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Approximation by matrix means on hexagonal domains in the generalized Hölder metric

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Abstract. In this paper the degree of approximation of the function f, which is periodic with respect to the hexagon lattice by matrix means $T_n^{(A)}(f)$ of its hexagonal Fourier series in the generalized Hölder metric, where A is a lower triangular infinite matrix of nonnegative real numbers with nonincreasing row is estimated.

1. Introduction

Studying approximation properties of $2\pi-$periodic functions by trigonometric Fourier series on the real line or on the unit circle, is an established field as illustrated in the classical treatise of Zygmund [24]. Much progress was made by using prominent summability methods such as partial sums and means of Fourier series (Cesàro, Abel-Poisson means, Riesz, Nörlund, matrix means etc.) which are used in studies to deal with approximation properties of functions. There are quite a number of excellent references on results of these studies (see e.g. [1, 3, 10, 11, 18, 19, 21]).

Approximating of multivariable functions is also important in approximation theory. The degree of approximation of bivariate functions by double means of double Fourier series and conjugate double Fourier series has been obtained, for the first time by F. Mòricz and X. L. Shi ([17]), who studied the rate of uniform approximation of functions belonging to the Lipschitz class and for those belonging to the Zygmund class, by rectangular double Cesàro means of the rectangular partial sums of double Fourier series. These results have been generalized by F. Mòricz and B. E. Rhoades (see [15]) obtaining the rate of uniform approximation of functions belonging to the Lipschitz class and for those belonging to the Zygmund class, using double Nörlund means of the rectangular partial sums of double Fourier series. In these studies the territory of classical (multiple) Fourier series for the simplest spectral sets, cubes in Euclidean space, which is the tensor product case considered and they studied by assuming that the functions are $2\pi-$periodic in each of their variables (see e.g. [15–17, 21], and [23, 24]). In comparison to the usual Fourier series for periodic functions in both variables on the plane, the periodicity of the Fourier series on a hexagonal domain is defined in terms of the hexagon lattice which offers the densest packing of the plane with unit circles. Because of the fact that the approximation quality on the hexagonal lattice is consistently better than orthogonal lattices, when choosing lattices with the same sampling density (see in [2]), we wish to think of

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the case of non-tensor product domain in the Euclidean plane $\mathbb{R}^2$ where another definition of periodicity is needed and thereby the functions that we consider are periodic under the translation of hexagonal lattice.

Now we first collect many of the known results about the the definition and basic properties of hexagonal Fourier series, and functions periodic with respect to the hexagon lattice. The definitions of lattices, generator matrices and spectral sets, and more detailed information about Fourier analysis on lattices can be found in [13] and [22]. This will be a useful source of information on the subject and many of these results are used in the later sections of this paper. Throughout this paper approximation properties of bivariate functions will be studied. So it is enough to give basic information about hexagonal lattice and hexagonal Fourier series in two dimension.

In this paper we will study in the Euclidean plane $\mathbb{R}^2$ and we will use hexagon lattice and regular hexagon as the spectral set instead of the standart lattice $\mathbb{Z}^2$ and rectangular domain $[-\frac{1}{2}, \frac{1}{2}]^2$. Hereby we will generalize the approach beyond the box domain and increase the approximation quality. So we can begin the work with choosing the generator matrix as

$$H = \begin{bmatrix} \sqrt{3} & 0 \\ -1 & 2 \end{bmatrix}.$$

Now if one want to carry out the discrete analysis with lattice $H\mathbb{Z}^2$ in a domain, this domain need to be fixed the containing 0 and tiled with the lattice $H\mathbb{Z}^2$ (see, for example [6,8]). Therefore the spectral set Ω_H of the hexagonal lattice $H\mathbb{Z}^2$ choosen as

$$\Omega_H = \left\{(x_1, x_2) \in \mathbb{R}^2 : -1 \le x_2, \frac{\sqrt{3}}{2}x_1 \pm \frac{1}{2}x_2 < 1\right\}.$$

For dealing with symmetry along the three direction, it is more convenient to use the 3-direction homogeneous coordinates (t_1, t_2, t_3) instead of the usual two coordinates and this coordinates need to satisfy $t_1 + t_2 + t_3 = 0$ for periodicity. Now if one define

$$t_1 := -\frac{x_2}{2} + \frac{\sqrt{3}x_1}{2},\ t_2 := x_2,\ t_3 := -\frac{x_2}{2} - \frac{\sqrt{3}x_1}{2}, \tag{1}$$

the hexagon Ω_H becomes basic hexagon (minimum periodic domain)

$$\Omega = \left\{(t_1, t_2, t_3) \in \mathbb{R}^3 : -1 \le t_1, t_2, -t_3 < 1,\ t_1 + t_2 + t_3 = 0\right\},$$

which is the intersection of the plane $t_1 + t_2 + t_3 = 0$ with the cube $[-1, 1]^3$.

The plane of homogeneous coordinates $t_1 + t_2 + t_3 = 0$, denoted by $\mathbb{R}^3_H$ are written by using bold letters $\mathbf{t}$ throughout the paper, that is

$$\mathbb{R}^3_H = \left\{\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}^3 : t_1 + t_2 + t_3 = 0\right\}.$$

The Jacobian determinant of the change of variables in (1) gives that

$$x = (x_1, x_2) \to \mathbf{t} = (t_1, t_2, t_3) \quad \text{is} \quad dx_1 dx_2 = \frac{2\sqrt{3}}{3} dt_1 dt_2.$$

The periodicity with respect to the hexagonal lattice H for a function f is defined by

$$f(x + Hk) = f(x),\ \left(k \in \mathbb{Z}^2\right).$$

In this case f can be also called H−periodic.

Let define $\mathbf{t} \equiv \mathbf{s}\,(\mathrm{mod}3)$ as

$$t_1 - s_1 \equiv t_2 - s_2 \equiv t_3 - s_3 \quad (\mathrm{mod}3)\,.$$

By using this definition we can give a periodicity state in homogeneous coordinates, as f is $H-$periodic if and only if

$$f(\mathbf{t}) = f(\mathbf{t}+\mathbf{s}) \quad \text{whenever} \quad \mathbf{s} \equiv \mathbf{0}(\quad \mathrm{mod}\ 3).$$

The equality

$$\int_{\Omega} f(\mathbf{t}+\mathbf{s})\,d\mathbf{t} = \int_{\Omega} f(\mathbf{t})\,d\mathbf{t},\ \left(\mathbf{s} \in \mathbb{R}_H^3\right).$$

follows from the definition of the periodicity ([22]).

Let $|\Omega|$ denote the area of Ω. The inner product defined by

$$\langle f, g\rangle_H = \frac{1}{|\Omega_H|}\int_{\Omega_H} f(x_1,x_2)\,\overline{g(x_1,x_2)}dx_1dx_2 = \frac{1}{|\Omega|}\int_{\Omega} f(\mathbf{t})\,\overline{g(\mathbf{t})}d\mathbf{t}, \tag{2}$$

renders $L^2(\Omega)$ becoming a Hilbert space.

Let $\mathbb{Z}_H^3$ denote $\mathbb{Z} \cap \mathbb{R}_H^3$ and $\langle \mathbf{j}, \mathbf{t}\rangle$ is the Euclidean inner product of $\mathbf{j}$ and $\mathbf{t}$, then the functions $\phi_{\mathbf{j}}(\mathbf{t})$ are $H-$periodic and the set

$$\left\{\phi_{\mathbf{j}} : \phi_{\mathbf{j}}(\mathbf{t}) := e^{\frac{2\pi i}{3}\langle \mathbf{j},\mathbf{t}\rangle},\ ,\mathbf{j} \in \mathbb{Z}_H^3,\ \mathbf{t} \in \mathbb{R}_H^3\right\}$$

is an orthonormal basis of $L^2(\Omega)$ with respect to the inner product (2) (See in [4]).The completeness can be seen by well known Stone theorem e.g. see ([3]).

The space defined for every natural number n by

$$\mathcal{H}_n := \operatorname{span}\left\{\phi_{\mathbf{j}} : \mathbf{j} \in \mathbb{H}_n\right\},\ (n \in \mathbb{N})$$

will be a finite dimensional space and the dimension of $\mathcal{H}_n$ is $\#\mathbb{H}_n = 3n^2 + 3n + 1$. And the member of the set are called the hexagonal trigonometric polynomials.

It should be noted here that the space $\mathcal{H}_n$'s indexes are choosen from the symmetric point subset of $\mathbb{Z}_H^3$, that is

$$\mathbb{H}_n := \left\{\mathbf{j} = (j_1, j_2, j_3) \in \mathbb{Z}_H^3 : -n \le j_1, j_2, j_3 \le n\right\}.$$

which consists of all integer points inside the hexagon $n\overline{\Omega}$ so that one can ensure the symmetry of inner product (2) on Ω).

