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Existence of solutions for a class of Boundary value problems involving Riemann Liouville derivative with respect to a function

A. Nouf[a], W. M. Shammakh[a], A. Ghanmi[b]

[a]*Department of Mathematics, Faculty of Sciences, University of Jeddah, Jeddah, Saudi Arabia.*
[b]*Faculté des Sciences de Tunis, LR10ES09 Modélisation mathématique, analyse harmonique et théorie du potentiel, Université de Tunis El Manar, Tunis 2092, Tunisie.*

Abstract. In this article, we study some class of fractional boundary value problem involving generalized Riemann Liouville derivative with respect to a function and the p-Laplace operator. Precisely, using variational methods combined with the mountain pass theorem, we prove that such problem has a nontrivial weak solution. Our main result significantly complement and improves some previous papers in the literature.

1. Introduction

Recently, fractional calculus has been attracted the attention of many authors. This is due to its importance and applications in many fields such as physics, chemistry, aerodynamics, electro dynamics of complex medium, polymer rheology (see [3, 4, 11, 12, 16, 20, 22, 24]). Concequently, there has been significant development in ordinary and partial differential equations involving different fractional operators. For details and examples, one can see the monographs [1, 2, 6, 13–15, 18, 21] and references therein.
By means of the mountain pass theorem, Torres [25] studied the existence of solutions for the following problem

$$\begin{cases} -{}_tD_1^{\alpha}\,{}_0D_t^{\alpha}u(t) = f(t,u(t)),\ t \in (0,T) \\ u(0) = u(T) = 0, \end{cases} \tag{1}$$

and obtained the existence of at least one nontrivial solution, where ${}_tD_1^{\alpha}$ and ${}_0D_t^{\alpha}$ are the right and left Riemann Liouville fractional derivatives. We notes that the first paper studiying such prblem by using the varitional aproach is the paper of Jiao and Zhou [17]. After this, many authors studied several works by using different medhods we refere the readers to [6–9, 15] and the references therein. Precisly, César [7] concidered the following p-Laplacian Dirichlet problem with mixed derivatives

$$\begin{cases} -{}_tD_1^{\alpha}\left(\varphi_p\left({}_0D_t^{\alpha}u(t)\right)\right) = f(t,u(t)),\ t \in (0,T) \\ u(0) = u(T) = 0, \end{cases} \tag{2}$$

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Communicated by Maria Alessandra Ragusa
Email addresses: NOUF6316@hotmail.com (A. Nouf), wmshammakh@uj.edu.sa (W. M. Shammakh), abdeljabbar.ghanmi@lamsin.rnu.tn (A. Ghanmi)

where $0 < \frac{1}{p} < \alpha < 1$ and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Under suitable assymptions on the nonlinearity f and using direct variational medhod combined with the mountain pass theorem, the author prove that problem (2) has at least one nontrivial weak solution.

Given this large number of definitions of fractional operators introduced so far, several researchers are still looking for how to choose the best fractional derivative to discuss certain objectives. one of these operators is the fractional Liouville operator with respect to another function which is introduced for the first time by Samko et al. [23], and developed after that by Kilbas [18]. Very recently, Almeida in [5] has found other properties in this direction. As we know, there are a few articles dealing with these types of operators that we cite for example [5, 26].

Motivated by the above mentioned works, in this paper, we want to contribute with the development of this new area on fractional differential equations theory. Precisely, we will study the existence of nontrivial weak solutions for the following fractional boundary value problem involving the p-Laplace operator and the ψ-Riemann Liouville derivative

$$\begin{cases} M(u(t))\, {}_tD_T^{\alpha,\psi}\left(\varphi_p\left({}_0D_t^{\alpha,\psi}u(t)\right)\right) = \lambda f(t,u(t)) + g(t,u(t)), t \in (0,T), \\ u(0) = u(T) = 0, \end{cases} \tag{3}$$

where λ is a positive parameter, $0 < \frac{1}{p} < \alpha \leq 1$, the function $M : \mathbb{R} \to \mathbb{R}$ is defined by

$$M(u(t)) = \left(a + b\int_0^T \left|{}_0D_t^{\alpha,\psi}u(t)\right|^p dt\right)^{p-1}, \text{ with } a \geq 1 \text{ and } b > 1.$$

While, the functions $f, g : [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous, moreover, g is positively homogeneous of degree $q-1$, which means that for all $s > 0$ and $(t,u) \in [0,T] \times \mathbb{R}$, we have $g(t,su) = s^{q-1}g(t,u)$.

