Four dimensional matrix mappings on double summable spaces

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Abstract. In a previous paper [9], some classes of triangular matrix transformations between the series spaces summable by the absolute weighted summability methods were characterized. In the present paper, we extend these classes to four dimensional matrices and double summability methods.

1. Introduction

Consider an infinite single series $\Sigma x_v$ of complex or real numbers with partial sums $s_n$ and let $\sigma^\alpha_n$ denote the $n$-th term of the Cesàro mean of order $\alpha > -1$ of the sequence $(s_n)$. The series $\Sigma x_v$ is summable $|C,\alpha|^k$, $k \geq 1$, in Flett’s notation (see [4]), if $\left( n^{1-1/k} \Delta \sigma^\alpha_n \right) \in \ell_k$, where $\ell_k$ is the set of absolutely $k$-summable sequences. Further let $\left( \phi_n \right)$ be a sequence of positive numbers and $(p_n)$ be a sequences of positive numbers satisfying

$$P_n = p_0 + p_1 + \ldots + p_n \to \infty \text{ as } n \to \infty, \quad P_{-1} = p_{-1} = 0.$$  \hfill (1)

By $T_n$, we denote the $n$-th term of weighted mean $\left( \overline{N}, p_n \right)$ of the sequence of $(s_n)$, i.e.

$$T_n = \sum_{v=0}^{n} p_v s_v / P_n.$$ 

The series $\Sigma x_v$ is said to be summable $\left| \overline{N}, p_n, \phi_n \right|_k$, $k \geq 1$, if (see [15]) $\left( \phi_n n^{1-1/k} \Delta T_n \right) \in \ell_k$, which reduces to the methods $\left| \overline{N}, p_n \right|_k$ and $\left| R, p_n \right|_k$ for $\phi_n = p_n/p_n$ and $\phi_n = n$ (see [2] and [12], respectively).

For $k \geq 1$, the space $\left| \overline{N}, p_n, \phi_n \right|_k$, the set of all series summable by the method $\left| \overline{N}, p_n, \phi_n \right|_k$, is a Banach space (see [9], [14]) according to the norm

$$\|x\|_{\overline{N}, p_n, \phi_n} = \left( |x_0|^k + \sum_{n=1}^{\infty} \phi^{k-1}_n \left( \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} x_v \right)^{1/k} \right).$$

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Further, a series \( \sum x_n \) is summable \( [N, p, \phi_n] \) if a sequence \( x = (x_n) \in [N, p] \), and the space \([N, p] \) is the same as the spaces \([N, p] \) and \([N, q] \) for \( \phi_n = p_n/p_n \) and \( \gamma_n = n \) (see [14] and [12], respectively).

We denote the set of all infinite triangular matrices which map a single sequence space \( X \) to another sequence space \( Y \) by \( (X, Y) \). The following characterizations of matrix classes are well known (see [9]), which include some known corollaries and applications for particular matrices (see [3, 5, 10-14, 16]).

Throughout the paper \( k' \) will denote the conjugate of \( k \), i.e., \( 1/k + 1/k' = 1 \) for \( k > 1, 1/k' = 0 \) for \( k = 1 \).

**Theorem 1.1.** Let \((p_n)\) and \((q_n)\) be positive sequences satisfying (1). Further, let \( A = (a_{m,n}) \) be an infinite triangular matrix and \((\phi_n)\) be a sequence of positive numbers. Then, \( A \in \left( [N, p], [N, q] \right) \), for the case \( 1 \leq k < \infty \), if and only if

\[
\frac{p_n a_{n}}{p_n a_{n}} \phi_n^{1/k} a_{n} = O(1)
\]

(2)

\[
\sum_{n=1}^{\infty} q_n^{-1} a_{n} \sum_{r=1}^{n} q_{r-1} \left( a_{r} - a_{r+1} \right) = O\left( \left( \frac{p_n}{p_n} \right)^{1/k} \right)
\]

(3)

\[
\sum_{n=1}^{\infty} q_n^{-1} a_{n} \sum_{r=1}^{n} q_{r-1} a_{r+1} = O\left( \left( \frac{p_n}{p_n} \right)^{1/k} \right)
\]

(4)

where

\[
\mu_n = \frac{q_n}{Q_n Q_{n-1}}, \quad n \geq 1
\]

(5)

**Theorem 1.2.** Let \((p_n)\) and \((q_n)\) be positive sequences satisfying (1). Further, let \( A = (a_{m,n}) \) be an infinite triangular matrix and \((\phi_n)\) be a sequence of positive numbers. Then, \( A \in \left( [N, p], [N, q] \right) \), for the case \( 1 < k < \infty \), if and only if

\[
\sum_{n=1}^{\infty} p_n^{-k} \left( \sum_{n=1}^{\infty} \sum_{r=1}^{n} q_{r-1} \left( p_{r} a_{r} - p_{r+1} a_{r+1} \right) \right)^{k'} < \infty
\]

(6)

where \( \mu_n \) is defined by (5).

In the present paper we establish Theorem 1.1 and Theorem 1.2 for four dimensional matrices and double summability, which extend earlier factor and inclusion results on absolute weighted summability to double summability.

