \right)^{\frac{1}{p}}, \quad G_2 = 2^{-k_0\lambda} \left( \sum_{k=0}^{\infty} 2^{k_0(0)p} \|f_0\chi_k\|_{L^{p,1}^q} \right)^{\frac{1}{p}}, \quad G_3 = 2^{-k_0\lambda} \left( \sum_{k=0}^{\infty} 2^{k_0(0)p} \|f_0\chi_k\|_{L^{p,1}^q} \right)^{\frac{1}{p}}. \]
By either \( \alpha(0) = \lambda > 0 \) or \( q_+ = q_- \), we infer
\[ \alpha(0) + \min(\lambda - \alpha(0), q_- / q_+, (\lambda - \alpha(0))q_+) - \lambda = 0. \]
Hence, by (8), one has
\[ G_1 \leq 2^{-k_0\lambda} \left( \sum_{k=0}^{k_0} 2^{k_0(0)p} 2^{\max(k(\lambda-\alpha(0))q, (\lambda-\alpha(0))q, / q_+)} \right)^{\frac{1}{p}} \]
\[ \leq 2^{-k_0\lambda} \left( \sum_{k=0}^{\infty} 2^{k_0(0)p} \left( \min(\lambda - \alpha(0), q_+, (\lambda-\alpha(0))q_+ / q_+) \right) \right)^{\frac{1}{p}} \]
\[ \leq 2^{k_0(0)+\min(\lambda-\alpha(0)), q_+, (\lambda-\alpha(0))q_+/ q_+} = 1. \]
(10)
By estimating as (10) above, we also have
\[ G_2 \leq 2^{-k_0\lambda} 2^{-\lambda(0)+\min(\lambda(0)-\alpha(0), q_+, (\lambda(0)-\alpha(0))q_+/ q_+)} \leq 2^{-k_0\lambda}. \]
By applying the inequality (8) again, we deduce
\[ G_3 \leq 2^{-k_0} \left( \sum_{k=1}^{k_0} 2^{|p(\alpha_\infty + \max\{(\lambda - \alpha(0)), q_-/q_+\}, (\lambda - \alpha(0)), q_-/q_-\} = 0,}
\]
\[ \leq \begin{cases} 2^{-k_0} \left( k_0^{1/p} + 1 \right), & \text{if } \alpha_\infty + \max\{(\lambda - \alpha(0)), q_-/q_+\}, (\lambda - \alpha(0)), q_-/q_-\} = 0, \\

2^{-k_0} + 2^{-k_0}(\lambda - \alpha(0)), q_-/q_-\} = 0, \\

\end{cases}
\]
\[ \leq 2^{-k_0}(k_0^{1/p} + 1) + 2^{-k_0}(\lambda - \alpha(0)), q_-/q_-\} = 0. \tag{12} \]

By the assumptions \( \lambda = \alpha(0) > 0 \) or \( q_+ = q_- \), we get
\[ \lambda = \alpha_\infty + \max\{(\lambda - \alpha(0)), q_-/q_+\}, (\lambda - \alpha(0)), q_-/q_-\} = 0. \]

From this, by (9)-(12), we derive
\[ \|f_0\|_{MK_{p,q}(\omega)} \leq \max\left\{ \sup_{k_0 \geq 0, k_0 \in \mathbb{Z}} \left( 2^{-k_0} \left( k_0^{1/p} + 2 \right) \right) \right\} < \infty. \]

In addition, we get
\[ U_{\psi,\alpha,d}(f_0)(x) = \left( \int_{[0,1]^d} \psi(t) |s(t)|^{\alpha_\infty - 1/\sqrt{2}n + 1} \cdot \left( \frac{\omega(x)}{\omega(s(t))} \right) dt \right)^{\alpha_\infty - 1/\sqrt{2}n + 1} \geq C_{1,\inf} f_0(x). \]

This leads to
\[ C_{1,\inf} \leq \|U_{\psi,\alpha,d}f_0\|_{MK_{p,q}(\omega)} < \infty. \]

Therefore, the proof of this theorem is completed. \( \square \)

Next, we obtain the boundedness for the Hardy–Cesáro operators on the weighted Herz spaces with variable exponent as follows.