Now, we put

$$F(x,s) = \int_0^s f(x,t)dt, \text{ and } G(x,s) = \int_0^s g(x,t)dt,$$

and we assume the following hypotheses:

($\mathbf{H_1}$) There exists $\delta > p, \mu > 0$ such that

$$0 < \delta F(t,u) \leq uf(t,u) \tag{4}$$

$$\forall (t,u) \in [0,T] \times \mathbb{R}\ \ , |u| \geq \mu.$$

and there exists $C_0 > 0$, such that

$$|F(t,u)| \leq C_0|u|^\delta. \tag{5}$$

($\mathbf{H_2}$) $G : [0,T] \times \mathbb{R} \to \mathbb{R}$ is positively homogeneous of degree q, tha is

$$G(t,su) = s^q G(t,u), \ \ \forall\ (s,t,u) \in (0,\infty) \times [0,T] \times \mathbb{R}.$$

We notes that from condition ($\mathbf{H_2}$), we get the so-called Euler identity:

$$ug(t,u) = qG(t,u). \tag{6}$$

Moreover, there exists $C_1 > 0$, such that

$$|G(t,u)| \leq C_1|u|^q. \tag{7}$$

Our main results is the following.

Theorem 1.1. *Assume that hypotheses* ($\mathbf{H_1}$) *and* ($\mathbf{H_2}$) *are satisfied. If* $\frac{1}{p} < \alpha < 1$, *and* $1 < p < p^2 < \min(\delta, q)$, *then there exists* $\lambda_0 > 0$, *such that for any* $\lambda \in (0, \lambda_0)$, *problem* (3) *has at least one nontrivial weak solution.*

Note that our main result here is new and generalizes some known results in the literature.

This paper is organized as follows. In Section 2, some preliminaries on the fractional calculus are presented. In Section 3, we set up the variational framework of problem and give some necessary lemmas. Finally, Section 4 presents the main result and its proof.

2. Preliminaries and variational setting

In this section, we present some preliminaries and background theory on the concept of Riemann Liouville operator with respect to another function which will be used in the rest of this paper. First let us start by introduce the definition of the fractional integral in the sens of Kilbas et al. [18] and Samko et al. [23]. Throughout this section, α and β denote positive real numbers, Γ denotes the Euler gamma function, and if $-\infty \leq a < b \leq \infty$, then $[a, b]$ denotes a finite or infinite interval in the real line.

Definition 2.1. *([18, 23]) Let u be an integrable function defined on $[a, b]$, and $\psi \in C^1([a, b], \mathbb{R})$ be an increasing function such that $\psi'(x) \neq 0$, for all $x \in [a, b]$. The left and right fractional integrals of a function u with respect to another function ψ are defined respectively as follows:*

$$I_{a^+}^{\alpha,\psi} u(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} u(t) dt,$$

and

$$I_{b^-}^{\alpha,\psi} u(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t)(\psi(t) - \psi(x))^{\alpha-1} u(t) dt.$$

Definition 2.2. *([18, 23]) Let u be an integrable function defined on $[a, b]$, and $\psi \in C^1([a, b], \mathbb{R})$ be an increasing function such that $\psi'(x) \neq 0$, for all $x \in [a, b]$. The left and right fractional derivatives of a function u with respect to another function ψ are defined respectively by:*

$$\begin{aligned} D_{a^+}^{\alpha,\psi} u(x) &:= \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n I_{a^+}^{n-\alpha,\psi} u(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \int_a^x \psi'(t)(\psi(x) - \psi(t))^{n-\alpha-1} u(t) dt, \end{aligned}$$

and

$$\begin{aligned} D_{b^-}^{\alpha,\psi} u(x) &:= \left(-\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n I_{b^-}^{n-\alpha,\psi} u(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(-\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \int_x^b \psi'(t)(\psi(t) - \psi(x))^{n-\alpha-1} u(t) dt, \end{aligned}$$

where $n = [\alpha] + 1$.

Now, by interchanging the order of integration by the Dirichlet formula in the particular case Fubini theorem, we can prove the following integration by parts for the ψ-Riemann-Liouville fractional integral:

$$\int_a^b I_{a^+}^{\alpha,\psi} u(x) v(x)\, dx = \int_a^b u(x) \psi'(x) I_{b^-}^{\alpha,\psi} \left(\frac{v(x)}{\psi'(x)} \right) dx.$$

For more details, one can see Equation (16) in [29].
Also, we need the following result which called fractional integration by parts:

Lemma 2.3. *Let $\psi \in C^1([a, b], \mathbb{R})$ be an increasing function such that $\psi'(x) \neq 0$, for all $x \in [a, b]$. If u is an absolutely countinous function on $[a, b]$ and v is of class C^1 on $[a, b]$ such that $v(a) = v(b) = 0$. Then we have*

$$\int_a^b D_{a^+}^{\alpha,\psi} u(x) v(x)\, dx = \int_a^b u(x) \psi'(x) D_{b^-}^{\alpha,\psi} \left(\frac{v(x)}{\psi'(x)} \right) dx \tag{8}$$

Proof. The proof is a direct consequence of Theorem 2.4 in citevsj. □

For $1 \le r \le \infty$, $L^r(a,b)$ denotes the set of all measurable function u on $[a,b]$, such that $\int_a^b |u(t)|^r dt < \infty$. Put

$$\|u\|_{L^r(a,b)} = \left(\int_a^b |u(t)|^r dt\right)^{\frac{1}{r}}, \text{ and } \|u\|_\infty = ess \sup_{a\le t\le b} |u(t)|.$$

