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Induced topologies on certain Banach algebras

M. Eshaghi[a], A. Ghaffari[a], M. B. Sahabi[a]

[a]*Department of Mathematics, Semnan University, P. O. Box 35195-363, Semnan, Iran*

Abstract. Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity bounded by 1. Two new topologies τ_{so} and τ_{wo} are introduced on $\mathcal{A}$. We study these topologies and compare them with each other and with the norm topology. The properties of τ_{so} and τ_{wo} are then studied further and we pay attention to the group algebra $L^1(G)$ of a locally compact group G. Various necessary and sufficient conditions are found for a locally compact group G to be finite.

1. Introduction and Notations

Let $\mathcal{A}$ be a Banach algebra. Terminologies and notations not explained in this section will be explained or referenced in the next section. Given a subspace X of $\mathcal{A}$, and a functional f on X, we will variously denote the value of f on $x \in X$ by $f(x)$ and $\langle f, x\rangle$. If X is any normed space, let's agree to denote by ball X the closed unite ball in X. The first Arens multiplication is defined as follows in three steps. For a, b in $\mathcal{A}$, f in $\mathcal{A}^*$ and E, F in $\mathcal{A}^{**}$, the elements fa, Ff of $\mathcal{A}^*$ and EF of $\mathcal{A}^{**}$ are defined as follows:

$$\langle fa, b\rangle = \langle f, ab\rangle,\ \langle Ff, a\rangle = \langle F, fa\rangle \text{ and } \langle EF, f\rangle = \langle E, Ff\rangle.$$

As is well-known [2], the second dual $\mathcal{A}^{**}$ of $\mathcal{A}$ endowed with the first Arens multiplication is a Banach algebra. The basic properties of this multiplication are as follows. For F fixed in $\mathcal{A}^{**}$, the mapping $E \mapsto EF$ is weak*-weak* continuous. For E fixed in $\mathcal{A}^{**}$, the mapping $F \mapsto EF$ is general not continuous unless E is in $\mathcal{A}$ (for more information see [2]). Whence the topological center of $\mathcal{A}^{**}$ with respect to this multiplication is defined as follows

$$Z_t(\mathcal{A}^{**}) = \{E \in \mathcal{A}^{**};\ F \mapsto EF \text{ is weak}^*\text{-weak}^* \text{ continuous on } \mathcal{A}^{**}\}.$$

When $\mathcal{A}$ has a bounded right approximate identity $\{e_\alpha\}$, any weak* cluster point E of $\{e_\alpha\}$ in $\mathcal{A}^{**}$ satisfies firstly $aE = a$ and then $FE = F$ for all $F \in \mathcal{A}^{**}$. Thus E is a right identity for $\mathcal{A}^{**}$, but not usually an identity for $\mathcal{A}^{**}$; however for $a \in \mathcal{A}^*$ it is true that $Ea = a$.
By $\mathcal{A}^*\mathcal{A}$ we denote the subspace of $\mathcal{A}^*$ consisting of the functionals of the form fa; for all f in $\mathcal{A}^*$ and a in $\mathcal{A}$. When $\mathcal{A}$ has a bounded approximate identity the Cohen-Hewitt factorization theorem shows that $\mathcal{A}^*\mathcal{A}$ is a norm closed linear subspace of $\mathcal{A}^*$. The dual of the space $\mathcal{A}^*\mathcal{A}$ equipped with the multiplication induced

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Email addresses: meshaghi@semnan.ac.ir (M. Eshaghi), aghaffari@semnan.ac.ir (A. Ghaffari), b-sahabi@yahoo.com (M. B. Sahabi)

by that of $\mathcal{A}^{**}$ is also a Banach algebra (for more details see [1]).
In [10], Peralta et al., among the other things, studied properties of the strong* topology and in particular compare in to the w-right topology. They characterize the class of Banach spaces X for which the topologies $s^*(X)$ and $\rho(X)$ coincide on bounded sets of X, see also [3].
In this paper, we continue our work [5] in the study of a Banach algebra $\mathcal{A}$ defined with respect to two new topologies on $\mathcal{A}$. Our purpose is to introduce two new topologies on a Banach algebra. We study these topologies and compare those two topologies with each other and with norm topology. Finally, we will shift our attention to group algebras and begin some discussion on group algebras.