**Theorem 3.2.** Let \( \alpha(0) = \alpha_\infty \) and the assumptions of Theorem 3.1 hold.

(i) If
\[ C_{2,\sup} = \int_{[0,1]^d} \psi(t) \cdot K_{\psi,\alpha}(t) \cdot|s(t)|^{\alpha(0)} dt < \infty, \]
then \( U_{\psi,\alpha,d} \) is a bounded operator from \( K_{\psi,\alpha}(\alpha_\infty, p) \) to itself.

(ii) If \( U_{\psi,\alpha,d} \) is a bounded operator from \( K_{\psi,\alpha}(\alpha_\infty, p) \) to itself, and either \( \alpha_\infty = 0 \) or \( q_+ = q_- \), then
\[ C_{2,\inf} = \int_{[0,1]^d} \psi(t) \cdot K_{\psi,\alpha}(t) \cdot|s(t)|^{\alpha(0)} dt < \infty. \]

Moreover,
\[ C_{2,\inf} \leq \|U_{\psi,\alpha,d}\|_{K_{\psi,\alpha}(\alpha_\infty, p) \rightarrow K_{\psi,\alpha}(\alpha_\infty, p)}. \]

**Proof.** First, we will consider the proof for the case (i). By Theorem 2.7 and the relation \( \alpha(0) = \alpha_\infty \), we infer
\[ \|U_{\psi,\alpha,d}(f)\|_{K_{\psi,\alpha}(\alpha_\infty, p)} \leq \left( \sum_{k=-\infty}^{\infty} 2^{k\alpha(0)p} \|U_{\psi,\alpha,d}(f)\|_{K_{\psi,\alpha}(\alpha_\infty, p)} \right)^{1/\sqrt{2}} \cdot \|U_{\psi,\alpha,d}(f)\|_{L_1(\omega)} \]: \( \mathcal{U}. \)

\[ \]
By having the inequality (3) and using the Minkowski inequality, we deduce

\[
U \leq \int_{[-1,1]^d} \psi(t) \mathcal{K}_0(t) \left( \sum_{k=-\infty}^{\infty} 2^{k\alpha(0)p} \left( \left\| f \chi_{k+1} \right\|_{L^p}^p + \left\| f \chi_{k+1} \right\|_{L^p}^p \right) \right) dt \\
\leq \int_{[-1,1]^d} \psi(t) \mathcal{K}_0(t) \left( \sum_{k=-\infty}^{\infty} 2^{k\alpha(0)p} \left\| f \chi_{k+1} \right\|_{L^p}^p \right) dt. 
\]

(14)

On the other hand, by $|s(t)| \approx 2^{0(0)}$, we derive

\[
\left( \sum_{k=-\infty}^{\infty} 2^{k\alpha(0)p} \left\| f \chi_{k+1} \right\|_{L^p}^p \right) \leq \left( \sum_{r=-\infty}^{\infty} 2^{(r-0)+0(0)p} \left\| f \chi_{r} \right\|_{L^p}^p \right) \leq 2^{-0(0)} \left\| f \right\|_{L^p(K)_{\delta,\omega}}. 
\]

By making as above, we also have

\[
\left( \sum_{k=-\infty}^{\infty} 2^{k\alpha(0)p} \left\| f \chi_{k+1} \right\|_{L^p}^p \right) \leq \left| s(t) \right|^{-0(0)} \left\| f \right\|_{L^p(K)_{\delta,\omega}}. 
\]

From these, by (13) and (14), it is clear to see that

\[
\left\| U_{\psi,K,\delta}(f) \right\|_{L^p(K)_{\delta,\omega}} \leq C_{2,\sup} \left\| f \right\|_{L^p(K)_{\delta,\omega}}. 
\]

Therefore, the proof of this case is achieved.