The Dirichlet kernel D_n is defined by

$$D_n(\mathbf{t}) := \sum_{\mathbf{j}\in\mathbb{H}_n} \phi_{\mathbf{j}}(\mathbf{t}),$$

and it has the compact formula

$$D_n(\mathbf{t}) = \Theta_n(\mathbf{t}) - \Theta_{n-1}(\mathbf{t}), \tag{3}$$

where $n \geq 1$ and

$$\Theta_n(\mathbf{t}) = \frac{\sin\frac{(n+1)(t_1-t_2)\pi}{3}\sin\frac{(n+1)(t_2-t_3)\pi}{3}\sin\frac{(n+1)(t_3-t_1)\pi}{3}}{\sin\frac{(t_1-t_2)\pi}{3}\sin\frac{(t_2-t_3)\pi}{3}\sin\frac{(t_3-t_1)\pi}{3}}, \quad \left(\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}_H^3\right) \tag{4}$$

which proved in [20].
Using the set-up above, the hexagonal Fourier series of an $H-$periodic function $f \in L^1(\Omega)$ is defined by

$$f(\mathbf{t}) \sim \sum_{\mathbf{j}\in\mathbb{Z}_H^3} \widehat{f_{\mathbf{j}}}\phi_{\mathbf{j}}(\mathbf{t}), \tag{5}$$

where

$$\widehat{f_{\mathbf{j}}} = \frac{1}{|\Omega|}\int_{\Omega} f(\mathbf{t})\, e^{-\frac{2\pi i}{3}\langle \mathbf{j},\mathbf{t}\rangle} d\mathbf{t}, \quad \left(\mathbf{j} \in \mathbb{Z}_H^3\right).$$

Therefore the nth partial sums of the series (5) are defined by

$$S_n(f)(\mathbf{t}) := \sum_{\mathbf{j}\in\mathbb{H}_n} \widehat{f_{\mathbf{j}}}\phi_{\mathbf{j}}(\mathbf{t}).$$

The nth partial sum has the next integral representation.

$$S_n(f)(\mathbf{t}) = \frac{1}{|\Omega|}\int_{\Omega} f(\mathbf{t}-\mathbf{s})\, D_n(\mathbf{s})\, d\mathbf{s}.$$

Li, Sun and Xu are the first authors who considered discrete Fourier analysis on lattices in their study ([13]) of the Lagrange interpolation and cubature formulas by trigonometric functions on a regular hexagon and on an equilateral triangle which is the case of hexagon lattice was used to deal with . Afterwards, the theory of approximation is further extended by the author Xu by replacing domain and periodicity and studying Cesàro and Abel summability of Fourier series over the regular hexagon, and proving that the $(C, 1)$ and Abel-Poisson means of hexagonal Fourier series of a continuous periodic function converge uniformly to this function. So the construction above can be found in their paper [22], [20] and [13]. A good deal of emphasis is on ideas related to the degree of approximation of $H-$periodic continuous functions by Cesàro, Riesz, and Nörlund means of their hexagonal Fourier series was investigated by the author in uniform norm and in the Hölder norm [5–9]. But, for our purposes the most important reference is [9], who looked at the order of matrix means of functions belonging to the Hölder class $H^{\alpha}\left(\overline{\Omega}\right)$, $0 < \alpha \leq 1$ in particular, in the uniform norm and Hölder norm $\|\cdot\|_{\beta}$, where $0 \leq \beta < \alpha$.

The aim of this paper is two-fold: 1) to estimate the order of approximation by matrix means of hexagonal Fourier series of functions belong to the generalized Hölder class $H^{\omega_\alpha}(\overline{\Omega})$ and generalize the results of [9]. 2) To give some special cases of matrix mean such as Nörlund and Riesz means. As we shall show below, many results resemble closely to those of Fourier analysis and approximation on $2\pi-$periodic functions.

2. Main Results

As a prelude to the main results of this paper, we will introduce the generalized Hölder classes of continuous functions on the hexagonal domains and Guven's original idea which we will further extend. The Banach space of $H-$periodic complex valued continuous functions on $\mathbb{R}_H^3$ which is equipped with the uniform norm

$$\|f\|_{C(\overline{\Omega})} = \sup\left\{\left|f(\mathbf{t})\right| : \mathbf{t} \in \overline{\Omega}\right\}.$$

will be denoted by $C_H\left(\overline{\Omega}\right)$. And the subspace which consist of all functions $f \in C_H\left(\overline{\Omega}\right)$ for which

$$\sup_{\mathbf{t}\neq\mathbf{s}} \frac{\left|f(\mathbf{t}) - f(\mathbf{s})\right|}{\|\mathbf{t}-\mathbf{s}\|^{\alpha}} < \infty, \quad \|\mathbf{t}\| = \max\{|t_1|, |t_2|, |t_3|\},$$

is called the Hölder space and denoted by $H^{\alpha}\left(\overline{\Omega}\right)$ $(0 < \alpha \leq 1)$.
Let $\omega : [0,\infty) \to [0,\infty)$ be a non-decreasing continuous function. If ω satisfies the condition

$$\omega(0) = 0, \ \omega(t_1 + t_2) \leq \omega(t_1) + \omega(t_2),$$

then ω is called a modulus of continuity. And any modulus of continuity satisfies

$$\omega(\lambda\delta) \leq (1+\lambda)\omega(\delta) \tag{6}$$

for $\lambda > 0$. In addition to that the next inequality is an important for this paper, such that

$$\frac{\omega_\alpha(\delta_2)}{\delta_2} \leq 2\frac{\omega_\alpha(\delta_1)}{\delta_1} \quad (\delta_1 < \delta_2), \tag{7}$$

which is obtained from (6).

The generalized Hölder class $H^{\omega}\left(\overline{\Omega}\right)$ is the set of functions $f \in C_H\left(\overline{\Omega}\right)$ for which

$$\Lambda^{\omega}(f) := \sup_{\mathbf{t}\neq\mathbf{s}} \frac{\left|f(\mathbf{t}) - f(\mathbf{s})\right|}{\omega(\|\mathbf{t}-\mathbf{s}\|)} < \infty,$$

and the norm on $H^{\omega}\left(\overline{\Omega}\right)$ given by

$$\left\|f\right\|_{\omega} := \left\|f\right\|_{C(\overline{\Omega})} + \Lambda^{\omega}(f)$$

makes $H^{\omega}\left(\overline{\Omega}\right)$ a Banach space for any modulus of continuity ω. Further if $\omega(\delta) = \delta^{\alpha}, 0 < \alpha \leq 1$, then we write $H^{\alpha}\left(\overline{\Omega}\right)$ instead of $H^{\omega}\left(\overline{\Omega}\right)$ and $\left\|f\right\|_{\alpha}$ instead of $\left\|f\right\|_{\omega}$.

A certain class of moduli of continuity introduced in [11] by L. Leindler:

Let $\mathcal{M}_\alpha$, $0 \leq \alpha \leq 1$ denote the class of moduli of continuity ω_α having the following properties:
(*i*) for any $\alpha' > \alpha$ there exists a natural number $\mu = \mu(\alpha')$ such that

$$2^{\mu\alpha'}\omega_\alpha\left(2^{-n-\mu}\right) > 2\omega_\alpha\left(2^{-n}\right), \ (n = 1, 2, ...),$$

(*ii*) for every natural number ν, there exists a natural number $N(\nu)$ such that

$$2^{\nu\alpha}\omega_\alpha\left(2^{-n-\nu}\right) \leq 2\omega_\alpha\left(2^{-n}\right), \ (n > N(\nu)).$$

It is clear that $\omega(\delta) = \delta^{\alpha} \in \mathcal{M}_\alpha$, but also $\omega_\alpha(\delta)$ is an extension of $\omega(\delta) = \delta^{\alpha}$. As a result, $H^{\omega_\alpha}\left(\overline{\Omega}\right)$ is larger than $H^{\alpha}\left(\overline{\Omega}\right)$ in general.
Additionally Leindler give an another fact, that is,

$$\gamma(t) = \gamma_{\alpha,\beta}(t) := \frac{\omega_\alpha(t)}{\omega_\beta(t)}$$

is non-decreasing function when $0 \leq \beta < \alpha \leq 1$, $\omega_\beta \in \mathcal{M}_\beta$ and $\omega_\alpha \in \mathcal{M}_\alpha$ in his another paper [12].

The A-transform of the sequence $(S_n(f))$ of partial sums the series (5) is defined by

$$T_n^{(A)}(f)(\mathbf{t}) := \sum_{k=0}^{n} a_{n,k} S_k(f)(\mathbf{t}) \quad (n \in \mathbb{N}).$$

where $A = (a_{n,k})(n, k = 0, 1, ...)$ be a lower triangular infinite matrix of real numbers.
In this paper we assume that the conditions

$$a_{n,k} \geq 0 \quad (n = 0, 1, ..., 0 \leq k \leq n) \tag{8}$$

$$a_{n,k} \geq a_{n,k+1} \quad (n = 0, 1, ..., 0 \leq k \leq n-1) \tag{9}$$

and

$$\sum_{k=0}^{n} a_{n,k} = 1 \quad (n = 0, 1, ...) \tag{10}$$

are satisfied by the lower triangular matrix $A = (a_{n,k})$.
Also, we use the notations

$$A_{n,k} := \sum_{v=0}^{k} a_{n,v} \quad (0 \leq k \leq n), \quad A_n(u) := A_{n,[u]}, \quad a_n(u) := a_{n,[u]} \quad (u > 0),$$

where $[u]$ denotes the integer part of u.
In addition to that the relation $x \lesssim y$ will mean that there exists an absolute constant $c > 0$ such that $x \leq cy$ holds for quantities x and y throughout the paper.
Main results of this section are the following:

Theorem 2.1. *Let $0 \leq \beta < \alpha \leq 1$, $\omega_\beta \in \mathcal{M}_\beta$, $\omega_\alpha \in \mathcal{M}_\alpha$ and $f \in H^{\omega_\alpha}\left(\overline{\Omega}\right)$. If the conditions (8), (9), and (10) satisfied by a lower triangular infinite matrix of real numbers $A = (a_{n,k})$ $(n, k = 0, 1, ...)$, then*

$$\left\| f - T_n^{(A)}(f) \right\|_{C_H(\overline{\Omega})} \lesssim a_{n,0} \log\left(\frac{1}{a_{n,0}}\right) \begin{cases} (n+1) \sum\limits_{k=1}^{n} \gamma\left(\frac{1}{k}\right) \frac{A_{n,k}}{k}, & \text{if } \alpha < 1 \text{ or } \beta > 0 \\ (n+1)\log(n+1) \sum\limits_{k=1}^{n} \gamma\left(\frac{1}{k}\right) \frac{A_{n,k}}{k}, & \text{if } \alpha = 1 \text{ and } \beta = 0 \end{cases} \tag{11}$$

holds for $n \geq 2$.