2. Induced topologies on Banach algebras

Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity bounded by 1, and for a in $\mathcal{A}$ define $T_a : \mathcal{A}^* \to \mathcal{A}^*\mathcal{A}$ by $T_a(f) = fa$. Then $\mathcal{A}$ can be embedded into $\mathcal{B}(\mathcal{A}^*, \mathcal{A}^*\mathcal{A})$ by a linear map T so that $T(a) = T_a$. Indeed it is obvious that $\|T_a\| \leq \|a\|$. Now let $\{e_\alpha\}$ be a bounded approximate identity bounded by 1. For any $\epsilon > 0$, there exists $f \in \mathcal{A}^*$ such that $\|f\| \leq 1$ and $\|a\| \leq |\langle f, a\rangle| + \epsilon$. Since $\{e_\alpha\}$ is a bounded approximate identity, we may choose $e_\alpha \in \{e_\alpha\}$ such that $\|ae_\alpha - a\| < \epsilon$. Therefore

$$\begin{aligned} \|a\| &\leq |\langle f, a\rangle| + \epsilon \leq |\langle f, ae_\alpha\rangle| + 2\epsilon = |\langle fa, e_\alpha\rangle| + 2\epsilon \\ &= |\langle T_a(f), e_\alpha\rangle| + 2\epsilon \leq \|T_a(f)\| + 2\epsilon \leq \|T_a\| + 2\epsilon. \end{aligned}$$

As $\epsilon > 0$ may be chosen arbitrarily, $\|a\| \leq \|T_a\|$. Since $\mathcal{B}(\mathcal{A}^*, \mathcal{A}^*\mathcal{A})$ carries naturally the strong operator topology (weak operator topology), T allows us to consider the induced topology on $\mathcal{A}$, which we denote by τ_{so} (τ_{wo}). From the definition we immediately derive $\tau_{wo} \leq \tau_{so} \leq \tau_{\|.\|}$.

Proposition 2.1. *Let $\mathcal{A}$ have a bounded approximate identity bounded by 1 and suppose E is a convex set in $\mathcal{A}$. Then the τ_{wo}-closure $\overline{E}^{wo}$ of E is equal to its original closure $\overline{E}$.*

Proof. Every τ_{wo}-neighborhood of 0 in $\mathcal{A}$ contains a set of the form

$$\bigcap_{i=1}^{m}\bigcap_{j=1}^{n}\{a \in \mathcal{A}; |\langle F_i, T_a(f_j)\rangle| < \epsilon\}$$

where $F_i \in \mathcal{A}^{**}$, $f_j \in \mathcal{A}^*$ and $\epsilon > 0$. $\overline{E}^{wo}$ is τ_{wo}-closed, hence originally closed, so that $\overline{E} \subseteq \overline{E}^{wo}$. To obtain the opposite inclusion, we first prove that $\overline{E}^{so} \subseteq \overline{E}$. Choose $a \in \overline{E}^{so}$. There exists a net $\{a_\beta\}$ in E converging to a in the τ_{so}-topology. We will prove that $\{a_\beta\}$ converges to a in the weak topology of $\mathcal{A}$. Fix any $f \in \mathcal{A}^*$. Obviously $\{fa_\beta\}$ converges to fa in the norm topology. Let $\{e_\alpha\}$ be a bounded approximate identity bounded by 1 for $\mathcal{A}$. Let $\epsilon > 0$ be given. There exists β_0 such that $\|fa_\beta - fa\| < \epsilon$ for all $\beta \geq \beta_0$. Fix $\beta \geq \beta_0$. Consider a fixed element e_α in $\{e_\alpha\}$ such that

$$\|a_\beta - a_\beta e_\alpha\| < \epsilon, \ \|ae_\alpha - a\| < \epsilon.$$

We have

$$\begin{aligned} |\langle f, a_\beta - a\rangle| &\leq |\langle f, a_\beta - a_\beta e_\alpha\rangle| + |\langle f, a_\beta e_\alpha - ae_\alpha\rangle| + |\langle f, ae_\alpha - a\rangle| \\ &\leq \|f\|\|a_\beta - a_\beta e_\alpha\| + \|fa_\beta - fa\|\|e_\alpha\| + \|f\|\|ae_\alpha - a\| \\ &< 2\|f\|\epsilon + \epsilon. \end{aligned}$$

This shows that $\{a_\beta\}$ converges to a in the weak topology. On the other hand, the weak closure $\overline{E}^{w}$ of E is equal to its original closure $\overline{E}$, see Theorem 3.12 in [11]. This shows that $\overline{E}^{so} \subseteq \overline{E}$.