Remark 2.4. *([19, 29]) If $0 < \alpha \le 1$, $r \ge 1$ and $q = \frac{p}{p-1}$, then for each $\varphi \in L^r(a,b)$, we have:*

(i) $I_{a^+}^{\alpha,\psi}\varphi$ is bounded in $L^r(a,b)$, moreover we have

$$\|I_{a^+}^{\alpha,\psi}\varphi\|_{L^r(a,b)} \le \frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)}\|\varphi\|_{L^r(a,b)}.$$

(ii) If $\frac{1}{r} < \alpha < 1$, then $I_{a^+}^{\alpha,\psi}$ is Hölder continuous on $[a,b]$ with exponent $\alpha - \frac{1}{r}$.

(iii) If $\frac{1}{r} < \alpha < 1$, then $\lim_{t\to a} I_{a^+}^{\alpha,\psi}\varphi(t) = 0$. That is $I_{a^+}^{\alpha,\psi}\varphi$ can be continuously extended by zero in $t = a$. So, $I_{a^+}^{\alpha,\psi}\varphi$ is continuous on $[a,b]$, moreover, we get

$$\|I_{a^+}^{\alpha,\psi}\varphi\|_\infty \le \frac{(\psi(b)-\psi(a))^{\alpha-\frac{1}{r}}}{\Gamma(\alpha)\left((\alpha-1)q+1\right)^{\frac{1}{q}}}\|\varphi\|_{L^r(a,b)}.$$

To show the existence of solutions to the problem (3), we will use the following theorem.

Theorem 2.5. *(Mountain pass theorem) Let E be a real Banach space and $J \in C^1(E,\mathbb{R})$ satisfying the palai smail condition. Assume that that*

(i) $J(0) = 0$,

(ii) There is $\rho > 0$ and $\sigma > 0$ such that $J(z) \ge \sigma$ for all $z \in E$ with $\|z\| = \rho$.

(iii) There exists $z_1 \in E$ with $\|z_1\| \ge \rho$ such that $J(z_1) < 0$.

Then ϕ_λ possesses a critical value $c \ge \sigma$. Moreover, c can be characterized as

$$c = \inf_{\gamma\in\Gamma}\max_{z\in[0,1]} \phi_\lambda(\gamma(z)),$$

where $\Gamma = \{\gamma \in C([0,1],E_p) : \gamma(0) = 0, \gamma(1) = z_1\}$.

We notes that The functional J satisfies the Palais-Smale condition if any Palais-Smale sequence has a strongly convergent subsequnce. That is $u_m \in E$ is such that $J(u_m)$ is bounded and $J'_\lambda(u_m)$ converges to 0 in $E_p'^{\alpha,\psi}$, then $.u_m$ has a convergent subsequence.

3. The proof of the main result

In this section, in order to apply the mountain pass theorem, we begin by introduce the fractional derivative space and some other interesting results. we denote by $C^\infty([0,T],\mathbb{R})$ the set of all functions u with $u(0) = u(T) = 0$. In order to formulate the variational setting to problem (3), we define the fractional derivative space $E_p^{\alpha,\psi}$ by the closure of $C_0^\infty([0,T],\mathbb{R})$ with respect to the norm

$$\|u\|_{E_p^{\alpha,\psi}} = \left(\|u\|^p_{L^p(0,T)} + \|{}_0D_t^{\alpha,\psi}u\|^p_{L^p(0,T)}\right)^{\frac{1}{p}}.$$

Note that, we have

$$E_p^{\alpha,\psi} = \left\{v \in L^p([0,T]) :\ {}_0D_t^{\alpha,\psi}v \in L^p([0,T],\mathbb{R}), v(0) = v(T) = 0\right\}.$$

Remark 3.1. *([19, 29]) Assume that $\frac{1}{p} < \alpha < 1$, the following statements hold true:*

(i) The space $E_p^{\alpha,\psi}$ is compactly embedded in $C([0,T],\mathbb{R})$. So, if φ_k is a sequence which converges weakly to φ in $E_p^{\alpha,\psi}$, then φ_k converge strongly to φ in $C([0,T],\mathbb{R})$.

(ii) The space $E_p^{\alpha,\psi}$ is uniformly convex, reflexive and separable Banach space.