Theorem 2.2. *Let $0 \leq \beta < \alpha \leq 1$, $\omega_\beta \in \mathcal{M}_\beta$, $\omega_\alpha \in \mathcal{M}_\alpha$ and $f \in H^{\omega_\alpha}\left(\overline{\Omega}\right)$. If the conditions (8), (9), and (10) satisfied by a lower triangular infinite matrix of real numbers $A = (a_{n,k})$ $(n, k = 0, 1, ...)$, then*

$$\left\| f - T_n^{(A)}(f) \right\|_{\omega_\beta} \lesssim a_{n,0} \log\left(\frac{1}{a_{n,0}}\right) \begin{cases} (n+1) \sum\limits_{k=1}^{n} \gamma\left(\frac{1}{k}\right) \frac{A_{n,k}}{k}, & \text{if } \alpha < 1 \text{ or } \beta > 0 \\ (n+1)\log(n+1) \sum\limits_{k=1}^{n} \gamma\left(\frac{1}{k}\right) \frac{A_{n,k}}{k}, & \text{if } \alpha = 1 \text{ and } \beta = 0 \end{cases} \tag{12}$$

holds for $n \geq 2$.

3. Proofs of main results

Proof. [Proof of Theorem 2.1] If one take $f \in H^{\omega_\alpha}\left(\overline{\Omega}\right)$, then it is clear that

$$\left| f(\mathbf{t}) - f(\mathbf{t} - \mathbf{u}) \right| \leq c\omega_\alpha (\|\mathbf{u}\|).$$

Thus we get

$$\left|f(\mathbf{t})-T_n^{(A)}(f)(\mathbf{t})\right| \lesssim \frac{1}{|\Omega|}\int_{\Omega}\omega_\alpha(\|\mathbf{u}\|)\left|\sum_{k=0}^{n}a_{n,k}D_k(\mathbf{u})\right|d\mathbf{u},$$

since

$$\left|f(\mathbf{t})-T_n^{(A)}(f)(\mathbf{t})\right| \leq \frac{1}{|\Omega|}\int_{\Omega}\left|f(\mathbf{t})-f(\mathbf{t}-\mathbf{u})\right|\left|\sum_{k=0}^{n}a_{n,k}D_k(\mathbf{u})\right|d\mathbf{u}.$$

Let set $\Theta_{-1}(u):=0$, by (3) we have

$$\int_{\Omega}\omega_\alpha(\|\mathbf{u}\|)\left|\sum_{k=0}^{n}a_{n,k}D_k(\mathbf{u})\right|d\mathbf{u}=\int_{\Omega}\omega_\alpha(\|\mathbf{u}\|)\left|\sum_{k=0}^{n}a_{n,k}\left(\Theta_n(\mathbf{u})-\Theta_{n-1}(\mathbf{u})\right)\right|d\mathbf{u}.$$

Since the function

$$t\to\omega_\alpha(\|\mathbf{t}\|)\left|\sum_{k=0}^{n}a_{n,k}\left(\Theta_k(\mathbf{t})-\Theta_{k-1}(\mathbf{t})\right)\right|,$$

is a symmetric function of t_1,t_2,t_3 it is sufficient to consider the integral over one triangle of the six equilateral triangles in $\overline{\Omega}$.

$$\begin{aligned}\Delta\ :\ &=\left\{\mathbf{t}=(t_1,t_2,t_3)\in\mathbb{R}_H^3:0\leq t_1,t_2,-t_3\leq 1\right\}\\ &=\ \left\{(t_1,t_2):t_1\geq 0,\ t_2\geq 0,\ t_1+t_2\leq 1\right\},\end{aligned}$$

By considering formula (4), we get

$$\int_{\Delta}\omega_\alpha(\|\mathbf{t}\|)\left|\sum_{k=0}^{n}a_{n,k}\left(\Theta_k(\mathbf{t})-\Theta_{k-1}(\mathbf{t})\right)\right|d\mathbf{t},$$

$$=\int_{\Delta}\omega_\alpha(t_1+t_2)\left|\sum_{k=0}^{n}a_{n,k}\left(\begin{array}{c}\frac{\sin\frac{(k+1)(t_1-t_2)\pi}{3}\sin\frac{(k+1)(t_2-t_3)\pi}{3}\sin\frac{(k+1)(t_3-t_1)\pi}{3}}{\sin\frac{(t_1-t_2)\pi}{3}\sin\frac{(t_2-t_3)\pi}{3}\sin\frac{(t_3-t_1)\pi}{3}}\\-\frac{\sin\frac{k(t_1-t_2)\pi}{3}\sin\frac{k(t_2-t_3)\pi}{3}\sin\frac{k(t_3-t_1)\pi}{3}}{\sin\frac{(t_1-t_2)\pi}{3}\sin\frac{(t_2-t_3)\pi}{3}\sin\frac{(t_3-t_1)\pi}{3}}\end{array}\right)\right|d\mathbf{t}.$$

By using the change of variables

$$s_1:=\frac{t_1-t_3}{3}=\frac{2t_1+t_2}{3},\ s_2:=\frac{t_2-t_3}{3}=\frac{t_1+2t_2}{3} \tag{13}$$

the integral becomes

$$3\int_{\widetilde{\Delta}}\omega_\alpha(s_1+s_2)\left|\sum_{k=0}^{n}a_{n,k}\left(\begin{array}{c}\frac{\sin((k+1)(s_1-s_2)\pi)\sin((k+1)s_2\pi)\sin((k+1)(-s_1\pi))}{\sin((s_1-s_2)\pi)\sin(s_2\pi)\sin((-s_1\pi))}\\-\frac{\sin(k(s_1-s_2)\pi)\sin(ks_2\pi)\sin(k(-s_1\pi))}{\sin((s_1-s_2)\pi)\sin(s_2\pi)\sin((-s_1)\pi)}\end{array}\right)\right|ds_1ds_2,$$

where $\widetilde{\Delta}$ is the image of Δ in the plane, that is

$$\widetilde{\Delta}:=\left\{(s_1,s_2):0\leq s_1\leq 2s_2,\ 0\leq s_2\leq 2s_1,\ s_1+s_2\leq 1\right\}.$$

Since the integrated function is symmetric with respect to s_1 and s_2, it is sufficient to estimate the integral over the triangle

$$\Delta^* := \left\{(s_1, s_2) \in \widetilde{\Delta} : s_1 \le s_2\right\} = \{(s_1, s_2) : s_1 \le s_2 \le 2s_1,\ s_1 + s_2 \le 1\}.$$

which is the half of $\widetilde{\Delta}$. Again by using the change of variables

$$s_1 := \frac{u_1 - u_2}{2},\ s_2 := \frac{u_1 + u_2}{2} \tag{14}$$

one can transform the triangle Δ^* to another triangle

$$\Gamma := \left\{(u_1, u_2) : 0 \le u_2 \le \frac{u_1}{3},\ 0 \le u_1 \le 1\right\}.$$

So now our necessity is estimating the integral

$$I_n = \int_\Gamma \omega_\alpha(u_1) \left|\sum_{k=0}^{n} a_{n,k} D_k^*(u_1, u_2)\right| du_1 du_2,$$

where

$$D_k^*(u_1, u_2) := \frac{\sin((k+1)(u_2)\pi)\sin(k+1)\left(\frac{u_1+u_2}{2}\pi\right)\sin\left((k+1)\frac{u_1-u_2}{2}\pi\right)}{\sin((u_2)\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)} - \frac{\sin(k(u_2)\pi)\sin\left(k\frac{u_1+u_2}{2}\pi\right)\sin\left(k\frac{u_1-u_2}{2}\pi\right)}{\sin((u_2)\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)}.$$

We obtain

$$D_k^*(u_1, u_2) = D_{k,1}^*(u_1, u_2) + D_{k,2}^*(u_1, u_2) + D_{k,3}^*(u_1, u_2) \tag{15}$$

where

$$D_{k,1}^*(u_1, u_2) := 2\cos\left(\left(k + \frac{1}{2}\right)u_2\pi\right) \times \frac{\sin\left(\left(\frac{1}{2}u_2\right)\pi\right)\sin\left((k+1)\frac{u_1+u_2}{2}\pi\right)\sin\left((k+1)\frac{u_1-u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)}$$

$$D_{k,2}^*(u_1, u_2) := 2\cos\left(\left(k + \frac{1}{2}\right)\frac{u_1 + u_2}{2}\pi\right) \times \frac{\sin((ku_2)\pi)\sin\left(\frac{1}{2}\frac{u_1+u_2}{2}\pi\right)\sin\left((k+1)\frac{u_1-u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)}$$

and

$$D_{k,3}^*(u_1, u_2) := 2\cos\left(\left(k + \frac{1}{2}\right)\frac{u_1 - u_2}{2}\pi\right) \times \frac{\sin((ku_2)\pi)\sin\left(k\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{1}{2}\frac{u_1-u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)}.$$

by elementary trigonometric identities.
Let divide the triangle into parts as Γ as $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\begin{aligned}
\Gamma_1 \ &: \ = \left\{(u_1, u_2) \in \Gamma : u_1 \le \frac{1}{n+1}\right\}, \\
\Gamma_2 \ &: \ = \left\{(u_1, u_2) \in \Gamma : \frac{1}{n+1} \le u_1,\ u_2 \le \frac{1}{3(n+1)}\right\}, \\
\Gamma_3 \ &: \ = \left\{(u_1, u_2) \in \Gamma : \frac{1}{n+1} \le u_1,\ \frac{1}{3(n+1)} \le u_2\right\}.
\end{aligned}$$