We next prove that $\overline{E}^{wo} \subseteq \overline{E}^{so}$. Let $a_0 \notin \overline{E}^{so}$. Part (b) of the separation Theorem 3.4 in [11] shows that there exist a τ_{so}-continuous linear functional L on $\mathcal{A}$ and $\gamma \in \mathbb{R}$ such that, for every $a \in E$,

$$ReL(a_0) < \gamma < ReL(a).$$

There exists a finite subset $\{f_1, \cdots, f_n\}$ of $\mathcal{A}^*$ and an $\epsilon > 0$ such that $\sup\{\|T_a(f_i)\|, 1 \le i \le n\} < \epsilon$, $a \in \mathcal{A}$ implies $|L(a)| < 1$. Consider the linear space $\mathcal{A}^* \times \cdots \times \mathcal{A}^*$ of all n-tuples $(f_1, \cdots, f_n)$ with $f_i \in \mathcal{A}^*$, $i = 1, \cdots, n$. Norm in $\mathcal{A}^* \times \cdots \times \mathcal{A}^*$ is

$$\|(f_1, \cdots, f_n)\| = \max\{\|f_1\|, \cdots, \|f_n\|\}.$$

Define $\pi : \mathcal{A} \to \mathcal{A}^* \times \cdots \times \mathcal{A}^*$ by $\pi(a) = (T_a(f_1), \cdots, T_a(f_n))$. If $a \in \mathcal{A}$ and $\pi(a) = 0$, $|L(ma)| < 1$ for all $m \in \mathbb{N}$. This shows that $L(a) = 0$. Hence $\lambda(\pi(a)) = L(a)$ define a linear functional λ on $\pi(\mathcal{A})$. Extend λ to a linear functional Λ on $\mathcal{A}^* \times \cdots \times \mathcal{A}^*$. This means that there exist $F_i \in \mathcal{A}^{**}$ such that $\Lambda(f_1, \cdots, f_n) = \sum_{i=1}^n \langle F_i, f_i\rangle$. Therefore

$$L(a) = \lambda(\pi(a)) = \sum_{i=1}^n \langle F_i, f_i a\rangle.$$

The set $\{a \in \mathcal{A}; Re \sum_{i=1}^n \langle F_i, f_i a\rangle = ReL(a) < \gamma\}$ is therefore a τ_{wo}-neighborhood of a_0 that dose not intersect E. Thus a_0 is not in $\overline{E}^{wo}$. This proves $\overline{E}^{wo} \subseteq \overline{E}$. □

Example 2.2. *Consider $G = \mathbb{Z}$, the additive group of the integers. For each $n \in \mathbb{N}$, let $l^1(\mathbb{Z})_n$ be a copy of $l^1(\mathbb{Z})$ and $\mathcal{A} = (\sum_{n=1}^\infty \bigoplus l^1(\mathbb{Z})_n)_0$ be their c_0-sum. It is easy to see that $\{e_n\}$ (($e_n = (1, 1, \cdots, 1, \cdots)$, 1 occurs n times) is a bounded approximate identity for $\mathcal{A}$. For each*

$$f = \{f_n\} \in \mathcal{A}^* = \Big(\sum_{n=1}^\infty \bigoplus l^1(\mathbb{Z})_n^*\Big)_1,$$

we have

$$\|fe_n - f\| = \sum_{k=n+1}^\infty \|f_k\| \to 0, \text{ as } n \to \infty.$$

$\mathcal{A}$ is a non unital Banach algebra. If $\{e_n\}$ converges to some $e \in \mathcal{A}$, then e is unite element of $\mathcal{A}$. This is a contradiction. We immediately conclude that the τ_{so}-topology is strictly weaker than the norm-topology.