### 2. Absolute double weighted summability

For any double sequence \((x_{rs})\) and four dimensional sequence \((y_{m,n,rs})\), we write for \( m, n, r, s \geq 0 \),

\[
\begin{align*}
\Delta_1 x_{rs} &= x_{r,s} - x_{r-1,s} & \Delta_2 x_{rs} &= x_{r,s} - x_{r,s-1} \\
\Delta_3 x_{rs} &= \Delta_2 (\Delta_1 x_{rs}), & x_{r,0} &= x_{0,0} = 0 \\
\Delta_1 y_{m,n,rs} &= y_{m,n,rs} - y_{m,n,r,s-1} & \Delta_2 y_{m,n,rs} &= y_{m,n,rs} - y_{m,n,r,s-1} \\
\Delta_3 y_{m,n,rs} &= \Delta_2 (\Delta_1 y_{m,n,rs}), & y_{m,n,0,s} &= y_{m,n,0,s-1} = 0,
\end{align*}
\]

We use the notations \( \sum_{r,s=0}^{\infty} \) and \( \sum_{r,s=0}^{n} \) instead of \( \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \) and \( \sum_{r=0}^{n} \sum_{s=0}^{n} \), respectively, and also
Let us denote the double weighted mean \( P \) the double series \( \sum_{r,s} \), we shall say that the series \( P \) let

\[
\mu'_{mn} = \begin{cases} 
\sum_{r,s} p_r q_s, & n = 0, m \geq 1 \\
\sum_{r,s} \frac{p_r q_s}{Q_{r,s}}, & m = 0, n \geq 1 \\
\sum_{r,s} \frac{p_r q_s}{Q_{r,s} Q_{r+s}}, & m \geq 1, n \geq 1.
\end{cases}
\]

(7)

Let \( \sum_{r,s=0}^{\infty} x_{rs} \) be an infinite double series with partial sums \( s_{mn} \), i.e.,

\[
s_{mn} = \sum_{r,s=0}^{m,n} x_{rs}
\]

Let us denote the double weighted mean \( \< N, p_m, p_n \> \) of the double sequence \( (s_{mn}) \) by

\[
T_{mn} = \frac{1}{p_m q_n} \sum_{r,s=0}^{m,n} p_r q_s s_{rs}
\]

we shall say that the series \( \sum_{r,s=0}^{\infty} x_{rs} \) is called summable \( \| N, p_m, q_n ; \gamma_{mn} \|_k \), \( k \geq 1 \), if

\[
\sum_{m,n=0}^{\infty} \gamma_{mn}^{k-1} |\Delta_{21} T_{mn}|^k < \infty.
\]

(9)

It may be noticed this method reduces to the methods \( \| N, p_m, q_n \|_k \), \( | R, p_m, q_n |_k \) and \( | C, 1, 1 |_k \) for \( \gamma_{mn} = P_m Q_n / p_m q_n \), \( \gamma_{mn} = mn \) and \( p_n = q_n = 1 \), respectively, \([8],[6-7] \).

Now, by \( \| N, p_m, q_n \|_k \), we introduce the set of all double series summable by the method \( \| N, p_m, q_n ; \gamma_{mn} \|_k \). Then, the double series \( \sum_{r,s=0}^{\infty} x_{rs} \) is summable \( \| N, p_m, q_n ; \gamma_{mn} \|_k \) if and only if a double sequence \( x = (x_{rs}) \in \| N, p_m, q_n \|_k \). Further, since, for \( m, n \geq 0 \)

\[
T_{mn} = \frac{1}{p_m q_n} \sum_{r,s=0}^{m,n} p_r q_s s_{rs} = \frac{1}{p_m q_n} \sum_{r,s=0}^{m,n} p_r q_s \sum_{r,s=0}^{m,n} x_{rs}
\]

\[
= \frac{1}{p_m q_n} \sum_{r,s=0}^{m,n} x_{rs} \sum_{r,s=0}^{m,n} p_r q_s
\]

\[
= \frac{1}{p_m q_n} \sum_{r,s=0}^{m,n} x_{rs} (p_m - p_{r-1})(q_n - q_{s-1})
\]

\[
= \sum_{r,s=0}^{m,n} x_{rs} \left( 1 - \frac{p_{r-1}}{p_m} \right) \left( 1 - \frac{q_{s-1}}{q_n} \right),
\]

it is easily seen that \( \Delta_1 T_{00} = \Delta_2 T_{00} = \Delta_2 T_{00} = x_{00} \) and, for \( m, n \geq 1 \),

\[
\Delta_1 T_{m0} = \frac{p_m}{p_m p_{m-1}} \sum_{r=1}^{m} P_{r-1} x_{r0}
\]

\[
\Delta_2 T_{0n} = \frac{q_n}{Q_n Q_{n-1}} \sum_{s=1}^{n} Q_{s-1} x_{0s}
\]

\[
\Delta_2 T_{mn} = \frac{p_m q_n}{p_m p_{m-1} Q_n Q_{n-1}} \sum_{r,s=1}^{m,n} P_{r-1} Q_{s-1} x_{rs}.
\]

(10)
Define the following space which plays an important role in this paper
\[ \pi \left[ \mathcal{N}_{p,l}^r \right] = \left\{ x = (x_r) \in \mathcal{N}_{p,l}^r : x_{r0} = x_{s0} = 0 \text{ for } r, s \geq 0 \right\} \]
Hence it is routine to verify that \( \pi \left[ \mathcal{N}_{p,l}^r \right] \) and \( \pi \left[ \mathcal{N}_{p,l}^s \right] \) are a Banach space according to the norm
\[ ||x||_{\mathcal{N}_{p,l}^r} = \left( \sum_{n,m=0}^{\infty} \gamma_{mn}^{k-1} |A_{21} T_{mn} k| \right)^{1/k} . \] (11)