Now, let us prove the case (ii). For any $\varepsilon > 0$, we choose the functions $f_\varepsilon$ as follows

\[
f_\varepsilon(x) = \begin{cases} 0, & \text{if } |x| \leq 1, \\ |x|^{-\alpha_0} - \varepsilon^{-\alpha_0} \omega(x)^{-1}, & \text{otherwise}. \end{cases}
\]

This leads to

\[
\left\| f_\varepsilon \right\|_{L^p(K)_{\delta,\omega}} > 0, \text{ for all } \varepsilon > 0.
\]

For $\varepsilon$ small enough, we get

\[
F_q(\lambda_\varepsilon, \omega, \chi_k) = \int_{\mathbb{R}^d} \chi_k^{(-\alpha_0-\varepsilon)}(x) dx = \int_{2^{k-1} \leq \xi < 2^k} \int_{\mathbb{R}^d} r^{(-\alpha_0-\varepsilon)}(r, x)^{-1} d\sigma(x) dr.
\]

This gives

\[
2^{\min[-k(\alpha_0+c)q_+, -k(\alpha_0+c)q_-]} \leq F_q(f_\varepsilon, \omega, \chi_k) \leq 2^{\max[-k(\alpha_0+c)q_+, -k(\alpha_0+c)q_-]}. 
\]

Hence, by the inequality (1), we derive

\[
2^{\min[-k(\alpha_0+c)q_+/q_+, -k(\alpha_0+c)q_/q_-]} \leq \left\| f_\varepsilon \chi_k \right\|_{L^p(K)_{\delta,\omega}} \leq 2^{\max[-k(\alpha_0+c)q_+/q_+, -k(\alpha_0+c)q_/q_-]}. 
\]

Now, we put

\[
\zeta_\varepsilon = \alpha_0 - \min((\alpha_0 + \varepsilon)q_+/q_+, (\alpha_0 + \varepsilon)q_-/q_-) \text{ and } \beta_\varepsilon = \alpha_0 - \max((\alpha_0 + \varepsilon)q_+/q_+, (\alpha_0 + \varepsilon)q_-/q_-). 
\]

By applying the conditions $\alpha_0 = 0$ or $q_+ = q_-$, we get $\zeta_\varepsilon, \beta_\varepsilon < 0$. Furthermore, it follows that

\[
\lim_{\varepsilon \to 0^+} \zeta_\varepsilon = 0, \lim_{\varepsilon \to 0^+} \beta_\varepsilon = 0 \text{ and } \lim_{\varepsilon \to 0^+} \frac{\beta_\varepsilon}{\zeta_\varepsilon} = \begin{cases} 1, & \text{if } q_+ = q_-, \\ q_+^2/q_-^2, & \text{if } \alpha_0 = 0. \end{cases}
\]

(15)
As a consequence, by Theorem 2.7, for each \( r \in \mathbb{Z}^+ \), we obtain
\[
\left\| f_k \right\|_{L^p_k} \leq \left\{ \sum_{k=1}^{\infty} 2^{k_0^r - p} \left\| f_k x_k \right\|_{L^p_k} \right\}^{1/p} \leq \left\{ \sum_{k=1}^{\infty} 2^{k_0^r} \right\}^{1/p} \leq \frac{2^{k_0^r}}{(1 - 2^{k_0^r})^{1/p}},
\]
and
\[
\left\{ \sum_{k=r}^{\infty} 2^{k_0^r - p} \left\| f_k x_k \right\|_{L^p_k} \right\}^{1/p} \geq \left\{ \sum_{k=r}^{\infty} 2^{k_0^r} \right\}^{1/p} \geq \frac{2^{k_0^r}}{(1 - 2^{k_0^r})^{1/p}}.
\]
(16)