Thus $I_n = I_{n,1} + I_{n,2} + I_{n,3}$, where

$$I_{n,j} := \int\limits_{\Gamma_j} \omega_\alpha(u_1) \left| \sum_{k=0}^{n} a_{n,k} D_k^*(u_1,u_2) \right| du_1 du_2, \quad (j = 1, 2, 3).$$

The inequalities

$$\left| \frac{\sin nt}{\sin t} \right| \le n \tag{16}$$

and

$$\sin t \ge \frac{2}{\pi} t \left(0 \le t \le \frac{\pi}{2}\right), \tag{17}$$

is needed for our proofs to estimate the integrals $I_{n,1}, I_{n,2}$ and $I_{n,3}$. By (16), we have

$$\begin{aligned} I_{n,1} &= \int\limits_{\Gamma_1} \omega_\alpha(u_1) \left| \sum_{k=0}^{n} a_{n,k} D_k^*(u_1,u_2) \right| du_1 du_2 \lesssim \int\limits_{\Gamma_1} \omega_\alpha(u_1) \left(\sum_{k=0}^{n} (k+1)^2 a_{n,k} \right) du_1 du_2 \\ &\le (n+1)^2 \int\limits_0^{\frac{1}{3(n+1)}} \int\limits_{3u_2}^{\frac{1}{n+1}} \omega_\alpha(u_1)\, du_1 du_2 \le (n+1)^2 \omega_\alpha\left(\frac{1}{n+1}\right) \int\limits_0^{1/3(n+1)} \left(\frac{1}{n+1} - 3u_2\right) du_2 \\ &\le (n+1)^2 \omega_\alpha\left(\frac{1}{n+1}\right) \frac{1}{6(n+1)^2} \lesssim \omega_\alpha\left(\frac{1}{n+1}\right) \\ &\le \gamma\left(\frac{1}{n+1}\right) \omega_\beta\left(\frac{1}{n+1}\right) \le \gamma\left(\frac{1}{n+1}\right) \omega_\beta(1) \le \gamma\left(\frac{1}{n+1}\right). \end{aligned}$$

Let divide Γ_2 into two parts

$$\begin{aligned} \Gamma_2' &: = \left\{ (u_1,u_2) \in \Gamma : \ u_2 \le \frac{a_{n,0}}{3(n+1)} \right\}, \\ \Gamma_2'' &: = \left\{ (u_1,u_2) \in \Gamma : \ u_2 \ge \frac{a_{n,0}}{3(n+1)} \right\}, \end{aligned}$$

for estimate $I_{n,2}$.

And let consider the Lemma 1 of [6] we have

$$\int\limits_{1/n}^{1} \frac{\omega_\alpha(t)}{t^2 \omega_\beta(t)} dt \lesssim \begin{cases} n\gamma\left(\frac{1}{n}\right), & \textit{if } \alpha < 1 \textit{ or } \beta > 0 \\ n\gamma\left(\frac{1}{n}\right) \log n, & \textit{if } \alpha = 1 \textit{ and } \beta = 0. \end{cases} \tag{18}$$

for every natural number $n \ge 2$.
Therefore by (17),(15) and (18),

$$\begin{aligned} \int\limits_{\Gamma_2'} \omega_\alpha(u_1) \left| \sum_{k=0}^{n} a_{n,k} D_{k,1}^*(u_1,u_2) \right| du_1 du_2 &\lesssim \int\limits_0^{\frac{a_{n,0}}{3(n+1)}} \int\limits_{\frac{1}{n+1}}^{1} \frac{\omega_\alpha(u_1)}{u_1^2} du_1 du_2 \lesssim \int\limits_0^{\frac{a_{n,0}}{3(n+1)}} \omega_\beta(1) \int\limits_{\frac{1}{n+1}}^{1} \frac{\omega_\alpha(u_1)}{\omega_\beta(u_1)\, u_1^2} du_1 du_2 \\ &\lesssim \frac{a_{n,0}}{3(n+1)} \begin{cases} (n+1)\gamma\left(\frac{1}{n+1}\right), & \text{if } \alpha < 1 \text{ or } \beta > 0 \\ (n+1)\gamma\left(\frac{1}{n+1}\right) \log(n+1), & \text{if } \alpha = 1 \text{ and } \beta = 0 \end{cases} \\ &\lesssim a_{n,0} \begin{cases} \gamma\left(\frac{1}{n+1}\right), & \text{if } \alpha < 1 \text{ or } \beta > 0 \\ \gamma\left(\frac{1}{n+1}\right) \log(n+1), & \text{if } \alpha = 1 \text{ and } \beta = 0. \end{cases} \end{aligned}$$

By (16),(17) and (18) we obtain

$$\int\limits_{\Gamma_2'} \omega_\alpha(u_1)\left|\sum_{k=0}^{n} a_{n,k} D_{k,j}^*(u_1,u_2)\right| du_1 du_2 \lesssim n \int\limits_{0}^{\frac{a_{n,0}}{3(n+1)}} \int\limits_{\frac{1}{n+1}}^{1} \frac{\omega_\alpha(u_1)}{u_1} du_1 du_2 \lesssim n \frac{a_{n,0}}{3(n+1)} \int\limits_{\frac{1}{n+1}}^{1} \frac{\omega_\alpha(u_1)}{u_1} du_1$$

$$\lesssim a_{n,0} \int\limits_{\frac{1}{n+1}}^{1} \frac{\omega_\alpha(u_1)}{u_1^2} du_1 \lesssim a_{n,0}\omega_\beta(1) \int\limits_{\frac{1}{n+1}}^{1} \frac{\omega_\alpha(u_1)}{u_1^2 \omega_\beta(u_1)} du_1$$

$$\lesssim a_{n,0} \begin{cases} (n+1)\gamma\left(\frac{1}{n+1}\right), & \text{if } \alpha < 1 \text{ or } \beta > 0 \\ (n+1)\gamma\left(\frac{1}{n+1}\right)\log(n+1), & \text{if } \alpha = 1 \text{ and } \beta = 0. \end{cases}$$

for $j = 2, 3$.
Also the expression

$$D_k^*(u_1,u_2) = H_{k,1}(u_1,u_2) + H_{k,2}(u_1,u_2) + H_{k,3}(u_1,u_2) \tag{19}$$

where

$$H_{k,1}(u_1,u_2) = \frac{1}{2} \frac{\cos(2k+1)u_2\pi}{\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)}$$

$$H_{k,2}(u_1,u_2) = -\frac{1}{2} \frac{\cos\left((2k+1)\frac{u_1+u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1-u_2}{2}\pi\right)}$$

$$H_{k,3}(u_1,u_2) = \frac{1}{2} \frac{\cos\left((2k+1)\frac{u_1-u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)}$$

can be considered because of the well known equality

$$\sin 2x + \sin 2y + \sin 2z = -4\sin x \sin y \sin z$$

for $x + y + z = 0$.
We get

$$\left|\sum_{k=0}^{n} a_{n,k}\cos(2k+1)t\right| \lesssim A_n\left(\frac{1}{t}\right) + a_n\left(\frac{1}{t}\right)\frac{1}{\sin t} \quad (0 < t < \pi) \tag{20}$$

and

$$\left|\sum_{k=0}^{n} a_{n,k}\cos(2k+1)t\right| \lesssim A_n\left(\frac{1}{t}\right) \quad (0 < t < \frac{\pi}{2}). \tag{21}$$

by the method used in ([14], p.179).
We obtain

$$\left|\sum_{k=0}^{n} a_{n,k} H_{k,1}(u_1,u_2)\right| \lesssim \frac{1}{u_1^2} A_n\left(\frac{1}{\pi u_2}\right) \tag{22}$$

and

$$\left|\sum_{k=0}^{n} a_{n,k} H_{k,3}(u_1,u_2)\right| \lesssim \frac{1}{u_1 u_2} A_n\left(\frac{3}{\pi u_1}\right) \tag{23}$$

by (21) for $(u_1, u_2) \in \Gamma_2'' \cup \Gamma_3$. Also, the fact

$$\sin\left(\frac{u_1\pi}{2}\right) \lesssim \sin\left(\frac{(u_1+u_2)\pi}{2}\right) \tag{24}$$

and the relation (20) yield

$$\left|\sum_{k=0}^{n} a_{n,k} H_{k,2}(u_1,u_2)\right| \lesssim \frac{1}{u_1u_2} A_n\left(\frac{3}{\pi u_1}\right).$$

for by $(u_1, u_2) \in \Gamma_2'' \cup \Gamma_3$. We can compute the integral below by considering (17) and (18), that is

$$\int_{\Gamma_2''} \omega_\alpha(u_1)\left|\sum_{k=0}^{n} a_{n,k} H_{k,1}(u_1,u_2)\right| du_1du_2 = \int_{\frac{a_{n,0}}{3(n+1)}}^{\frac{1}{3(n+1)}} \int_{\frac{1}{n+1}}^{1} \frac{\omega_\alpha(u_1)}{u_1^2} du_1du_2$$

$$\lesssim \int_{\frac{a_{n,0}}{3(n+1)}}^{\frac{1}{3(n+1)}} \omega_\beta(1) \int_{\frac{1}{n+1}}^{1} \frac{\omega_\alpha(u_1)}{u_1^2\omega_\beta(u_1)} du_1du_2 \lesssim \left(\frac{1-a_{n,0}}{3(n+1)}\right) \int_{\frac{1}{n+1}}^{1} \frac{\omega_\alpha(u_1)}{u_1^2\omega_\beta(u_1)} du_1$$

$$\lesssim \begin{cases} \gamma\left(\frac{1}{n+1}\right), & \text{if } \alpha < 1 \text{ or } \beta > 0 \\ \gamma\left(\frac{1}{n+1}\right)\log(n+1), & \text{if } \alpha = 1 \text{ and } \beta = 0. \end{cases}$$