Also, there is a close relationship between the spaces \( \mathcal{N}_{p,l}^r \) and \( \mathcal{L}_k \), i.e., \( (x_r) \in \mathcal{N}_{p,l}^r \) if and only if \( \left( \frac{1}{r'} A_{21} T_{mn} \right) \in \mathcal{L}_k \), where \( \mathcal{L}_k \) is the set of all double sequences \( (x_r) \) of complex numbers such that \( \sum_{r,s=0}^{\infty} |x_{rs}|^{k} < \infty \), the case \( k = 1 \) of which reduces to the space \( \mathcal{L} \) studied by Zeltser [18]. The space \( \mathcal{L}_k \), \( 1 \leq k < \infty \), is a Banach space [1] according to the natural norm
\[ ||x||_{\mathcal{L}_k} = \left( \sum_{r,s=0}^{\infty} |x_{rs}|^{k} \right)^{1/k} \]
and the space \( \mathcal{L}_\infty \) of all bounded double sequences is also a Banach space with the norm \( ||x||_{\mathcal{L}_\infty} = \sup_{r,s} |x_{rs}| \).

Let \( x = (x_r) \) be a double sequence. If for every \( \varepsilon > 0 \) there exists a natural integer \( n_0(\varepsilon) \) and real number \( l \) such that \( |x_{rs} - l| < \varepsilon \) for all \( r, s \geq n_0(\varepsilon) \), then, the double sequence \( x = (x_r) \) is said to be convergent in the Peringsheim’s sense. Also, a double series \( \sum_{r,s=0}^{\infty} x_{rs} \) is convergent if and only if the double sequence of partial sums of series is convergent.

Let \( U \) and \( V \) be double sequence spaces and \( A = (a_{mrs}) \) be a four dimensional infinite matrix of complex (or, real) numbers. Then, \( A \) defines a matrix transformation from \( U \) to \( V \), written \( A \in (U, V) \), if for every sequence \( x = (x_r) \in U \) the \( A \)-transform \( A(x) = (A_{mrs}(x)) \) of \( x \) is well defined and belongs to \( V \), where
\[ A_{mrs}(x) = \sum_{r,s=0}^{\infty} a_{mrs} x_{rs} \]
provided the double series in the right hand side converges for \( m, n \geq 0 \).

The transpose \( A^t = (a_{trmn}) \) of the matrix \( A = (a_{mrs}) \) is defined by
\[ A^t_{rs}(x) = \sum_{m,n=0}^{\infty} a_{mrs} x_{mn} \text{ for } m, n \geq 0. \]

The \( \beta \)-dual \( U_{\beta} \) of the space \( U \) is the set of all double sequences \( (b_{rs}) \) such that \( \sum_{r,s=0}^{\infty} b_{rs} x_{rs} \) converges for all \( x \in U \).

An infinite four dimensional matrix \( A = (a_{mrs}) \) is called triangular if \( a_{mrs} = 0 \) for \( r > m \) or \( s > n \).

We require the following lemmas for the proof of our theorems.

**Lemma 2.1.** ([18]). If \( T \) is a linear mapping from a Banach space \( X \) into a Banach space \( Y \), then \( T \) is continuous if and only if it is bounded, i.e., there exists a constant \( L \) such that
\[ ||T(x)||_Y \leq L ||x||_X \text{ for all } x \in X \]

**Lemma 2.2.** Let \( 1 < k < \infty \) and \( A = (a_{mni}) \) be an infinite four dimensional matrix. Define \( W_k(A) \) and \( w_k(A) \) by
\[ W_k(A) = \sum_{r,s=0}^{\infty} \left( \sum_{m,n=0}^{\infty} |a_{mrs}| \right)^k, \]
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\[ w_k(A) = \sup_{M, N} \sum_{0}^{\infty} \left| \sum_{(m, n) \in M \times N} a_{mn} x_{rs} \right| k \]

where the supremum is taken through all finite subsets \( M \) and \( N \) of the natural numbers. Then, the following statements are equivalent:

(i) \( W_k(A) < \infty \)  \quad (ii) \( A \in (\mathcal{L}_k, \mathcal{L}) \)

(iii) \( A' \in (\mathcal{L}_{\infty}, \mathcal{L}_k) \)  \quad (iv) \( w_k(A) < \infty \).

**Proof.** To prove the lemma, it is enough to show that (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv) \( \Rightarrow \) (i).

(i) \( \Rightarrow \) (ii). Assume (i) holds. Then, for all \( x \in \mathcal{L}_k \), it follows from Hölder’s inequality that

\[
\|A(x)\|_L = \sum_{m,n=0}^{\infty} \sum_{r,s=0}^{\infty} |a_{mn} x_{rs}| \leq \sum_{r,s=0}^{\infty} \sum_{m,n=0}^{\infty} |a_{mn} x_{rs}| \leq \left( \sum_{r,s=0}^{\infty} \sum_{m,n=0}^{\infty} |a_{mn} x_{rs}|^k \right)^{1/k} ||x||_{\mathcal{L}_k}
\]

\[
\leq (W_k(A))^{1/k} ||x||_{\mathcal{L}_k} < \infty,
\]

which gives (ii).