By setting
\[
V_x^p = \{ t \in [0,1]^d : |s(t)x| \geq 1 \} \quad \text{and} \quad U_x^p = \{ t \in [0,1]^d : |s(t)| \geq \varepsilon \}.
\]
This gives
\[
U_x^p \subset V_x^p, \quad \forall x \in B(0, \varepsilon^{-1}).
\]
Instantly, for \( 0 < \varepsilon \leq 1 \), we get
\[
U_{\psi,\alpha,\varepsilon}(f)(x) \geq \int_{V_x^p} \psi(t)|s(t)x|^{-\alpha_0} \alpha(s(t)x)^{-1} dt
\]
\[
\geq \left( \int_{U_x^p} \psi(t)|s(t)|^{-\alpha_0} \alpha(s(t)x)^{-1} dt \right)|x|^{-\alpha_0} \alpha(x)^{-1} X_{B(0,\varepsilon^{-1})}(x)
\]
\[
\geq \left( \int_{U_x^p} \psi(t)|s(t)|^{-\alpha_0} \alpha(s(t)x)^{-1} dt \right)f_x(x)X_{B(0,\varepsilon^{-1})}(x).
\]
Hence, by using Theorem 2.7 again and (16) with \( k_0 \) is the smallest integer number so that \( 2^{k_0} \geq \varepsilon^{-1} \geq 1 \), we infer
\[
\left\| U_{\psi,\alpha,\varepsilon}(f) \right\|_{K_{q_0}L^p} \geq \left( \int_{U_x^p} \psi(t)|s(t)|^{-\alpha_0} \alpha(s(t)x)^{-1} dt \right)^{1/p} \left\{ \sum_{k=k_0}^{\infty} 2^{k_0^r - p} \left\| f_k x_k \right\|_{L^p_k} \right\}^{1/p}
\]
\[
\geq \left( \frac{1 - 2^{k_0^r}}{(1 - 2^{k_0^r})^{1/p}} \right)^{1/p} \varepsilon^{-\varepsilon} \left( \int_{U_x^p} \psi(t)|s(t)|^{-\alpha_0} \alpha(s(t)x)^{-1} dt \right)^{1/p} \left\| f \right\|_{K_{q_0}L^p}
\]
\[
\geq \left( \frac{1 - 2^{k_0^r}}{(1 - 2^{k_0^r})^{1/p}} \right)^{1/p} \varepsilon^{-\varepsilon} \left( \int_{U_x^p} \psi(t)|s(t)|^{-\alpha_0} \alpha(s(t)x)^{-1} dt \right)^{1/p} \left\| f \right\|_{K_{q_0}L^p}.
\]
(17)

On the other hand, by (15) and \( \lim_{\varepsilon \to 0^+} \varepsilon^{-\beta_0} = 1 \), we deduce
\[
\lim_{\varepsilon \to 0^+} \frac{(1 - 2^{k_0^r})^{1/p} \varepsilon^{-\beta_0}}{(1 - 2^{k_0^r})^{1/p} 2^{\alpha_0}} \varepsilon^{-\varepsilon} = \begin{cases} 1, & \text{if } q_+ = q_-, \\ \left( q_+^2 / q_-^2 \right)^{1/p}, & \text{if } \alpha_0 = 0. \end{cases}
\]

From this, by (17) and the dominated convergence theorem of Lebesgue, it is clear to see that
\[
C_{2,\text{int}} \leq \left\| U_{\psi,\alpha,\varepsilon}(f) \right\|_{K_{q_0}L^p} \leq \infty.
\]
Thus, the proof for this case is finished.