For $j = 2, 3$ (23) and (24) give

$$\int_{\Gamma_2''} \omega_\alpha(u_1)\left|\sum_{k=0}^{n} a_{n,k} H_{k,j}(u_1,u_2)\right| du_1du_2 = \int_{\frac{a_{n,0}}{3(n+1)}}^{\frac{1}{3(n+1)}} \int_{\frac{1}{n+1}}^{1} \frac{\omega_\alpha(u_1)}{u_1u_2} A_n\left(\frac{3}{\pi u_1}\right) du_1du_2$$

$$= \log\left(\frac{1}{a_{n,0}}\right) \int_{\frac{1}{n+1}}^{1} \frac{\omega_\alpha(u_1)}{u_1} A_n\left(\frac{3}{\pi u_1}\right) du_1 = \log\left(\frac{1}{a_{n,0}}\right) \int_{\frac{3}{\pi}}^{\frac{3}{\pi}(n+1)} \frac{\omega_\alpha\left(\frac{3}{\pi t}\right)}{t} A_n(t)\, dt$$

$$= \log\left(\frac{1}{a_{n,0}}\right) \sum_{k=1}^{n} \int_{\frac{3}{\pi}k}^{\frac{3}{\pi}(k+1)} \frac{\omega_\alpha\left(\frac{3}{\pi t}\right)}{t} A_n(t)\, dt \le \log\left(\frac{1}{a_{n,0}}\right) \sum_{k=1}^{n} \frac{\omega_\alpha\left(\frac{1}{k}\right)}{k} A_n\left(\frac{3}{\pi}(k+1)\right)$$

$$\le \log\left(\frac{1}{a_{n,0}}\right) \sum_{k=1}^{n} \frac{\omega_\alpha\left(\frac{1}{k}\right)}{k} A_{n,k+1} \lesssim \log\left(\frac{1}{a_{n,0}}\right) \sum_{k=1}^{n} \frac{\omega_\alpha\left(\frac{1}{k}\right)}{k} A_{n,k}.$$

Therefore the estimate can be given as

$$I_{n,2} \lesssim \begin{cases} a_{n,0}(n+1)\gamma\left(\frac{1}{n+1}\right) + \log\left(\frac{1}{a_{n,0}}\right) \sum\limits_{k=1}^{n} \frac{\omega_\alpha\left(\frac{1}{k}\right)}{k} A_{n,k}, & \text{if } \alpha < 1 \text{ or } \beta > 0 \\ a_{n,0}(n+1)\left(\gamma\left(\frac{1}{n+1}\right) + \sum\limits_{k=1}^{n} \frac{\omega_\alpha\left(\frac{1}{k}\right)}{k} A_{n,k}\right) \log\left(\frac{1}{a_{n,0}}\right)\log(n+1), & \text{if } \alpha = 1 \text{ and } \beta = 0. \end{cases}$$

By considering (22) and (7), we obtain

$$\int\limits_{\Gamma_3} \omega_\alpha(u_1)\left|\sum_{k=0}^{n} a_{n,k} H_{k,1}(u_1,u_2)\right| du_1 du_2 = \int\limits_{\frac{1}{3(n+1)}}^{\frac{1}{3}} \int\limits_{3u_2}^{1} \frac{\omega_\alpha(u_1)}{u_1^2} A_n\left(\frac{1}{\pi u_2}\right) du_1 du_2$$

$$= \frac{2}{3} \int\limits_{\frac{1}{3(n+1)}}^{\frac{1}{3}} \int\limits_{3u_2}^{1} \frac{\omega_\alpha(3u_2)}{u_1 u_2} A_n\left(\frac{1}{\pi u_2}\right) du_1 du_2 = \frac{2}{3} \int\limits_{\frac{1}{3(n+1)}}^{\frac{1}{3}} \frac{\omega_\alpha(3u_2)}{u_2} \log\left(\frac{1}{3u_2}\right) A_n\left(\frac{1}{\pi u_2}\right) du_2$$

$$\leq \log(n+1) \int\limits_{\frac{1}{3(n+1)}}^{\frac{1}{3}} \frac{\omega_\alpha(3u_2)}{u_2} A_n\left(\frac{1}{\pi u_2}\right) du_2 = \log(n+1) \int\limits_{\frac{3}{\pi}}^{\frac{3}{\pi}(n+1)} \frac{\omega_\alpha\left(\frac{3}{\pi t}\right)}{t} A_n(t)\, dt$$

$$= \log(n+1) \sum_{k=1}^{n} \int\limits_{\frac{3}{\pi}k}^{\frac{3}{\pi}(k+1)} \frac{\omega_\alpha\left(\frac{3}{\pi t}\right)}{t} A_n(t)\, dt = \log(n+1) \sum_{k=1}^{n} \frac{\omega_\alpha\left(\frac{1}{k}\right)}{k} A_n\left(\frac{3}{\pi}(k+1)\right)$$

$$\leq \log(n+1) \sum_{k=1}^{n} \frac{\omega_\alpha\left(\frac{1}{k}\right)}{k} A_{n,k}.$$

Also for $j = 2, 3$, we have

$$\int\limits_{\Gamma_3} \omega_\alpha(u_1)\left|\sum_{k=1}^{n} a_{n,k} H_{k,j}(u_1,u_2)\right| du_1 du_2 \lesssim \int\limits_{\frac{1}{n+1}}^{1} \int\limits_{\frac{1}{3(n+1)}}^{\frac{u_1}{3}} \frac{\omega_\alpha(u_1)}{u_1 u_2} A_n\left(\frac{3}{\pi u_1}\right) du_1 du_2$$

$$= \int\limits_{\frac{1}{n+1}}^{1} \frac{\omega_\alpha(u_1)}{u_1} \log((n+1)u_1) A_n\left(\frac{3}{\pi u_1}\right) du_1 \leq \log(n+1) \int\limits_{\frac{1}{n+1}}^{1} \frac{\omega_\alpha(u_1)}{u_1} A_n\left(\frac{3}{\pi u_1}\right) du_1$$

$$= \log(n+1) \int\limits_{\frac{3}{\pi}}^{\frac{3}{\pi}(n+1)} \frac{\omega_\alpha\left(\frac{3}{\pi t}\right)}{t} A_n(t)\, dt = \log(n+1) \sum_{k=1}^{n} \int\limits_{\frac{3}{\pi}k}^{\frac{3}{\pi}(k+1)} \frac{\omega_\alpha\left(\frac{3}{\pi t}\right)}{t} A_n(t)\, dt$$

$$= \log(n+1) \sum_{k=1}^{n} \frac{\omega_\alpha\left(\frac{1}{k}\right)}{k} A_n\left(\frac{3}{\pi}(k+1)\right) \leq \log(n+1) \sum_{k=1}^{n} \frac{\omega_\alpha\left(\frac{1}{k}\right)}{k} A_{n,k}$$

by (23) and (24). Therefore, we have the inequality

$$I_{n,3} \quad \lesssim \log(n+1) \sum_{k=1}^{n} \omega_\alpha\left(\tfrac{1}{k}\right) \tfrac{A_{n,k}}{k}.$$

We can write the inequality below, that is

$$\omega_\alpha\left(\frac{1}{n+1}\right) \leq \omega_\alpha\left(\frac{1}{n}\right) = \frac{n\omega_\alpha\left(\frac{1}{n}\right)}{n} = \sum_{k=1}^{n} \omega_\alpha\left(\frac{1}{n}\right) \frac{A_{n,n}}{n} = \sum_{k=1}^{n} \omega_\alpha\left(\frac{1}{k}\right) \frac{A_{n,k}}{k}$$

and

$$\gamma\left(\frac{1}{n+1}\right) \leq \gamma\left(\frac{1}{n}\right) = \gamma\left(\frac{1}{n}\right) \frac{n}{n} = \sum_{k=1}^{n} \gamma\left(\frac{1}{k}\right) \frac{A_{n,n}}{n} \leq \sum_{k=1}^{n} \gamma\left(\frac{1}{k}\right) \frac{A_{n,k}}{k}$$

since the sequence $\frac{A_{n,k}}{k}$ is non-increasing with respect to k.
The proof finishes by considering (11) and the last estimate. □

Proof. [Proof of Theorem 2.2.] Let $e_n(t) := f(t) - T_n^{(A)}(f)(t)$. Therefore we can write

$$\left\|f - T_n^{(A)}(f)\right\|_{H^{\omega}(\overline{\Omega})} = \left\|f - T_n^{(A)}(f)\right\|_{C_H(\overline{\Omega})} + \Lambda^{\omega_\beta}(e_n). \tag{25}$$