Definition 3.2. *(see [27, 28])A Banach space E is said to have the Kadec Klee property if whenever $\{u_k\}$ is a sequence in E that converges weakly to $u_* \in E$ and $\|u_k\| \to \|u_*\|$, as $k \to \infty$, then, $\{u_k\}$ converges strongly to u_*.*

Remark 3.3. *(See [10]) Every uniformly convex real Banach space is reflexiv, strictly convex and has the Kadec Klee property.*

Proposition 3.4. *If $\frac{1}{p} < \alpha$ and $q = \frac{p}{p-1}$, then for each $\varphi \in E_p^{\alpha,\psi}$ one has:*

(i)

$$\|\varphi\|_{L^r(a,b)} \leq \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\|D_{a^+}^{\alpha,\psi}\varphi\|_{L^r(a,b)}. \tag{9}$$

(ii)

$$\|\varphi\|_{\infty} \leq \frac{(\psi(b)-\psi(a))^{\alpha-\frac{1}{r}}}{\Gamma(\alpha)\left((\alpha-1)q+1\right)^{\frac{1}{q}}}\|D_{a^+}^{\alpha,\psi}\varphi\|_{L^r(a,b)}. \tag{10}$$

Proof. The proof of Proposition 3.4 is a direct consequence of Proposition 2.1 in [29]. □

Remark 3.5. *From, Equation (9), the space $E_p^{\alpha,\psi}$ can be equipped with the following equivalent norm:*

$$\|u\|_{\alpha,\psi} = \|{}_0D_t^{\alpha,\psi}u\|_{L^p(0,T)}.$$

Moreover, from (10), if $\frac{1}{p} < \alpha$, then $E_p^{\alpha,\psi}$ is continuously injected into $C([0,T],\mathbb{R})$.

Now, we are in a position to define the notion of solution.

Definition 3.6. *we say that u is a weak solution of problem (p) if for every $v \in E_p^{\alpha,\psi}$ we have :*

$$M(u(t))\int_0^T |{}_0D_t^{\alpha,\psi}u(t)|^{p-2}\,{}_0D_t^{\alpha,\psi}u(t)\,{}_0D_t^{\alpha,\psi}v(t)dt - \lambda\int_0^T f(t,u(t))v(t)dt - \int_0^T g(t,u(t))v(t)dt = 0$$

Since we use the variational method, it is natural to define the associate functional $\phi_\lambda : E_p^{\alpha,\psi} \to \mathbb{R}$, which is defined by:

$$\phi_\lambda(u) = \frac{1}{bp^2}\left(a + b\|u\|_{\alpha,\psi}^p\right)^p - \lambda\int_0^T F(t,u(t))dt - \int_0^T G(t,u(t))dt - \frac{a^p}{bp^2}.$$

Since F and G are continuous, it is not difficult to show that $\phi_\lambda \in C^1(E_p^{\alpha,\psi},\mathbb{R})$, moreover for all $u, v \in E_p^{\alpha,\psi}$, we have

$$\begin{aligned} \langle \phi_\lambda'(u), v\rangle &= \left(a + b\|u\|_{\alpha,\psi}^p\right)^{p-1}\int_0^T \varphi_p({}_0D_t^{\alpha,\psi}u(t))\,{}_0D_t^{\alpha,\psi}v(t)dt \\ &- \lambda\int_0^T f(t,u(t))v(t)dt - \int_0^T g(t,u(t))v(t)dt \end{aligned} \tag{11}$$

So, critical points of ϕ_λ are solutions of problem (3).
In order to prove that ϕ_λ satisfies the mountain pass geometry, we need to prove the following lemmas.

Lemma 3.7. *If conditions* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *are satisfied, and* $\min(\delta, q) > p$, *then There is* $\rho > 0$ *and* $\sigma > 0$ *such that* $\phi_\lambda(z) \geq \sigma$ *for all* $z \in E_p^{\alpha,\psi}$ *with* $\|z\| = \rho$.

Proof. Let $z \in E_p^{\alpha,\psi}$, then, by equations (5),(7), proposition 3.4 and Remark 3.5, we obtain

$$\begin{aligned}
\phi_\lambda(z) &= \frac{1}{bp^2}\left(a + b\|z\|_{\alpha,\psi}^p\right)^p - \lambda\int_0^T F(t, z(t))dt - \int_0^T G(t, z(t))dt - \frac{a^p}{bp^2}\\
&\geq \frac{1}{bp^2}\left(a + b\|z\|_{\alpha,\psi}^p\right)^p - \lambda C_0\int_0^T |z|^\delta dt - C_1\int_0^T |z|^q dt - \frac{a^p}{bp^2}\\
&\geq \frac{1}{bp^2}\left(a + b\|z\|_{\alpha,\psi}^p\right)^p - \lambda C_0 T\left(\frac{(\psi(T) - \psi(0))^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)((\alpha - 1)q + 1)^{\frac{1}{q}}}\right)^\delta \|z\|_{\alpha,\psi}^\delta\\
&- C_1 T\left(\frac{(\psi(T) - \psi(0))^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)((\alpha - 1)q + 1)^{\frac{1}{q}}}\right)^q \|z\|_{\alpha,\psi}^q - \frac{a^p}{bp^2}.
\end{aligned}$$