(ii) \( \Rightarrow \) (iii). Suppose \( A \in (\mathcal{L}_k, \mathcal{L}) \). Then, since \( \mathcal{L}_k \) is a Banach space for \( k \geq 1 \), by Lemma 2.1, there exists a constant \( L \) such that

\[
\|A(x)\|_L = \sum_{m,n=0}^{\infty} \sum_{r,s=0}^{\infty} a_{mn} x_{rs} \leq L \|x\|_{\mathcal{L}_k}
\]

for all \( x \in \mathcal{L}_k \). Also, it is observed by putting \( x_{rs} sgn a_{mn} \) instead of \( x_{rs} \) that

\[
\sum_{m,n=0}^{\infty} \sum_{r,s=0}^{\infty} |a_{mn} x_{rs}| \leq L \|x\|_{\mathcal{L}_k}
\]

Now, let \( u \in \mathcal{L}_{\infty} \) be given. Then, by (12),

\[
\left| \sum_{m,n=0}^{\infty} \sum_{r,s=0}^{\infty} u_{mn} a_{mn} x_{rs} \right| \leq ||u||_{\mathcal{L}_{\infty}} \sum_{m,n=0}^{\infty} \sum_{r,s=0}^{\infty} |a_{mn} x_{rs}| \leq L ||u||_{\mathcal{L}_{\infty}} \|x\|_{\mathcal{L}_k}
\]

In (13), taking \( x_{rs} = 1 \) for \( (r, s) = (i, j) \), and zero otherwise, it is easily seen that

\[
\left| \sum_{m,n=0}^{\infty} a_{mn} u_{mn} \right| \leq \sum_{m,n=0}^{\infty} |a_{mn} u_{mn}| \leq L \|u\|_{\mathcal{L}_{\infty}},
\]

which gives that \( A'(u) \) is defined for all \( r, s \geq 0 \), where the double sequence \( A'(u) = (A'_{rs}(u)) \) is given by

\[
A'_{rs}(u) = \sum_{m,n=0}^{\infty} a_{mn} u_{mn} : m, n \geq 0
\]
Again, it follows by considering (13) that
\[
\left| \sum_{r,s=0}^{\infty} A_r^s(u)x_{rs} \right| \leq L\|u\|_{L_\infty} \|x\|_{L_1}
\] (15)

which implies that the series in the left side hand of (14) converges. Therefore, since the dual of space \( L_k \) is the space \( L_k \) (see [1]), we obtain \( A_r^s(u) \in L_k \), i.e., \( A_r^s \in (L_\infty, L_k) \).

(iii) \( \Rightarrow \) (iv). If \( A_r^s \in (L_\infty, L_k) \), then, by Lemma 2.1, there exists a constant \( K \) such that \( \|A_r^s(x)\|_{L_k} \leq K \|x\|_{L_\infty} \) for all \( x \in L_\infty \), i.e.,
\[
\left( \sum_{r,s=0}^{\infty} \sum_{m,n=0}^{\infty} a_{mns}^r x_{mn} \right)^{1/k} \leq K \|x\|_{L_\infty}.
\] (16)

Let \( M \) and \( N \) be any finite subsets of all nature numbers. Take a sequence \( x = (x_{mn}) \) as \( x_{mn} = 1 \) for \((r,s) \in MXN\), and zero otherwise. Then, (16) is reduced to
\[
\left( \sum_{r,s=0}^{\infty} \sum_{(m,n)\in MXN} a_{mns}^r \right)^{1/k} \leq K
\]
which proves \( w_k(A) < \infty \).

(iii) \( \Rightarrow \) (iv). Suppose (iii) is satisfied and \( a_{mns} \) are real numbers. Then, for every finite subsets \( M \) and \( N \) of nature numbers,
\[
\sum_{r,s=0}^{\infty} \sum_{(m,n)\in MXN} a_{mns}^r \leq w_k(A).
\]

Let \( H^+ = \{(m, n) \in MXN : a_{mns} \geq 0 \} \) and \( H^- = \{(m, n) \in MXN : a_{mns} < 0 \} \). Then, by considering the inequality \( |a + b|^k \leq 2^k \left( |a|^k + |b|^k \right) \), where \( a \) and \( b \) are complex numbers, we have
\[
W_k(A) = \sum_{r,s=0}^{\infty} \left( \sum_{m,n=0}^{\infty} |a_{mns}| \right)^k
= \sum_{r,s=0}^{\infty} \left\{ \sum_{(m,n)\in H^+} a_{mns} + \sum_{(m,n)\in H^-} -a_{mns} \right\}^k
\leq 2^k \sum_{r,s=0}^{\infty} \left\{ \left( \sum_{(m,n)\in H^+} a_{mns} \right)^k + \left( \sum_{(m,n)\in H^-} a_{mns} \right)^k \right\}
\leq 2^{k+1} w_k(A).
\]

If \( a_{mns} \) is complex number for \( m, n, r, s \geq 0 \), it is easily seen that \( W_k(A) \leq 2^{2k+3} w_k(A) < \infty \), which implies (iv).

This step ends the proof.