Now we need to estimate the integral

$$J_n := \int\limits_{\Omega} \left|f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})\right| \left|\sum_{k=0}^{n} a_{n,k} D_k(\mathbf{u})\right| d\mathbf{u}. \tag{26}$$

since

$$|e_n(t) - e_n(s)| \leq \frac{1}{|\Omega|} \int\limits_{\Omega} \left|f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})\right| \left|\sum_{k=0}^{n} a_{n,k} D_k(\mathbf{u})\right| d\mathbf{u}. \tag{27}$$

If $f \in H^{\omega_\alpha}(\overline{\Omega})$, then

$$\left|f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})\right| \leq c\omega_\beta(\|\mathbf{t}-\mathbf{s}\|) \frac{\omega_\alpha(\|\mathbf{u}\|)}{\omega_\beta(\|\mathbf{u}\|)} \tag{28}$$

which is proved in [6].
Thus, by (28) we get

$$\begin{aligned} J_n &= \int\limits_{\Omega} \left|f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})\right| \left|\sum_{k=0}^{n} a_{n,k} D_k(\mathbf{u})\right| d\mathbf{u} \\ &\lesssim \omega_\beta(\|\mathbf{t}-\mathbf{s}\|) \int\limits_{\Omega} \frac{\omega_\alpha(\|\mathbf{u}\|)}{\omega_\beta(\|\mathbf{u}\|)} \left|\sum_{k=0}^{n} a_{n,k} D_k(\mathbf{u})\right| d\mathbf{u} \end{aligned}$$

As in proof of Theorem 2.1, it is sufficient to compute the last integral over the triangle Δ. We obtain

$$\begin{aligned} \int\limits_{\Delta} \frac{\omega_\alpha(\|\mathbf{t}\|)}{\omega_\beta(\|\mathbf{t}\|)} |\Theta_n(\mathbf{t})| d\mathbf{t} &= 3 \int\limits_{\Gamma} \frac{\omega_\alpha(u_1)}{\omega_\beta(u_1)} \left|\sum_{k=0}^{n} a_{n,k} D_k^*(u_1, u_2)\right| du_1 du_2 \\ &= 3 \left(\int\limits_{\Gamma_1} + \int\limits_{\Gamma_2} + \int\limits_{\Gamma_3} \right) \frac{\omega_\alpha(u_1)}{\omega_\beta(u_1)} \left|\sum_{k=0}^{n} a_{n,k} D_k^*(u_1, u_2)\right| du_1 du_2. \end{aligned}$$

by transformations (13) and (14). Also by considering (16),

$$\int\limits_{\Gamma_1} \frac{\omega_\alpha(u_1)}{\omega_\beta(u_1)} \left|\sum_{k=0}^{n} a_{n,k} D_k^*(u_1, u_2)\right| du_1 du_2 \leq (n+1)^2 \int\limits_{\Gamma_1} \frac{\omega_\alpha(u_1)}{\omega_\beta(u_1)} du_1 du_2$$

$$= (n+1)^2 \int\limits_{0}^{\frac{1}{3(n+1)}} \int\limits_{3u_2}^{\frac{1}{n+1}} \frac{\omega_\alpha(u_1)}{\omega_\beta(u_1)} du_1 du_2 = (n+1)^2 \int\limits_{0}^{\frac{1}{3(n+1)}} \left(\int\limits_{3u_2}^{\frac{1}{n+1}} \frac{\omega_\alpha(u_1)}{\omega_\beta(u_1)} du_1 \right) du_2$$

$$\leq (n+1)^2 \gamma\left(\frac{1}{n+1}\right) \int\limits_{0}^{\frac{1}{3(n+1)}} du_2 \leq (n+1)\gamma\left(\frac{1}{n+1}\right).$$

Thus we get

$$J_{n,1} \leq (n+1)\omega_\alpha(\|t-s\|)\,\gamma\left(\frac{1}{n+1}\right).$$

When Γ_2 is divided into two parts as in the proof of Theorem 2.1, by considering (15)

$$\Gamma_2' := \left\{(u_1,u_2)\in\Gamma:\ u_2 \leq \frac{a_{n,0}}{3(n+1)}\right\},$$
$$\Gamma_2'' := \left\{(u_1,u_2)\in\Gamma:\ u_2 \geq \frac{a_{n,0}}{3(n+1)}\right\},$$

we get $J_{n,2} = J_{n,2}' + J_{n,2}''$, where

$$J_{n,2}' = \omega_\alpha(\|t-s\|)\int_{\Gamma_2'}\frac{\omega_\alpha(u_1)}{\omega_\beta(u_1)}\left|\sum_{k=0}^{n}a_{n,k}D_k^*(u_1,u_2)\right|du_1du_2$$
$$= \int_{\Gamma_2'}\frac{\omega_\alpha(u_1)}{\omega_\beta(u_1)}\sum_{k=0}^{n}a_{n,k}\left(\left|D_{k,1}^*(u_1,u_2)\right| + \left|D_{k,2}^*(u_1,u_2)\right| + \left|D_{k,3}^*(u_1,u_2)\right|\right)du_1du_2,$$

and

$$J_{n,2}'' = \omega_\alpha(\|t-s\|)\int_{\Gamma_2''}\frac{\omega_\alpha(u_1)}{\omega_\beta(u_1)}\left|\sum_{k=0}^{n}a_{n,k}D_k^*(u_1,u_2)\right|du_1du_2$$
$$= \int_{\Gamma_2''}\frac{\omega_\alpha(u_1)}{\omega_\beta(u_1)}\sum_{k=0}^{n}a_{n,k}\left(\left|H_{k,1}(u_1,u_2)\right| + \left|H_{k,2}(u_1,u_2)\right| + \left|H_{k,3}(u_1,u_2)\right|\right)du_1du_2.$$

By considering (17) and (18), we obtain

$$\int_{\Gamma_2'}\frac{\omega_\alpha(u_1)}{\omega_\beta(u_1)}\left|\sum_{k=0}^{n}a_{n,k}D_{k,1}^*(u_1,u_2)\right|du_1du_2 \lesssim \int_0^{\frac{a_{n,0}}{3(n+1)}}\int_{\frac{1}{n+1}}^{1}\frac{\omega_\alpha(u_1)}{u_1^2\omega_\beta(u_1)}du_1du_2$$
$$\lesssim \frac{a_{n,0}}{3(n+1)}\begin{cases}(n+1)\gamma\left(\frac{1}{n+1}\right), & \text{if } \alpha<1 \text{ or } \beta>0\\ (n+1)\gamma\left(\frac{1}{n+1}\right)\log(n+1), & \text{if } \alpha=1 \text{ and } \beta=0.\end{cases}$$
$$\lesssim a_{n,0}\begin{cases}\gamma\left(\frac{1}{n+1}\right), & \text{if } \alpha<1 \text{ or } \beta>0\\ \gamma\left(\frac{1}{n+1}\right)\log(n+1), & \text{if } \alpha=1 \text{ and } \beta=0.\end{cases}$$

Also we get

$$\int_{\Gamma_2'} \frac{\omega_\alpha(u_1)}{\omega_\beta(u_1)} \left| \sum_{k=0}^{n} a_{n,k} D_{k,j}^*(u_1,u_2) \right| du_1 du_2 \lesssim n \int_0^{\frac{a_{n,0}}{3(n+1)}} \int_{\frac{1}{n+1}}^{1} \frac{\omega_\alpha(u_1)}{u_1 \omega_\beta(u_1)} du_1 du_2$$

$$\lesssim n \frac{a_{n,0}}{3(n+1)} \int_{\frac{1}{n+1}}^{1} \frac{\omega_\alpha(u_1)}{u_1 \omega_\beta(u_1)} du_1 \leq a_{n,0} \int_{\frac{1}{n+1}}^{1} \frac{\omega_\alpha(u_1)}{u_1^2 \omega_\beta(u_1)} du_1$$

$$\lesssim a_{n,0} \begin{cases} (n+1)\gamma\left(\frac{1}{n+1}\right), & \text{if } \alpha < 1 \text{ or } \beta > 0 \\ (n+1)\gamma\left(\frac{1}{n+1}\right)\log(n+1), & \text{if } \alpha = 1 \text{ and } \beta = 0. \end{cases}$$

for $j = 2, 3$. These last two estimates gives

$$J_{n,2}' \lesssim \omega_\beta(\|t-s\|)\, a_{n,0}(n+1) \begin{cases} \gamma\left(\frac{1}{n+1}\right), & \text{if } \alpha < 1 \text{ or } \beta > 0 \\ \gamma\left(\frac{1}{n+1}\right)\log(n+1), & \text{if } \alpha = 1 \text{ and } \beta = 0. \end{cases}$$

Now we need to compute the integral

$$J_{n,2}'' = \omega_\beta(\|t-s\|) \int_{\Gamma_2''} \frac{\omega_\alpha(u_1)}{\omega_\beta(u_1)} \left| \sum_{k=0}^{n} a_{n,k} D_k^*(u_1,u_2) \right| du_1 du_2$$

$$= \omega_\beta(\|t-s\|) \int_{\Gamma_2''} \frac{\omega_\alpha(u_1)}{\omega_\beta(u_1)} \sum_{k=0}^{n} a_{n,k} \left(\left|H_{k,1}(u_1,u_2)\right| + \left|H_{k,2}(u_1,u_2)\right| + \left|H_{k,3}(u_1,u_2)\right| \right) du_1 du_2.$$