Assume that $\|z\|_{\alpha,\psi} = \rho > 0$, then, using the elementary inequality

$$(x + y)^p \geq x^p + pyx^{p-1},$$

we obtain

$$\begin{aligned}
\phi_\lambda(z) &\geq \frac{1}{bp^2}(a + b\rho^p)^p - \lambda C_0 T\left(\frac{(\psi(T) - \psi(0))^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)((\alpha - 1)q + 1)^{\frac{1}{q}}}\right)^\delta \rho^\delta\\
&- C_1 T\left(\frac{(\psi(T) - \psi(0))^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)((\alpha - 1)q + 1)^{\frac{1}{q}}}\right)^q \rho^q - \frac{a^p}{bp^2}\\
&\geq \frac{\rho^p a^{p-1}}{p} - \lambda C_0 T\left(\frac{(\psi(T) - \psi(0))^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)((\alpha - 1)q + 1)^{\frac{1}{q}}}\right)^\delta \rho^\delta - C_1 T\left(\frac{(\psi(T) - \psi(0))^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)((\alpha - 1)q + 1)^{\frac{1}{q}}}\right)^q \rho^q\\
&\geq \rho^p \chi(\rho).
\end{aligned}$$

where

$$\chi(t) = \frac{a^{p-1}}{p} - \lambda C_0 T\left(\frac{(\psi(T) - \psi(0))^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)((\alpha - 1)q + 1)^{\frac{1}{q}}}\right)^\delta t^{\delta - p} - C_1 T\left(\frac{(\psi(T) - \psi(0))^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)((\alpha - 1)q + 1)^{\frac{1}{q}}}\right)^q t^{q-p}.$$

Since $\min(\delta, q) > p$ and

$$\lim_{\rho \to 0}\chi(\rho) = \frac{a^{p-1}}{p} > 0,$$

then, we can choose $\rho > 0$ small enough such that

$$\rho^p \chi(\rho) := \sigma > 0$$

So we have

$$\phi_\lambda(z) \geq \sigma > 0.$$

This completes the proof of Lemma 3.7 □

Lemma 3.8. *If hypotheses* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *are fulfilled, then, there exists* $z_1 \in E_p^{\alpha,\psi}$ *with* $\|z_1\| \geq \rho$ *such that* $\phi_\lambda(z_1) < 0$.

Proof. Let $\xi > 0$ large enough such that $a < \xi^p$, then using hypothesis $(\mathbf{H_2})$, Equation (5) and Proposition 3.4, we obtain for all $u \in E_p^{\alpha,\psi}$

$$\begin{aligned}
\phi_\lambda(\xi u) &= \frac{1}{bp^2}\left(a + b\|\xi u\|^p_{\alpha,\psi}\right)^p - \lambda\int_0^T F(t,\xi u(t))dt - \int_0^T G(t,\xi u(t))dt - \frac{a^p}{bp^2} \\
&\le \frac{\xi^{p^2}}{bp^2}\left(1 + b\|u\|^p_{\alpha,\psi}\right)^p - \xi^q\int_0^T G(t,u(t))dt - \frac{a^p}{bp^2}
\end{aligned}$$

Since we have $q > p^2$, then we have $\lim_{\xi\to\infty}\phi_\lambda(\xi u) = -\infty$. So we can choose $\xi_0 > a^{\frac{1}{p}}$ large enough such that $z_1 = \xi_0 u$, satisfy $\|z_1\| \ge \rho$ such that $\phi_\lambda(z_1) < 0$. $\square$

Lemma 3.9. *Suppose that the hypotheses* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *are satisfied, then, The functional* ϕ_λ *satisfies the Palais-Smale condition*

Proof. Let $\{u_k\} \in E_p^{\alpha,\psi}$ be a palai smail sequence, that is $\phi_\lambda(u_k)$ is bounded and $\phi'_\lambda(u_k)$ tends to zero as k tends to infinity. So, there exist $C_2 > 0$ and $C_3 > 0$ such that

$$|\phi_\lambda(u_k)| \le C_2, \text{ and } |\langle\phi'_\lambda(u_k), u_k\rangle| < C_3. \tag{12}$$

We begin by proving that $\{u_k\}$ is bounded. If not, up to a subsequence still denoted by $\{u_k\}$, we can assume that $\|u_k\| \to \infty$. From (11),we have

$$\begin{aligned}
\langle\phi'_\lambda(u_k), u_k\rangle &= \left(a + b\|u_k\|^p_{\alpha,\psi}\right)^{p-1}\|u_k\|^p - \lambda\int_0^T f(t,u_k(t))u_k(t)dt \\
&- \int_0^T g(t,u_k(t))u_k(t)dt.
\end{aligned}\tag{13}$$

by using (4),(6) and (12), we obtain

$$\begin{aligned}
C_2 &\ge \phi_\lambda(u_k) \\
&= \frac{1}{bp^2}\left(a + b\|u_k\|^p_{\alpha,\psi}\right)^p - \lambda\int_0^T F(t,u_k(t))dt - \int_0^T G(t,u_k(t))dt - \frac{a^p}{bp^2} \\
&= \frac{1}{bp^2}\left(a + b\|u_k\|^p_{\alpha,\psi}\right)^p - \frac{\lambda}{\delta}\int_0^T f(t,u_k(t))u_k dt - \frac{1}{q}\int_0^T g(t,u_k(t))u_k dt - \frac{a^p}{bp^2} \\
&\ge \frac{1}{bp^2}\left(a + b\|u_k\|^p_{\alpha,\psi}\right)^p - \frac{\lambda}{\max(\delta,q)}\int_0^T f(t,u_k(t))u_k dt \\
&\quad -\frac{1}{\max(\delta,q)}\int_0^T g(t,u_k(t))u_k dt - \frac{a^p}{bp^2}
\end{aligned}\tag{14}$$