Now, we state the boundedness of Hardy–Cesàro operators on weighted central Morrey spaces.
Theorem 3.3. Let \( q(s^{-1}(t)) = q(\cdot) \) for almost everywhere \( t \in \text{supp}(\psi) \), \( \kappa > 0 \) and \( \omega(x) = |x|^{\gamma} \) with \( \gamma > -n \).
(i) If
\[
C_{3, \text{max}} = \int_{[0,1]} \psi(t)|s(t)|^{(\kappa+n)-\gamma} \max\{|s(t)|^\frac{\gamma}{2}, |s(t)|^{\frac{r}{2}}\} dt < \infty,
\]
then the operator \( U_{\psi,s,d} \) is bounded from \( M^q_{\omega} \) to itself.
(ii) If \( U_{\psi,s,d} \) is bounded from \( M^q_{\omega} \) to itself and \( q_+ = q_- \), we have
\[
C_3 = \int_{[0,1]} \psi(t)|s(t)|^{(\kappa+n)-\gamma} \frac{dt}{t} < \infty.
\]
Moreover,
\[
||U_{\psi,s,d}||_{M^q_{\omega}} \approx C_3.
\]

Proof. (i) By the same arguments as (3) above, one has
\[
||U_{\psi,s,d}(f)||_{L^q_{\psi}(\mathcal{H})} \leq \int_{[0,1]} \psi(t) K_{s,\theta}(t) \cdot ||f||_{L^q_{\psi}(\mathcal{H})} \cdot dt,
\]
where \( \ell = \ell(t) \in \mathbb{Z} \) such that \( 2^{\ell-1} < |s(t)| \leq 2^{\ell} \). Accordingly, we infer
\[
||U_{\psi,s,d}(f)||_{M^q_{\omega}(\mathcal{H})} = \sup_{k \in \mathbb{Z}} \frac{1}{\omega(B_k)} ||U_{\psi,s,d}(f)||_{L^q_{\psi}(\mathcal{H})} \leq \sup_{k \in \mathbb{Z}} \left( \int_{[0,1]} \psi(t) K_{s,\theta}(t) \cdot \omega(B_{k+\ell})^\kappa \cdot \omega(B_k)^\kappa \cdot dt \right) ||f||_{M^q_{\omega}(\mathcal{H})}.
\]

(18)
Here
\[
K_{s,\theta}(t) = \text{ess sup}_{x \in \mathbb{R}^d} \frac{\omega(s^{-1}(t)z)}{\omega(z)} \cdot \max\{|s(t)|^\frac{\gamma}{2}, |s(t)|^{\frac{r}{2}}\} = |s(t)|^\gamma \max\{|s(t)|^\frac{\gamma}{2}, |s(t)|^{\frac{r}{2}}\}.
\]
And by \( |s(t)| \approx 2^{\ell} \), we have
\[
\frac{\omega(B_{k+\ell})}{\omega(B_k)} \approx \frac{2^{(k+\ell)(\kappa+n)}}{2^{k(\kappa+n)}} \approx |s(t)|^{\kappa+n}.
\]
From these, by (18), we obtain
\[
||U_{\psi,s,d}(f)||_{M^q_{\omega}(\mathcal{H})} \leq C_{3, \text{max}} \cdot ||f||_{M^q_{\omega}(\mathcal{H})}.
\]
This gives the proof of the case (i) is ended.

(ii) Suppose that \( U_{\psi,s,d} \) is a bounded operator from \( M^q_{\omega}(\mathcal{H}) \) to itself and \( q_+ = q_- \). Let us choose \( h_0(x) = |x|^{(n+\gamma)-\frac{\gamma}{2}} \cdot \omega(x)^{\frac{1}{2}} \).

It is not hard to see that
\[
||h_0||_{L^q_{\psi}(\mathcal{H})}^q = \left( \int_{\mathbb{R}^d} |x|^{q(n+\gamma)-\gamma} dx \right)^{1/q} \approx 2^{k(x(n+\gamma))}.
\]
Thus
\[
||h_0||_{M^q_{\omega}(\mathcal{H})} = \sup_{k \in \mathbb{Z}} \frac{1}{\omega(B_k)} ||h_0||_{L^q_{\psi}(\mathcal{H})} \approx 1.
\]
Besides, we deduce
\[
U_{\psi,s,d}(h_0)(x) = \left( \int_{[0,1]} \psi(t)|s(t)|^{(\kappa+n)-\gamma} \cdot \frac{dt}{t} \right) |x|^{(n+\gamma)-\gamma} \cdot \omega(x)^{\frac{1}{2}} = C_3 h_0(x).
\]
Proof. By estimating as in the proof of Theorem 3.3(i), we immediately have
\[ C_3 \leq \| U_{\psi, s, d} \|_{B_{\infty, \infty}^{(s) - \psi, d} \to B_{\infty, \infty}^{(s) - \psi, d}} < \infty. \]