If we consider (17) and (18), we get

$$\int_{\Gamma_2''} \frac{\omega_\alpha(u_1)}{\omega_\beta(u_1)} \left| \sum_{k=0}^{n} a_{n,k} H_{k,1}(u_1,u_2) \right| du_1 du_2 = \int_{\frac{a_{n,0}}{3(n+1)}}^{\frac{1}{3(n+1)}} \int_{\frac{1}{n+1}}^{1} \frac{\omega_\alpha(u_1)}{u_1^2 \omega_\beta(u_1)} du_1 du_2$$

$$\lesssim \left(\frac{1-a_{n,0}}{3(n+1)} \right) \int_{\frac{1}{n+1}}^{1} \frac{\omega_\alpha(u_1)}{u_1^2 \omega_\beta(u_1)} du_1 du_2 \lesssim \begin{cases} \gamma\left(\frac{1}{n+1}\right), & \text{if } \alpha < 1 \text{ or } \beta > 0 \\ \gamma\left(\frac{1}{n+1}\right)\log(n+1), & \text{if } \alpha = 1 \text{ and } \beta = 0. \end{cases}$$

(23) and (24) give

$$\int_{\Gamma_2''}\frac{\omega_\alpha(u_1)}{\omega_\beta(u_1)}\left|\sum_{k=0}^{n}a_{n,k}H_{k,j}(u_1,u_2)\right|du_1du_2=\int_{\frac{a_{n,0}}{3(n+1)}}^{\frac{1}{3(n+1)}}\int_{\frac{1}{n+1}}^{1}\frac{\omega_\alpha(u_1)}{u_1u_2\omega_\beta(u_1)}A_n\left(\frac{3}{\pi u_1}\right)du_1du_2$$

$$=\log\left(\frac{1}{a_{n,0}}\right)\int_{\frac{1}{n+1}}^{1}\frac{\omega_\alpha(u_1)}{\omega_\beta(u_1)u_1}A_n\left(\frac{3}{\pi u_1}\right)du_1=\log\left(\frac{1}{a_{n,0}}\right)\int_{\frac{3}{\pi}}^{\frac{3}{\pi}(n+1)}\frac{\omega_\alpha\left(\frac{3}{\pi t}\right)}{\omega_\beta\left(\frac{3}{\pi t}\right)t}A_n(t)\,dt$$

$$=\log\left(\frac{1}{a_{n,0}}\right)\sum_{k=1}^{n}\int_{\frac{3}{\pi}k}^{\frac{3}{\pi}(k+1)}\frac{\omega_\alpha\left(\frac{3}{\pi t}\right)}{\omega_\beta\left(\frac{3}{\pi t}\right)t}A_n(t)\,dt\le\log\left(\frac{1}{a_{n,0}}\right)\sum_{k=1}^{n}\int_{\frac{3}{\pi}k}^{\frac{3}{\pi}(k+1)}\frac{\omega_\alpha(1/k)}{\omega_\beta(1/k)k}A_n\left(\frac{3}{\pi}(k+1)\right)$$

$$\le\log\left(\frac{1}{a_{n,0}}\right)\sum_{k=1}^{n}\int_{\frac{3}{\pi}k}^{\frac{3}{\pi}(k+1)}\frac{\omega_\alpha(1/k)}{\omega_\beta(1/k)k}A_{n,k+1}\lesssim\log\left(\frac{1}{a_{n,0}}\right)\sum_{k=1}^{n}\frac{\omega_\alpha(1/k)}{\omega_\beta(1/k)}\frac{A_{n,k}}{k}$$

for $j=2,3$. Thus, we obtain

$$J_{n,2}''\lesssim\omega_\beta(\|t-s\|)\begin{cases}\left(\gamma\left(\frac{1}{n+1}\right)+\sum\limits_{k=1}^{n}\gamma\left(\frac{1}{k}\right)\frac{A_{n,k}}{k}\right)\log\left(\frac{1}{a_{n,0}}\right), & \text{if } \alpha<1 \text{ or } \beta>0\\ \left(\gamma\left(\frac{1}{n+1}\right)+\sum\limits_{k=1}^{n}\gamma\left(\frac{1}{k}\right)\frac{A_{n,k}}{k}\right)\log\left(\frac{1}{a_{n,0}}\right)\log(n+1), & \text{if } \alpha=1 \text{ and } \beta=0\end{cases}$$

Thus we get

$$J_{n,2}\lesssim\omega_\beta(\|t-s\|)\begin{cases}a_{n,0}(n+1)\left(\gamma\left(\frac{1}{n+1}\right)+\sum\limits_{k=1}^{n}\gamma\left(\frac{1}{k}\right)\frac{A_{n,k}}{k}\right)\log\left(\frac{1}{a_{n,0}}\right), & \text{if } \alpha<1 \text{ or } \beta>0\\ a_{n,0}(n+1)\left(\gamma\left(\frac{1}{n+1}\right)+\sum\limits_{k=1}^{n}\gamma\left(\frac{1}{k}\right)\frac{A_{n,k}}{k}\right)\log\left(\frac{1}{a_{n,0}}\right)\log(n+1), & \text{if } \alpha=1 \text{ and } \beta=0.\end{cases}$$

Now we need to compute the integral

$$J_{n,3}=\omega_\beta(\|t-s\|)\int_{\Gamma_3}\frac{\omega_\alpha(u_1)}{\omega_\beta(u_1)}\left|\sum_{k=0}^{n}a_{n,k}D_k^*(u_1,u_2)\right|du_1du_2$$

$$=\omega_\beta(\|t-s\|)\int_{\Gamma_3}\frac{\omega_\alpha(u_1)}{\omega_\beta(u_1)}\sum_{k=0}^{n}a_{n,k}\left(\left|H_{k,1}(u_1,u_2)\right|+\left|H_{k,2}(u_1,u_2)\right|+\left|H_{k,3}(u_1,u_2)\right|\right)du_1du_2.$$

By considering (22) and (7), we have

$$\int_{\Gamma_3} \frac{\omega_\alpha(u_1)}{\omega_\beta(u_1)} \left| \sum_{k=0}^{n} a_{n,k} H_{k,1}(u_1,u_2) \right| du_1 du_2 = \int_{\frac{1}{3(n+1)}}^{\frac{1}{3}} \int_{3u_2}^{1} \frac{\omega_\alpha(u_1)}{u_1^2 \omega_\beta(u_1)} A_n\left(\frac{1}{\pi u_2}\right) du_1 du_2$$

$$\leq \frac{2}{3} \int_{\frac{1}{3(n+1)}}^{\frac{1}{3}} \int_{3u_2}^{1} \frac{\omega_\alpha(3u_2)}{u_1 u_2 \omega_\beta(3u_2)} A_n\left(\frac{1}{\pi u_2}\right) du_1 du_2 \leq \frac{2}{3} \int_{\frac{1}{3(n+1)}}^{\frac{1}{3}} \frac{\omega_\alpha(3u_2)}{u_2 \omega_\beta(3u_2)} \log\left(\frac{1}{3u_2}\right) A_n\left(\frac{1}{\pi u_2}\right) du_1 du_2$$

$$\leq \log(n+1) \int_{\frac{1}{3(n+1)}}^{\frac{1}{3}} \frac{\omega_\alpha(3u_2)}{u_2 \omega_\beta(3u_2)} A_n\left(\frac{1}{\pi u_2}\right) du_2 \leq \log(n+1) \int_{\frac{3}{\pi}}^{\frac{3}{\pi}(n+1)} \frac{\omega_\alpha\left(\frac{3}{\pi t}\right)}{\omega_\beta\left(\frac{3}{\pi t}\right) t} A_n(t)\, dt$$

$$= \log(n+1) \sum_{k=1}^{n} \int_{\frac{3}{\pi}k}^{\frac{3}{\pi}(k+1)} \frac{\omega_\alpha\left(\frac{3}{\pi t}\right)}{\omega_\beta\left(\frac{3}{\pi t}\right) t} A_n(t)\, dt \leq \log(n+1) \sum_{k=1}^{n} \int_{\frac{3}{\pi}k}^{\frac{3}{\pi}(k+1)} \frac{\omega_\alpha(1/k)}{\omega_\beta(1/k)\, k} A_n\left(\frac{3}{\pi}(k+1)\right)$$

$$\leq \log(n+1) \sum_{k=1}^{n} \int_{\frac{3}{\pi}k}^{\frac{3}{\pi}(k+1)} \frac{\omega_\alpha(1/k)}{\omega_\beta(1/k)\, k} A_{n,k+1} \lesssim \log(n+1) \sum_{k=1}^{n} \gamma\left(\frac{1}{k}\right) \frac{A_{n,k}}{k}$$