On the other hand,from (12) and (13), we get

$$\begin{aligned}
C_3 &> |\langle\phi'_\lambda(u_k), u_k\rangle| \\
&\ge -\langle\phi'_\lambda(u_k), u_k\rangle \\
&= -\left(a + b\|u_k\|^p_{\alpha,\psi}\right)^{p-1}\|u_k\|^p \\
&+ \lambda\int_0^T f(t,u_k(t))u_k(t)dt + \int_0^T g(t,u_k(t))u_k(t)dt.
\end{aligned}\tag{15}$$

By combining (14) with (15), we get

$$\begin{aligned}
\max(\delta, q)C_2 + C_3 &\geq \max(\delta, q)\frac{1}{bp^2}\left(a + b\|u_k\|^p_{\alpha,\psi}\right)^p - \left(a + b\|u_k\|^p_{\alpha,\psi}\right)^{p-1}\|u_k\|^p \\
&- (\max(\delta, q) - 1)\frac{a^p}{bp^2} \\
&= \left(a + b\|u_k\|^p_{\alpha,\psi}\right)^{p-1}\left[\max(\delta, q)\frac{1}{bp^2}(a + b\|u_k\|^p_{\alpha,\psi}) - \|u_k\|^p_{\alpha,\psi}\right] \\
&- -(\max(\delta, q) - 1)\frac{a^p}{bp^2} \\
&= \left(a + b\|u_k\|^p_{\alpha,\psi}\right)^{p-1}\left[\frac{a\max(\delta, q)}{bp^2} + (\frac{\max(\delta, q)}{p^2} - 1)\|u_k\|^p_{\alpha,\psi})\right] \\
&- -(\max(\delta, q) - 1)\frac{a^p}{bp^2}.
\end{aligned}$$

Since $\max(\delta, q) > p^2$, then by letting k tends to infinity we obtain a contradiction. So, $\{u_k\}$ is bounded. Therefore, from Remark 3.1, there exists $u_* \in E^{\alpha,\psi}_p$ such that, up to a subsequence, we have

$$\begin{cases} u_k \rightharpoonup u, \text{ weakly in } E^{\alpha,\psi}_p, \\ u_k \to u, \text{ in } C([0, T], \mathbb{R}). \end{cases}$$

From (11), we get

$$\begin{aligned}
\langle \phi'_\lambda(u_k) - \phi'_\lambda(u_*), u_k - u_*\rangle &= \left(a + b\|u_k\|^p_{\alpha,\psi}\right)^{p-1}\int_0^T \varphi_p({}_0D^{\alpha,\psi}_t u_k(t))_0D^{\alpha,\psi}_t(u_k(t) - u_*(t))dt \\
&\quad - \left(a + b\|u_*\|^p_{\alpha,\psi}\right)^{p-1}\int_0^T \varphi_p({}_0D^{\alpha,\psi}_t u_*(t))_0D^{\alpha,\psi}_t(u_k - u_*(t))dt \\
&\quad -\lambda\int_0^T \left(f(t, u_k(t) - f(t, u_*(t))\right)(u_k(t) - u_*(t))dt \\
&\quad -\lambda\int_0^T \left(g(t, u_k(t) - g(t, u_*(t))\right)(u_k(t) - u_*(t))dt \\
&= \left(a + b\|u_k\|^p_{\alpha,\psi}\right)^{p-1}\left(\|u_k\|^p_{\alpha,\psi} - \int_0^T \varphi_p({}_0D^{\alpha,\psi}_t u_k(t))_0D^{\alpha,\psi}_t u_*(t)dt\right) \\
&+ \left(a + b\|u_*\|^p_{\alpha,\psi}\right)^{p-1}\left(\|u_k\|^p_{\alpha,\psi} - \int_0^T \varphi_p({}_0D^{\alpha,\psi}_t u_*(t))_0D^{\alpha,\psi}_t u_k(t)dt\right) \\
&\quad -\lambda\int_0^T \left(f(t, u_k(t) - f(t, u_*(t))\right)(u_k(t) - u_*(t))dt \\
&\quad -\lambda\int_0^T \left(g(t, u_k(t) - g(t, u_*(t))\right)(u_k(t) - u_*(t))dt.
\end{aligned}$$

Using the Hölder inequality, we have

$$\int_0^T \varphi_p({}_0D^{\alpha,\psi}_t u_k(t))_0D^{\alpha,\psi}_t u_*(t)dt \leq \|u_k\|^{p-1}_{\alpha,\psi}\|u_*\|^p_{\alpha,\psi},$$

$$\int_0^T \varphi_p({}_0D^{\alpha,\psi}_t u_*(t))_0D^{\alpha,\psi}_t u_k(t)dt \leq \|u_*\|^{p-1}_{\alpha,\psi}\|u_k\|^p_{\alpha,\psi}.$$