3. Main Results

In this section we prove the following theorems.
Theorem 3.1. Let \((p_n), (q_n), (p'_n)\) and \((q'_n)\) be sequences of positive numbers satisfying (1). Further, let \(\gamma = (\gamma_n)\) be a double sequence of positive numbers and \(A = (a_{mnr})\) be a four dimensional triangle matrix and define the matrix \(B\) by

\[
b_{mnr} = \begin{cases} 
\sum_{i,j=0}^{m,n} P'_{r-1} Q'_{s-1} a_{ijrs}, & 1 \leq r \leq m, 1 \leq s \leq n \\
0, & r > m, \text{ or } s > n.
\end{cases}
\] (17)

Then, \(A \in \left( |N|, \pi |N|_{rs}|_{k} \right), \ 1 \leq k < \infty,\) if and only if

\[
\sum_{m,n=1}^{\infty} |\mu'_{mn} b_{mn,r+1,s+1}|^k = O(1)
\] (18)

\[
\sum_{m,n=1}^{\infty} |\mu'_{mn} \Delta_2 b_{mn,r+1,s+1}|^k = O \left( \left( \frac{q_n}{Q_n} \right)^k \right)
\] (19)

\[
\sum_{m,n=1}^{\infty} |\mu'_{mn} \Delta_3 b_{mn,r+1,s+1}|^k = O \left( \left( \frac{p_n}{P_n} \right)^k \right)
\] (20)

\[
\sum_{m,n=1}^{\infty} |\mu'_{mn} \Delta_2 b_{mn,r+1,s+1}|^k = O \left( \left( \frac{P_n Q_n}{p_n q_n} \right)^k \right)
\] (21)

where \(\mu'_{mn}\) is defined by (7).

Proof. Necessity. Let \(A \in \left( |N|, \pi |N|_{rs}|_{k} \right).\) Then, since \(N,\) and \(\pi |N|_{rs}|_{k}\) are Banach spaces, it is seen from Lemma 2.1 that \(A : |N| \rightarrow \pi |N|_{rs}|_{kk}\) defined by

\[
A_{mn}(x) = \sum_{r,s=0}^{m,n} a_{mnr} x_{rs}
\] (22)

is a bounded linear operator. So, there exists a constant \(M\) such that

\[
||A(x)||_{\pi |N|_{rs}|_{kk}} \leq M ||x||_{\pi |N|}
\] (23)

for all \(x = (x_{rs}) \in |N|\). Put \(t_{mn} = \Delta_2 T_{mn}\) for \(m,n \geq 0,\) where \(\Delta_2 T_{mn}\) is defined by (9). Then, \(t = (t_{mn}) \in L.\) Also, \(A(x) = (A_{rs}(x)) \in \pi |N|_{r's'}|_{kk}\) if and only if \(L'(x) = (L'_{mn}(x)) \in L'_k, i.e.,\)

\[
||A(x)||_{\pi |N|_{r's'}|_{kk}} = ||L'(x)||_{L'_k} = \left( \sum_{m,n=1}^{\infty} |\mu'_{mn}(x)|^k \right)^{1/k} < \infty
\] (24)

where

\[
L'_{mn}(x) = \mu'_{mn} \sum_{r,s=1}^{m,n} P'_{r-1} Q'_{s-1} A_{rs}(x).
\] (25)
Choose a sequence \( x = (x_{ij}) \in \mathbb{N}_{p,q} \) such that \( x_{rs} = 1, x_{ij} = 0 \) for \( i \neq r, j \neq s \). Then, using (9), we have, for \( m, n \geq 1 \),

\[
\ell_{mn} = \left\{ \begin{array}{ll}
0, & m < r, n < s \\
\frac{\mu_m \Delta_2 b_{mnr+1}}{\mu_{mn} b_{mnr}}, & m \geq r, n = s \\
\frac{\mu_m \Delta_2 b_{mnr+1}}{\mu_{mn} b_{mnr}}, & m \geq r, n > s
\end{array} \right., \quad \|x\|_{\mathbb{N}_{p,q}} = \|\ell\|_\ell = 1
\]

(26)

Also, it is easily seen that

\[
A_{mn}(x) = \left\{ \begin{array}{ll}
0, & m < r, n < s \\
a_{mnrs}, & m \geq r, n \geq s
\end{array} \right.
\]

which gives, by (24),

\[
L'_{mn}(x) = \left\{ \begin{array}{ll}
0, & m < r, n < s \\
\mu'_m b_{mnrs}, & m \geq r, n \geq s
\end{array} \right.
\]

and so

\[
\|A(x)\|_{\mathbb{N}'_{p,q}} = \left( \sum_{m,n=r,s}^\infty |\mu'_m b_{mnrs}|^k \right)^{1/k} \leq M^k
\]

(27)

Now, it follows by applying (26) and (27) to the inequality (23) that, for \( r, s \geq 1 \),

\[
\sum_{m,n=r,s}^\infty |\mu'_m b_{mnrs}|^k \leq M^k
\]

which is equivalent to (18).

Now take \( x_{rs} = 1, x_{r,s+1} = -1 \), and zero, otherwise. Then, by (10), we get

\[
\ell_{mn} = \left\{ \begin{array}{ll}
0, & m < r, n < s \\
\frac{\mu_m \Delta_2 b_{mnr+1}}{\mu_{mn} b_{mnr}}, & m \geq r, n = s \\
\frac{\mu_m \Delta_2 b_{mnr+1}}{\mu_{mn} b_{mnr}}, & m \geq r, n > s
\end{array} \right., \quad \|x\|_{\mathbb{N}_{p,q}} = \|\ell\|_\ell = \frac{2q_s}{Q_s}
\]

(28)