Hence, we achieve the proof of this case. \( \square \)

Finally, we present the boundedness for the Hardy-Cesàro operators on weighted local central Morrey spaces and weighted non-local central Morrey spaces with variable exponent as follows.

**Theorem 3.4.** Let \( q(s^{-1}(t)) = q(\cdot) \) for almost everywhere \( t \in \text{supp}(\psi) \), \( \kappa > 0 \) and \( \omega(x) = |x|^\gamma \) with \( \gamma > -n \).

(i) If \( C_{3, \text{max}} \) in Theorem 3.3 is finite, then the operator \( U_{\psi, s, d} \) is bounded from \( B_{q(s^{-1})}^{(s)} \) to itself.

(ii) If \( U_{\psi, s, d} \) is bounded from \( B_{q(s^{-1})}^{(s)} \) to itself, we then have
\[ C_{3, \text{min}} = \int_{|t| < 1} \psi(t)|s(t)|^{-\gamma + \kappa(n + \gamma)q_{-1}/\mu} \cdot \min \left\{ |s(t)|^{\frac{1}{\mu}}, |s(t)|^{\frac{1}{\kappa}} \right\} dt < \infty. \]

Furthermore,\[ \tilde{C}_{3, \text{min}} \leq \| U_{\psi, s, d} \|_{B_{q(s^{-1})}^{(s)} \to B_{q(s^{-1})}^{(s)}}. \]

**Proof.** By estimating as in the proof of Theorem 3.3(i), we immediately have
\[ \| U_{\psi, s, d} \|_{B_{q(s^{-1})}^{(s)} \to B_{q(s^{-1})}^{(s)}} \leq C_{3, \text{max}}. \]

Hence, we accomplish the proof of case (i). Next, let us prove the case (ii). For any \( m \in \mathbb{N}^* \), we set
\[ g_m(x) = \left| x^{\mu_m(n + \gamma) - \frac{1}{\mu}} \omega^{-1}(x) \right|. \]

with \( \mu_m = \left( 1 + \frac{1}{m} \right)^{\frac{q}{q-1}} \). Thus, it is easy to see that
\[ \| g_m \|_{B_{q(s^{-1})}^{(s)}} > 0, \quad \text{for any } m \in \mathbb{N}^*. \]

On the other hand, for any \( R \in (0, 1) \) and \( m \in \mathbb{N}^* \), we have
\[
F_q(g_m, \omega; B(0, R)) = \int_{B(0, R)} \left| x^{\mu_m(n + \gamma) - \frac{1}{\mu}} \omega^{-1}(x) \right| dx = \int_0^R \int_{S^{n-1}} t^{\mu_m(n + \gamma)q_{-1}/\mu} \omega'(t') dt' \, dr = \frac{R^{\mu_m(n + \gamma)q_{-1}}}{\mu_m(n + \gamma)q_{-1}}.
\]

Consequently, by using the inequality (1), we get
\[
\| g_m \|_{B_{q(s^{-1})}^{(s)}} \leq \max \left\{ \frac{R^{\mu_m(n + \gamma)}}{\mu_m(n + \gamma)q_{-1}/\mu}, \frac{R^{\mu_m(n + \gamma)q_{-1}}}{\mu_m(n + \gamma)q_{-1}/\mu} \right\} \leq \eta_m R^{\mu_m(n + \gamma)q_{-1}/\mu}, \quad \text{for all } R \in (0, 1) \text{ and } m \in \mathbb{N}^*.
\]

Here \( \eta_m = \max \left\{ \mu_m(n + \gamma)q_{-1}/\mu, \mu_m(n + \gamma)q_{-1}/\mu \right\} \).