Hence, we obtain

$$\begin{aligned}\langle \phi'_\lambda(u_k) - \phi'_\lambda(u_*), u_k - u_*\rangle \quad \geq \quad & \left(a + b\|u_k\|^p_{\alpha,\psi}\right)^{p-1}\left(\|u_k\|^p_{\alpha,\psi} - \|u_k\|^{p-1}_{\alpha,\psi}\|u_*\|_{\alpha,\psi}\right)\\ & + \left(a + b\|u_*\|^p_{\alpha,\psi}\right)^{p-1}\left(\|u_k\|^p_{\alpha,\psi} - \|u_*\|^{p-1}_{\alpha,\psi}\|u_k\|_{\alpha,\psi}\right)\\ & -\lambda\int_0^T \Big(f(t,u_k(t) - f(t,u_*(t)\Big)(u_k(t) - u_*(t))dt\\ & -\lambda\int_0^T \Big(g(t,u_k(t) - g(t,u_*(t)\Big)(u_k(t) - u_*(t))dt\\ \geq \quad & \left(a + b\|u_k\|^p_{\alpha,\psi}\right)^{p-1}\|u_k\|^{p-1}_{\alpha,\psi}\left(\|u_k\|_{\alpha,\psi} - \|u_*\|_{\alpha,\psi}\right)\\ & + \left(a + b\|u_*\|^p_{\alpha,\psi}\right)^{p-1}\|u_*\|^{p-1}_{\alpha,\psi}\left(\|u_k\|_{\alpha,\psi} - \|u_k\|_{\alpha,\psi}\right)\\ & -\lambda\int_0^T \Big(f(t,u_k(t) - f(t,u_*(t)\Big)(u_k(t) - u_*(t))dt\\ & -\lambda\int_0^T \Big(g(t,u_k(t) - g(t,u_*(t)\Big)(u_k(t) - u_*(t))dt. \end{aligned} \tag{16}$$

Since $u_k \to u,$ in $C([0,T],\mathbb{R})$, and $|f(t,u_k(t) - f(t,u_*(t)|$, $|g(t,u_k(t) - g(t,u_*(t)|$, are bounded, then we get

$$\int_0^T \Big(f(t,u_k(t) - f(t,u_*(t)\Big)(u_k(t) - u_*(t))dt \to 0, \text{ as } k \to \infty, \tag{17}$$

and

$$\int_0^T \Big(g(t,u_k(t) - g(t,u_*(t)\Big)(u_k(t) - u_*(t))dt \to 0, \text{ as } k \to \infty. \tag{18}$$

On the other hand, since $u_k \rightharpoonup u,$ weakly in $E_p^{\alpha,\psi}$, and $\phi'_\lambda(u_k) \to 0$, then we obtain

$$\langle \phi'_\lambda(u_k) - \phi'_\lambda(u_*), u_k - u_*\rangle \to 0, \text{ as } k \to \infty. \tag{19}$$

By combining Equations (17), (18), (19) with Equation (16), we obtain

$$\|u_k\|_{\alpha,\psi} \to \|u_*\|_{\alpha,\psi}, \text{ as } k \to \infty. \tag{20}$$

Finally, by combining Remarks 3.1, 3.3, Definition 3.2 with Equation (20), we obtain that $u_k \to u,$ strongly in $E_p^{\alpha,\psi}$. this ends the proof of Lemma 3.9. □

Now, we are ready to prove the main result of this paper.
Proof of Theorem 1.1 First of all, it is easy that $\phi_\lambda(0) = 0$.
Now, from Lemma 3.7, There exist $\rho > 0$ and $\sigma > 0$ such that for all $z \in E_p^{\alpha,\psi}$, if $\|z\| = \rho$, then we have

$$\phi_\lambda(z) \geq \sigma > 0. \tag{21}$$

On the other hand, from Lemma 3.8, there exists $z_1 \in E_p^{\alpha,\psi}$ satisfying

$$\|z_1\| \geq \rho, \text{ and } \phi_\lambda(z_1) < 0. \tag{22}$$

By combining Equations (21) and (22) with Lemma 3.9, we conclude that all hypothesis of the mountain pass theorem (Theorem 2.5) are satisfied. So, we can deduce the existence of a critical point u of ϕ_λ, which is a weak solution for problem (3). Moreover, Equation (21) implies that u is nontrivial. The proof of Theorem1.1, is now completed.