Further, we obtain

\[
A_{mn}(x) = \sum_{i,j=r,s}^{m,n} a_{mnij} x_{ij} = \left\{ \begin{array}{ll}
0, & n < s, m < r \\
-\Delta_2 b_{mnr+1}, & n \geq s, m \geq r
\end{array} \right.
\]

which implies, by (25),

\[
L'_{mn}(x) = \left\{ \begin{array}{ll}
0, & m < r, n < s \\
-\mu'_m \Delta_2 b_{mnr+1}, & m \geq r, n \geq r
\end{array} \right.
\]

and

\[
\|A(x)\|_{\mathbb{N}'_{p,q}} = \left( \sum_{m,n=r,s}^\infty |\mu'_m \Delta_2 b_{mnr+1}|^k \right)^{1/k} \leq M^k
\]

(29)

So, using (28) and (29), we have from (23) that (19) holds. Also, by taking \( x_{rs} = 1, x_{r+1,s} = -1 \), and zero, otherwise, then, similarly, (20) holds.
Finally, put $x_n = 1$, $x_{n+1} = -1$, $x_{r+1,s} = -1$, $x_{r+1,s+1} = 1$, and zero, otherwise. Then,

$$t_{mn} = \begin{cases} 
0, & m < r, n < s \ 
p_{n,q} p_{m,q}^{-1}, & n = s, m = r \ 
p_{m,r} p_{n,r}^{-1}, & n > s, m = r \ 
p_{n,s} p_{m,s}^{-1}, & n = s, m > r \ 
p_{m,s} p_{n,s}^{-1}, & n > s, m > r \ 
\end{cases}$$  \hfill (30)

and

$$A_{mn}(x) = \begin{cases} 
\Delta_{21} a_{mn,r+1,s+1}, & r \leq m, s \leq n \ 
0, & r > m, s > n. \end{cases}$$

This verifies

$$L_{mn}'(x) = \begin{cases} 
\mu_{mn} \Delta_{12} b_{mn,r+1,s+1}, & r \leq m, s \leq n \ 
0, & r > m, s > n. \end{cases}$$

and

$$\|x\|_{\mathbb{N}_{/r'}}^{1/k} = \left( \sum_{m,n=r,s}^{\infty} |\mu_{mn} \Delta_{12} b_{mnr+1s+1}|^k \right)^{1/k}. \hfill (31)$$

Therefore, considering (30) and (31), it follows from (23) that (21) holds.

**Sufficiency.** Given $x = (x_n) \in \mathbb{N}_{/r'}$. Then, $t = (t_{mn}) \in \mathcal{L}$, where $t_{mn} = \Delta_{21} T_{mn}$ for $m, n \geq 0$, as above. Now, we should show that $A(x) = (A_{mn}(x)) \in \pi |\mathbb{N}_{/r'}|_{k}$, i.e.,

$$\sum_{m,n=r,s}^{\infty} |L_{mn}'(x)|^k < \infty$$

where $L'(x) = (L_{mn}'(x))$ is defined by (25). To achieve this, by solving (10) for $x_{mn}$, we obtain, for $m, n \geq 1$,

$$x_{mn} = \frac{p_m q_n t_{mn} - p_{m-2} q_n t_{m-1,n}}{p_m q_n} - \frac{q_{n-1} p_{m,n-1} t_{m-1,n}}{p_{m-1} q_{n-1}} + \frac{p_{m-2} q_{n-2} t_{m-2,n-2}}{p_{m-1} q_{n-1}}.$$ \hfill (32)

Hence, since $B$ is a triangular matrix, a few calculations reveal that

$$L_{mn}'(x) = \mu_{mn} \sum_{i,j=1}^{m,n} P_{i-1,j-1} A_{ij}(x) = \mu_{mn} \sum_{r,s=1}^{m,n} x_{rs} \sum_{i,j=r,s}^{m,n} P_{i-1,j-1} A_{ijrs} = \mu_{mn} \sum_{r,s=1}^{m,n} b_{mnr+1s+1} x_{rs}.$$
which completes the proof of the sufficiency.
Now, it is obvious that $A(x) = (A_m(x)) \in \left[ N'_P\right]_{k_1}^\nu$, i.e.,
\[
\sum_{m,n=0}^\infty |L'_{mn}(x)|^k = \sum_{m=0}^\infty |L'_{m0}(x)|^k + \sum_{n=1}^\infty |L'_{0n}(x)|^k + \sum_{m,n=1}^\infty |L'_{mn}(x)|^k < \infty
\] (34)
if and only if
\[
\sum_{m=0}^\infty |L'_{m0}(x)|^k < \infty, \sum_{n=1}^\infty |L'_{0n}(x)|^k < \infty, \sum_{m,n=1}^\infty |L'_{mn}(x)|^k < \infty,
\]

where
\[
L'_{m0}(x) = \mu'_m \sum_{n=1}^m P'_{r-1}A_0(x)
\]
\[
L'_{0n}(x) = \mu'_n \sum_{n=1}^n Q'_{s-1}A_0(x)
\]
\[
L'_{mn}(x) = \mu'_m \sum_{r=1}^m \sum_{s=1}^n m_{rs} P'_{r-1}Q'_{s-1}A_r(x)
\]

So, if we define the matrices $A_1, A_2$ and $A_3$ by
\[
A_1 = (a_{0m0}), A_2 = (a_{0n0}), A_3 = (a_{mn}) \text{ for } m, n \geq 1
\] (35)
then, $A \in \left[ N'_P\right]_{k_1}^\nu$ if and only if $A_1 \in \left[ N'_P\right]_{k_1}^\nu$, $A_2 \in \left[ N'_P\right]_{k_1}^\nu$, and $A_3 \in \left[ N'_P\right]_{k_1}^\nu$, where $A = (a_{mn})$ is a triangle matrix for $m, n \geq 0$, $v_1 = (v_{m0})$ and $v_2 = (v_{0n})$.