Thus, for any \( m \in \mathbb{N}^* \), we obtain
\[
\| g_m \|_{B_{q(s^{-1})}^{(s)}} = \sup_{0 < R < 1} \left( \int_{B(0, R)} \left| x^{\mu_m(n + \gamma) - \frac{1}{\mu}} \omega^{-1}(x) \right| dx \right)^{1/q} = \eta_m R^{\mu_m(n + \gamma)q_{-1}/\mu} \| g_m \|_{B_{q(s^{-1})}^{(s)}} = \eta_m \sup_{0 < R < 1} R^{\mu_m(n + \gamma)q_{-1}/\mu} = \eta_m < \infty.
\]

Besides, one has
\[
U_{\psi, s, d}(g_m)(x) = \left( \int_{|t| < 1} \psi(t)|s(t)|^{\mu_m(n + \gamma) - \frac{1}{\mu}} dt \right) g_m(x) \geq C_{3, \text{min}} \cdot g_m(x).
\]
Hence, the proof for this case is solved.

Here

\[ C_{3,m,\text{min}} = \int_{[0,1]} \psi(t), |s(t)|^{p(t)x(n+\gamma)q} \min \{|s(t)|^{q}, |s(t)|^{\nu}\} dt. \]

From this, by \( U_{\psi,d} \) is a bounded operator from \( B_{\omega,\text{loc}}^{s,1} \) to itself, we infer

\[ C_{3,m,\text{min}} \leq \|U_{\psi,d}\|_{B_{\omega,\text{loc}}^{s,1} \rightarrow B_{\omega,\text{loc}}^{s,1}} < \infty. \]

By applying \( \lim_{m \to \infty} \mu_m = \frac{q}{q} \) and the dominated convergence theorem of Lebesque, we get

\[ \hat{C}_{3,\text{min}} \leq \|U_{\psi,d}\|_{B_{\omega,\text{loc}}^{s,1} \rightarrow B_{\omega,\text{loc}}^{s,1}} < \infty. \]

Hence, the proof for this case is solved. \( \square \)

**Theorem 3.5.** Let \( q(s^{-1}(t)) = q(t) \) for almost everywhere \( t \in \text{supp}(\psi) \), \( \kappa > 0 \) and \( \omega(x) = |x|^\gamma \) with \( \gamma > -n \).

(i) If \( C_{3,\text{max}} \) in Theorem 3.3 is finite, then the operator \( U_{\psi,d} \) is bounded from \( B_{\omega}^{s,1} \) to itself.

(ii) If \( U_{\psi,d} \) is bounded from \( B_{\omega}^{s,1} \) to itself, then we have

\[ C_{3,\text{min}} = \int_{[0,1]} \psi(t)|s(t)|^{-\gamma+q(n+\gamma)q_{-1}} \min \{|s(t)|^{q}, |s(t)|^{\nu}\} dt < \infty. \]

Moreover,

\[ C_{3,\text{min}}^* \leq \|U_{\psi,d}\|_{B_{\omega}^{s,1} \rightarrow B_{\omega}^{s,1}}. \]

**Proof.** From Theorem 3.3(i), let us obtain the proof of case (i). Now, we will prove case (ii). By choosing the functions \( g_m \) as in Theorem 3.4 with \( \mu_m = \left(1 - \frac{1}{m} \right)^{q/q} \), we also have

\[ \|g_m\|_{B_{\omega}^{s,1}} > 0, \text{ for any } m \in \mathbb{N}. \]