For (23) and (24), we have

$$\int_{\Gamma_3} \frac{\omega_\alpha(u_1)}{\omega_\beta(u_1)} \left| \sum_{k=0}^{n} a_{n,k} H_{k,j}(u_1,u_2) \right| du_1 du_2 = \int_{\frac{1}{n+1}}^{1} \int_{\frac{1}{3(n+1)}}^{\frac{u_1}{3}} \frac{\omega_\alpha(u_1)}{u_1 u_2 \omega_\beta(u_1)} A_n\left(\frac{3}{\pi u_1}\right) du_1 du_2$$

$$= \int_{\frac{1}{n+1}}^{1} \frac{\omega_\alpha(u_1)}{u_1 \omega_\beta(u_1)} \log((n+1)u_1)\, A_n\left(\frac{3}{\pi u_1}\right) du_1 \leq \log(n+1) \int_{\frac{1}{n+1}}^{1} \frac{\omega_\alpha(u_1)}{u_1 \omega_\beta(u_1)} A_n\left(\frac{3}{\pi u_1}\right) du_1$$

$$= \log(n+1) \int_{\frac{1}{n+1}}^{1} \frac{\omega_\alpha(u_1)}{\omega_\beta(u_1)\, u_1} A_n\left(\frac{3}{\pi u_1}\right) du_1 = \log((n+1)) \int_{\frac{3}{\pi}}^{\frac{3}{\pi}(n+1)} \frac{\omega_\alpha\left(\frac{3}{\pi t}\right)}{\omega_\beta\left(\frac{3}{\pi t}\right) t} A_n(t)\, dt$$

$$= \log(n+1) \sum_{k=1}^{n} \int_{\frac{3}{\pi}k}^{\frac{3}{\pi}(k+1)} \frac{\omega_\alpha\left(\frac{3}{\pi t}\right)}{\omega_\beta\left(\frac{3}{\pi t}\right) t} A_n(t)\, dt \leq \log(n+1) \sum_{k=1}^{n} \int_{\frac{3}{\pi}k}^{\frac{3}{\pi}(k+1)} \frac{\omega_\alpha\left(\frac{1}{k}\right)}{\omega_\beta\left(\frac{1}{k}\right) k} A_n\left(\frac{3}{\pi}(k+1)\right)$$

$$\leq \log(n+1) \sum_{k=1}^{n} \int_{\frac{3}{\pi}k}^{\frac{3}{\pi}(k+1)} \frac{\omega_\alpha\left(\frac{1}{k}\right)}{\omega_\beta\left(\frac{1}{k}\right) k} A_{n,k+1} \lesssim \log(n+1) \sum_{k=1}^{n} \gamma\left(\frac{1}{k}\right) \frac{A_{n,k}}{k}$$

for $j = 2, 3$. Thus, we get

$$J_{n,3} \lesssim \omega_\beta(\|\mathbf{t} - \mathbf{s}\|) \log(n+1) \sum_{k=1}^{n} \gamma\left(\frac{1}{k}\right) \frac{A_{n,k}}{k}.$$

Thus we get

$$\frac{|e_n(\mathbf{t})-e_n(\mathbf{s})|}{\omega_\beta(\|\mathbf{t}-\mathbf{s}\|)} \lesssim \begin{cases} a_{n,0}(n+1)\left(\gamma\left(\frac{1}{n+1}\right)+\sum\limits_{k=1}^{n}\gamma\left(\frac{1}{k}\right)\frac{A_{n,k}}{k}\right)\log\left(\frac{1}{a_{n,0}}\right), & \text{if } \alpha<1 \text{ or } \beta>0 \\ a_{n,0}(n+1)\left(\gamma\left(\frac{1}{n+1}\right)+\sum\limits_{k=1}^{n}\gamma\left(\frac{1}{k}\right)\frac{A_{n,k}}{k}\right)\log\left(\frac{1}{a_{n,0}}\right)\log(n+1), & \text{if } \alpha=1 \text{ and } \beta=0. \end{cases}$$

where $\mathbf{t}\neq\mathbf{s}$ which gives Λ^{ω_β}. The proof is finished by combining (11) and (25) . □

4. Nörlund and Riesz means

We close this paper by displaying some special cases of matrix mean such as Nörlund and Riesz means which we conclude from Theorem 2.1 and Theorem 2.2. We will not mention exactly the proofs but the interested reader can easily fill in the details.

Case I : Let $p=(p_k)$ be nonincreasing sequence of positive real numbers. If we take

$$a_{n,k} := \begin{cases} \frac{p_k}{P_n}, & 0\le k\le n \\ 0, & k>n \end{cases}$$

where $P_n := \sum\limits_{k=0}^{n} p_k$, then $A=(a_{n,k})$ satisfies (8), (9) and (10), and $T_n^{(A)}$ becomes the Riesz mean

$$R_n(p;f)=\frac{1}{P_n}\sum_{k=0}^{n}p_kS_k(f).$$

Theorem 2.1 gives

$$\left\|f-R_n(p;f)\right\|_{C_H(\overline{\Omega})} \lesssim \begin{cases} (n+1)\frac{p_0}{P_n^2}\log\frac{P_n}{p_0}\sum\limits_{k=1}^{n}\gamma\left(\frac{1}{k}\right)\frac{P_k}{k}, & \text{if } \alpha<1 \text{ or } \beta>0 \\ (n+1)\frac{p_0}{P_n^2}\log(n+1)\log\left(\frac{P_n}{p_0}\right)\sum\limits_{k=1}^{n}\gamma\left(\frac{1}{k}\right)\frac{P_k}{k}, & \text{if } \alpha=1 \text{ and } \beta=0 \end{cases} \tag{29}$$

and Theorem 2.2 yields

$$\left\|f-R_n(p;f)\right\|_{\omega_\beta} \lesssim \begin{cases} (n+1)\frac{p_0}{P_n^2}\log\left(\frac{P_n}{p_0}\right)\sum\limits_{k=1}^{n}\gamma\left(\frac{1}{k}\right)\frac{P_k}{k}, & \text{if } \alpha<1 \text{ or } \beta>0 \\ (n+1)\frac{p_0}{P_n^2}\log(n+1)\log\left(\frac{P_n}{p_0}\right)\sum\limits_{k=1}^{n}\gamma\left(\frac{1}{k}\right)\frac{P_k}{k}, & \text{if } \alpha=1 \text{ and } \beta=0 \end{cases} \tag{30}$$

for $f\in H^{\omega_\alpha}(\overline{\Omega})$.

Case II: Let $p=(p_k)$ be nondecreasing sequence of positive real numbers. In this case the matrix $A=(a_{n,k})$ with entries

$$a_{n,k} := \begin{cases} \frac{p_{n-k}}{P_n}, & 0\le k\le n \\ 0, & k>n. \end{cases}$$

where $P_n := \sum\limits_{k=0}^{n} p_k$, then $A=(a_{n,k})$ satisfies (8), (9) and (10), and $T_n^{(A)}$ becomes the Nörlund mean

$$N_n(p;f)=\frac{1}{P_n}\sum_{k=0}^{n}p_{n-k}S_k(f).$$

Theorem 2.1 gives

$$\left\|f - N_n(p;f)\right\|_{C_H(\overline{\Omega})} \lesssim \begin{cases} (n+1)\frac{p_n}{P_n^2}\log\left(\frac{P_n}{p_n}\right)\sum\limits_{k=1}^{n}\gamma\left(\frac{1}{k}\right)\frac{Q_{n,k}}{k}, & \text{if } \alpha<1 \text{ or } \beta>0 \\ (n+1)\frac{p_n}{P_n^2}\log(n+1)\log\left(\frac{P_n}{p_n}\right)\sum\limits_{k=1}^{n}\gamma\left(\frac{1}{k}\right)\frac{Q_{n,k}}{k}, & \text{if } \alpha=1 \text{ and } \beta=0 \end{cases} \quad (31)$$

and Theorem 2.2 yields

$$\left\|f - N_n(p;f)\right\|_{\omega_\beta} \lesssim \begin{cases} (n+1)\frac{p_n}{P_n^2}\log\left(\frac{P_n}{p_n}\right)\sum\limits_{k=1}^{n}\gamma\left(\frac{1}{k}\right)\frac{Q_{n,k}}{k}, & \text{if } \alpha<1 \text{ or } \beta>0 \\ (n+1)\frac{p_n}{P_n^2}\log(n+1)\log\left(\frac{P_n}{p_n}\right)\sum\limits_{k=1}^{n}\gamma\left(\frac{1}{k}\right)\frac{Q_{n,k}}{k}, & \text{if } \alpha=1 \text{ and } \beta=0 \end{cases} \quad (32)$$

for $f \in H^{\omega_\alpha}(\overline{\Omega})$ where $Q_{n,k} := \sum\limits_{v=n-k}^{n} p_v$.

Case III: If we take $p_k = 1 (k = 0, 1, ...)$, $R_n(p;f)$ and $N_n(p;f)$ become $(C,1)$ means $S_n^1(f)$ and both of (29) and (31) reduce to

$$\left\|f - S_n^1(f)\right\|_{C_H(\overline{\Omega})} \lesssim \frac{1}{n+1}\begin{cases} \log(n+1)\sum\limits_{k=1}^{n}\gamma\left(\frac{1}{k}\right), & \text{if } \alpha<1 \text{ or } \beta>0 \\ (\log(n+1))^2\sum\limits_{k=1}^{n}\gamma\left(\frac{1}{k}\right), & \text{if } \alpha=1 \text{ and } \beta=0 \end{cases}$$

for $f \in H^{\omega_\alpha}(\overline{\Omega})$ and both of (30) and (32) yields

$$\left\|f - S_n^1(f)\right\|_{\omega_\beta} \lesssim \frac{1}{n+1}\begin{cases} \log(n+1)\sum\limits_{k=1}^{n}\gamma\left(\frac{1}{k}\right), & \text{if } \alpha<1 \text{ or } \beta>0 \\ (\log(n+1))^2\sum\limits_{k=1}^{n}\gamma\left(\frac{1}{k}\right), & \text{if } \alpha=1 \text{ and } \beta=0 \end{cases}$$

for $f \in H^{\omega_\alpha}(\overline{\Omega})$ which proved in [6].

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