References

[1] R. Agarwal, M. Benchohra, S. Hamani, Boundary value problems for fractional differential equations, Georg. Math. J. 16(3)(2009), 401-411.
[2] O. Agrawal, J. Tenreiro Machado, J. Sabatier, Fractional derivatives and their application, Nonlinear dynamics, Springer-Verlag, Berlin, 2004.
[3] A. Ahmadova, N. I. Mahmudov, Asymptotic stability analysis of Riemann-Liouville fractional stochastic neutral differential equations, Miskolc Mathematical Notes, 22 (2)(2021), 503-520.
[4] A. O. Akdemir, A. Karaoğlan, M. A. Ragusa, E. Set, Fractional integral inequalities via Atangana-Baleanu operators for convex and concave functions, Journal of Function Spaces, vol. 2021, art. ID 1055434, (2021).
[5] R. Almeida, Further properties of osler′s generalized fractional integrals and derivatives with respect to another function, Rocky Mountain J. Math., 49(8)(2019), 2459-2493.
[6] K. Ben Ali, A. Ghanmi, K. Kefi, Existence of solutions for fractional differential equations with Dirichlet boundary conditions, Electron. J. Differ. Equ. 2016(2016), 1-11.
[7] T. L. César, Boundary value problem with fractional p-Laplacian operator, Advances in Nonlinear Analysis, 5, (2)(2016), 133-146.
[8] T. L. César, Existence and concentration of solutions for a nonlinear fractional Schödinger equation with steep potential well, Commun. Pure Appl. Anal., 15(2016), 535-547.
[9] T. L. César, Existence of solution for a general fractional advection-dispersion equation, Anal. Math. Phys., 9(3)(2019), 1303-1318.
[10] C.E. Chidume, S. I. Ikechukwu, A. Adamu, Inertial algorithm for approximating a common fixed point for a countable family of relatively nonexpansive maps, Fixed Point Theory Appl. (2018) 2018:9; https://doi.org/10.1186/s13663-018-0634-3
[11] W. Chen, G. Pang, A new definition of fractional Laplacian with application to modeling three-dimensional nonlocal heat conduction, J. Comput. Phys. 309 (2016), 350-367.
[12] Y. Cho, I. Kim, D. Sheen, A fractional-order model for MINMOD millennium, Math. Biosci. 262 (2015), 36-45.
[13] A. Ghanmi, M. Althobaiti, Existence results involving fractional Liouville derivative, Bol. Soc. Parana. Mat. 39(5)(2021), 93-102.
[14] A. Ghanmi, S. Horrigue Existence of positive solutions for a coupled system of nonlinear fractional differential equations, Ukr. Math. J. 71 (1)(2019), 39-49.
[15] A. Ghanmi, Z. Zhang, Nehari manifold and multiplicity results for a class of fractional boundary value problems with p-Laplacian, Bull. Korean Math. Soc. 56(5)(2019), 1297-1314.
[16] N.M. Grahovac, M. M. Žigič, Modelling of the hamstring muscle group by use of fractional derivatives, Comput. Math. Appl. 59 (5)(2010), 1695-1700.
[17] F. Jiao, Y. Zhou, Existence results for fractional boundary value problem via critical point theory, Intern. Journal of Bif. and Chaos, 22(4)(2012), 1-17.
[18] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, vol. 207, Elsevier Science B.V., Amsterdam, 2006.
[19] C. T. Ledesma, Mountain pass solution for a fractional boundary value problem, J Fract Calc Appl. 5(1)(2014), 1–10.
[20] R.L. Magin, M. Ovadia, Modeling the cardiac tissue electrode interface using fractional calculus, J. Vib. Control 14 (9)(2008), 1431-1442.
[21] K. Miller and B. Ross, An introduction to the fractional calculus and fractional differential equations, Wiley and Sons, New York, 1993.
[22] Y.A. Rossikhin, M.V. Shitikova, Analysis of two colliding fractionally damped spherical shells in modelling blunt human head impacts, Cent. Eur. J. Phys. 11(6) (2013), 760-778.
[23] S. G. Samko, A. A. Kilbas, O. I. Marichev, Fractional integrals and derivatives, theory and functions, (1993) , Gordon and Breach, Yverdon.
[24] E. Sen, Transmission problem for the Sturm-Liouville equation involving a retarded argument, Filomat, 35 (6)(2021), 2071-2080.
[25] C. Torres, Mountain pass solution for a fractional boundary value problem, J. Fract. Calculus Appli., 5(1) (2014), 1-10.
[26] J. V. da C. Sousa, E. C. de Oliveira, On the ψ-fractional integral and applications, Comp. Appl. Math. 38(4)(2019), https://doi.org/10.1007/s40314-019-0774-z.
[27] W. Takahashi, Convex Analysis and Approximation of Fixed Points, Yokohama Publishers, Yokohama (2000). (Japanese)
[28] W. Takahashi, Nonlinear Functional Analysis, Yokohama Publishers, Yokohama (2000).
[29] J. Vanterler da C. Sousa , Jiabin Zuo, Donal O'Regand, The Nehari manifold for a ψ-Hilfer fractional p-Laplacian, Applicable Analysis (2021), DOI: 10.1080/00036811.2021.1880569.
[30] J. V. da C. Sousa, L. S. Tavares, C. E. T. Ledesma, A variational approach for a problem involving a ψ-Hilfer fractional operator, J. Appl. Anal. Comput., 11(3)(2021), 1610-1630.