On the other hand, for any \( R \geq 1 \) and \( m \in \mathbb{N} \), we infer

\[ F_{\eta}(g_m,\omega,B(0, R)) = \int_{0}^{R} \int_{S^{n-1}} \rho^{\mu_m x(n+\gamma)q(x')} |\omega(x')|^{-1} d\sigma(x') dr \]

\[ = \int_{0}^{R} \int_{S^{n-1}} \rho^{\mu_m x(n+\gamma)q(x')} |\omega(x')|^{-1} d\sigma(x') dr + \int_{1}^{R} \int_{S^{n-1}} \rho^{\mu_m x(n+\gamma)q(x')} |\omega(x')|^{-1} d\sigma(x') dr \]

\[ \leq \int_{0}^{R} \rho^{\mu_m x(n+\gamma)q_{-1}} |\omega(x')|^{-1} dr + \int_{1}^{R} \rho^{\mu_m x(n+\gamma)q_{-1}} |\omega(x')|^{-1} dr \]

\[ = \frac{\rho^{\mu_m x(n+\gamma)q_{-1}}}{\mu_m k(n+\gamma)q_{-1}} + v_{m_{r}} \]

where \( v_m = \frac{1}{\mu_m (n+\gamma)q_{-1}} - \frac{1}{\mu_m (n+\gamma)q_{-1}} \geq 0 \). Thus, by applying the inequality (1) again, we deduce

\[ \|g_m\|_{L_{\omega,\text{loc}}^{s,1}(B(0,R))} \leq \max \left\{ \frac{\rho^{\mu_m x(n+\gamma)q_{-1}}}{\mu_m k(n+\gamma)q_{-1}} + v_{m_{r}}, \frac{\rho^{\mu_m x(n+\gamma)q_{-1}}}{\mu_m k(n+\gamma)q_{-1}} + v_{m_{r}} \right\} \]

\[ \leq \max \left\{ \frac{\rho^{\mu_m x(n+\gamma)q_{-1}}}{\mu_m k(n+\gamma)q_{-1}} + v_{m_{r}}, \frac{\rho^{\mu_m x(n+\gamma)q_{-1}}}{\mu_m k(n+\gamma)q_{-1}} + v_{m_{r}} \right\} \]

\[ \leq \eta_m \rho^{\mu_m x(n+\gamma)q_{-1}} + v_{m_{r}}, \text{ for any } R \geq 1 \text{ and } m \in \mathbb{N}. \]
with $\tilde{\eta}_m = \max \{1/m^{1/n_1}, 1/m^{1/n_2}\}$ and $\tilde{\nu}_m = \max \{1/m^{1/n_1}, 1/m^{1/n_2}\}$.

Hence, from the definition of non-local central Morrey spaces, one has

$$
\|g_m\|_{B^{p,1}_{q}} = \sup_{R \geq 1} \frac{1}{\omega(B(0,R))} \|g_m\|_{L^p(B(0,R))} \\
\leq \sup_{R \geq 1} R^{-\kappa(n+\gamma)} \left( \eta_m \cdot R^{\kappa(n+\gamma)/q} + \tilde{\nu}_m \right) \\
= \sup_{R \geq 1} \left\{ \eta_m \cdot R^{\kappa(n+\gamma)\mu/\gamma - 1} + \tilde{\nu}_m \cdot R^{-\kappa(n+\gamma)} \right\} \\
\leq \tilde{\eta}_m + \tilde{\nu}_m < \infty, \text{ for any } m \in \mathbb{N}^*.
$$

By the same arguments as in the final section of the proof of Theorem 3.4(ii) and $\lim_{m \to \infty} \eta_m = \frac{q}{p}$, we also obtain

$$
C_{3,\min}^{s} \leq \|U_{\phi,\Delta L}\|_{B^{p,1}_{q} \to B^{p,1}_{q}} < \infty.
$$

Therefore, we accomplish the proof of this case. \qed

Acknowledgments

The authors are grateful to the anonymous reviewer for the valuable suggestions and comments which lead to the improvement of the paper.

References