By identifying $A_1 = (a_{0m0}) \equiv (a_{mr}), A_2 = (a_{0n0}) \equiv (a_{mn})$, the main theorem is immediately deduced by Theorem 1.1 and Theorem 2.1 as follows.

**Theorem 3.2.** Let the sequences $(p_n)$, $(q_n)$, $(p'_n)$, $(q'_n)$, $(v_{mn})$, and the matrices $A, B$ be as in Theorem 3.1. Then, $A \in \left[ N'_P\right]_{k_1}^\nu$, if and only if conditions (18)–(21) and the following conditions are satisfied:
\[
\frac{P_p P_r}{q_r q_r} y^{1/k} a_{000} = O(1)
\]
\[
\sum_{n,r=1}^\infty \left| \mu'_n \sum_{r=0}^n P'_{r-1}A_1 a_{0r0,0r+1,0} \right|^k = O \left( \left( \frac{P_r}{P_r} \right)^k \right)
\]
\[
\sum_{n,r=1}^\infty \left| \mu'_n \sum_{r=0}^n P'_{r-1}a_{0r0,0r+1,0} \right|^k = O(1)
\]
\[
\frac{Q_r q'_r}{q_r q_r} y^{1/k} a_{000} = O(1)
\]
\[
\sum_{n,r=1}^\infty \left| \mu'_n \sum_{r=0}^n Q'_{r-1}A_2 a_{0r0,0r+1,0} \right|^k = O \left( \left( \frac{Q_r}{Q_r} \right)^k \right)
\]
\[
\sum_{n,r=1}^\infty \left| \mu'_n \sum_{r=0}^n Q'_{r-1}a_{0r0,0r+1,0} \right|^k = O(1)
\]
where \( \mu'_0 \) and \( \mu'_n \) are defined by (7). Now we qualify the converse of the matrix class in Theorem 3.2, which, although similar to the previous one, has a very different character.

**Theorem 3.3.** Let \((p_n),(q_n),(p'_n),(q'_n)\) and \((\gamma_{mn})\) be as in Theorem 3.1. Further, let \(A = (a_{mn})\) be a four dimensional triangle matrix and define the matrix \(B\) by (17) for \(m,n \geq 1\), and

\[
b_{mnr s} = \begin{cases} 
\sum_{j=0}^{n} Q'_{j-1} a_{0j0s}, & 1 \leq s \leq n, \ m = 0 \\
\sum_{j=0}^{r} P'_{j-1} a_{0j0s}, & 1 \leq r \leq m, \ n = 0 
\end{cases}
\]

Then, \(A \in \left(\mathbb{L}_{p}', \mathbb{L}_{Q'}\right), 1 < k < \infty,\) if and only if

\[
\sum_{s=1}^{\infty} \frac{1}{\gamma_{0s}} \left(\sum_{n=s}^{\infty} \mu'_0 \left|Q_0 \Delta_2 b_{0n,s+1} - b_{0n,s+1}\right|\right)^k < \infty
\]

(36)

\[
\sum_{r=1}^{\infty} \frac{1}{\gamma_{r0}} \left(\sum_{m=r}^{\infty} \mu'_m \left|P_r \Delta_1 b_{m0,r+1,0} - b_{m0,r+1,0}\right|\right)^k < \infty
\]

(37)

\[
\sum_{s=1}^{\infty} \frac{1}{\gamma_{rs}} \left(\sum_{m,n,m,r,s} \mu'_{mn} c_{mnrs}\right)^k < \infty
\]

(38)

where \(\mu'_{mn}\) and \(c_{mnrs}\) are given by (7) with \(k = 1\) and (21), respectively.

**Proof.** Assume that \(x = (x_{rs}) \in \left[\mathbb{L}_{p}'\right]_k\) and \(A(x)\) is \(A\)-transform sequence of \(x\). Let \(t_{m0} = \Delta_1 t_{m0}, t_{0n} = \Delta_2 t_{0n}\) and \(t_{mn} = \Delta_2 T_{mn}\) for \(m,n \geq 1\), where \(\Delta_1 T_{m0}, \Delta_2 T_{0n}\) and \(\Delta_2 T_{mn}\) are defined by (10). Further, put \(u_{m0} = \gamma_{m0}^{-1} t_{m0}, u_{0n} = \gamma_{0n}^{-1} t_{0n}\) and \(u_{mn} = \gamma_{mn}^{-1} t_{mn}\). Then, \(u = (u_{mn}) \in \mathbb{L}_k\), or, equivalently, \((u_{mn})\), \((u_{0n})\) \(\in \ell_k\) and \((u_{mn})\) \(\in \mathbb{L}_k\). Also, \(A(x) \in \left[\mathbb{L}_{Q'}\right]_k\), iff \(L'(x) = (L'_m(x)) \in \mathbb{L}_k\), or, equivalently, as in (34), \((L'_{0n}(x)), (L'_{mn}(x)) \in \ell_k\), and \((L'_{mn}(x)) \in \mathbb{L}_k\), where

\[
L'_{0n}(x) = \mu'_n \sum_{s=1}^{n} Q'_s A_{0s}(x)
\]

\[
L'_{m0}(x) = \mu'_m \sum_{r=1}^{m} P'_r A_{0r}(x)
\]

\[
L'_{mn}(x) = \mu'_{mn} \sum_{r,s=1}^{\infty} Q'_r Q'_{s-1} A_{rs}(x)
\]

It follows by solving (10) for \(x_{m0}\) and \(x_{0n}\) that

\[
x_{m0} = \frac{P_m}{\gamma_{m0}} t_{m0} - \frac{P_{m-2}}{\gamma_{m-1}} t_{m-1,0}, \quad x_{0n} = \frac{Q_n}{\gamma_{0n}} t_{0n} - \frac{Q_{n-2}}{\gamma_{n-1}} t_{0,n-1}
\]

(39)

Since \(A\) and \(B\) are triangular matrix and \(P_{-1} = Q_{-1} = 0\), it is easily written from (39) and (32) that, for \(m,n \geq 1\),

\[
L'_{0n}(x) = \mu'_n \sum_{s=1}^{n} Q'_s A_{0s}(x) = \mu'_n \sum_{j=0}^{n} b_{0n,j} x_{0j}
\]

\[
= \mu'_n \sum_{j=0}^{n} \left(\frac{Q_j}{\gamma_j} \Delta_2 b_{0n,j+1} - b_{0n,j+1}\right)^{-1/k} u_{0j} x_{0j}
\]
\[ L'_{m0}(x) = \mu'_m \sum_{r=1}^{\infty} P'_{r-1} A_0(x) = \mu'_m \sum_{r=0}^{\infty} b_{mr0} x_r \]

\[ = \mu'_m \sum_{r=0}^{\infty} \left( \frac{P_i}{P_j} \Delta_1 b_{mr0,j+1,0} - b_{mr0,j+1,0} \right) y_{j+1}^{-1/k} u_{j+1} \]

\[ L'_{mn}(x) = \mu'_{mn} \sum_{i,j=1}^{\infty} P'_{i-1} Q'_{j-1} A_{ij}(x) = \mu'_{mn} \sum_{i,j=1}^{\infty} P'_{i-1} Q'_{j-1} \sum_{s=1}^{\infty} a_{ijrs} x_s \]

\[ = \mu'_{mn} \sum_{r,s=1}^{\infty} x_{rs} \sum_{i,j=rs}^{\infty} P'_{i-1} Q'_{j-1} a_{ijrs} = \mu'_{mn} \sum_{r,s=1}^{\infty} b_{mnrs} x_s \]

\[ = \mu'_{mn} \sum_{r,s=1}^{\infty} \left( \frac{P_i Q_j}{P_j q_s} t_{rs} - \frac{P_{r-2} Q_{s-1}}{P_{r-1} q_s} t_{rs-1} + \frac{P_{r-2} Q_{s-2}}{P_{r-1} q_s} t_{rs-2} \right) u_{rs} \gamma_{rs}^{-1/k} \]

\[ = \mu'_{mn} \sum_{r,s=1}^{\infty} \left( \frac{P_i Q_j}{P_j q_s} b_{mn,r+1,s+1} - \frac{P_i Q_j}{P_j q_s} b_{mn,r,s+1} - \Delta_1 b_{mn,r+1,s+1} + \Delta_2 b_{mn,r+1,s+1} \right) u_{rs} \gamma_{rs}^{-1/k} \]

Hence, it can be expressed that

\[ L'_{mn}(u) = \sum_{r,s=1}^{\infty} d_{mnrs} u_{rs} \]

where

\[ d_{mnrs} = \begin{cases} \mu'_{mn} \left( \frac{P_i Q_j}{P_j q_s} \Delta_2 b_{mn0,s+1} - b_{mn0,s+1} \right) y_{j+1}^{-1/k} \gamma_{j+1}^{-1/k}, & 0 \leq s, n, m = r = 0 \\ \mu'_{mn} \left( \frac{P_i}{P_j} \Delta_1 b_{mn0,r+1,0} - b_{mn0,r+1,0} \right) y_{r+1}^{-1/k} \gamma_{r+1}^{-1/k}, & 0 \leq r, n, s = 0 \\ \mu'_{mn} c_{mnrs} y_{rs}^{-1/k}, & 1 \leq r \leq m, 1 \leq s \leq n \\ 0, & \text{otherwise.} \end{cases} \]

This gives that \( A \in \left[ \mathbb{N}^{p'_{12}}, \mathbb{N}^{p'_{12}} \right] \) if and only if \( D \in (L_q, L) \). Therefore, it follows from Lemma 2.2 that the conclusion of the theorem is valid if and only if \( W_k(A) < \infty \), or, equivalently,

\[ \sum_{k=0}^{\infty} \left( \sum_{r=0}^{\infty} |d_{m00}| \right)^k < \infty, \quad \sum_{r=0}^{\infty} \left( \sum_{m=0}^{\infty} |d_{m0r}| \right)^k < \infty \]

and

\[ \sum_{r,s=1}^{\infty} \left( \sum_{m,n=r}^{\infty} |d_{mnrs}| \right)^k < \infty, \]

which gives (36), (37) and (38).

This completes the proof.
References

[14] M.A. Sarıgöl, Necessary and sufficient conditions for the equivalence of the summability methods $\mathcal{N}(p_{n_1})$ and $\mathcal{C}_{1_{k}}$, Indian J. Pure Appl. Math. 22 (1991), 483-489.