Proposition 2.3. *Let $\mathcal{A}$ have a bounded approximate identity bounded by 1. Then the following assertions holds.*

(i) Every τ_{so}-continuous linear functional $L : \mathcal{A} \to \mathbb{C}$ is originally bounded, and vice versa.

(ii) Every τ_{so}-bounded set is originally bounded, and vice versa

Proof. Let L be a τ_{so}-continuous linear functional on $\mathcal{A}$. Let $\epsilon > 0$ be given. There exists a finite subset $\{f_1, \cdots, f_n\}$ of $\mathcal{A}^*$ and an $\delta > 0$ such that $\max\{\|T_a(f_i)\|; 1 \le i \le n\} < \delta$, $a \in \mathcal{A}$ implies $|L(a)| < \epsilon$. If $a \in \{a \in \mathcal{A}; \max\{\|f_1\|, \cdots, \|f_n\|\}\|a\| < \delta\}$, then $|L(a)| < \epsilon$, because

$$\{a \in \mathcal{A}; \max\{\|f_1\|, \cdots, \|f_n\|\}\|a\| < \delta\} \subseteq \bigcap_{i=1}^n \{a \in \mathcal{A}; \|T_a(f_i)\| < \delta\}.$$

This shows that L is norm continuous.

Conversely, suppose that $L \in \mathcal{A}^*$ is not τ_{so}-continuous. Let $\mathcal{F}$ denote the collection of all finite subsets F of $\mathcal{A}^*$, and for every $F \in \mathcal{F}$, $\epsilon > 0$, let

$$V_{(F,\epsilon)} = \{a \in \mathcal{A}; \|T_a(f)\| < \epsilon \text{ for every } f \in F\}.$$

For every $F \in \mathcal{F}$ and $\epsilon > 0$, there exists $a_{(F,\epsilon)} \in V_{(F,\epsilon)}$ such that $|L(a_{(F,\epsilon)})| > 1$ (which is not empty by the assumption). Partially order $\mathcal{D} = \{(F, \epsilon); F \in \mathcal{F}, \epsilon > 0\}$ by declaring $(F', \epsilon') \ge (F, \epsilon)$ to mean that $F \subseteq F'$ and $\epsilon' \le \epsilon$. We claim that $a_{(F,\epsilon)} \to 0$ in the weak topology on $\mathcal{A}$. Suppose that $f_0 \in \mathcal{A}^*$ and $\epsilon_0 > 0$. Let $(F, \epsilon) \ge (\{f_0\}, \epsilon_0)$. Thus $f_0 \in \mathcal{F}$ and $\epsilon \le \epsilon_0$. It follows that $\|f_0 a_{(F,\epsilon)}\| < \epsilon \le \epsilon_0$. Let $\{e_\alpha\}$ be an approximate identity for $\mathcal{A}$ of bound 1. We have $|\langle f_0 a_{(F,\epsilon)}, e_\alpha\rangle| < \epsilon \le \epsilon_0$ for all α, and so $|\langle f_0, a_{(F,\epsilon)}\rangle| \le \epsilon \le \epsilon_0$. This shows that $\langle f_0, a_{(F,\epsilon)}\rangle \to 0$. Hence $a_{(F,\epsilon)} \to 0$ weakly. On the other hand, $|L(a_{(F,\epsilon)})| > 1$ for all $F \in \mathcal{F}$ and $\epsilon > 0$ which is a contradiction.

Suppose that E is a norm bounded set of $\mathcal{A}$. There exists $M < \infty$ such that $\|a\| \leq M$ for all $a \in E$. For each f in $\mathcal{A}^*$ define $\rho_f : \mathcal{A} \to [0, \infty)$ by $\rho_f(a) = \|T_a(f)\|$. Then ρ_f is a seminorm. It is easy to see that $\Omega = \{\rho_f; f \in \mathcal{A}^*\}$ separates the points of $\mathcal{A}$ and make $\mathcal{A}$ into a locally convex space. The topology defined by theses seminorms is τ_{so}-topology on $\mathcal{A}$. On the other hand, $\|\rho_f(a)\| \leq \|f\|\|a\|$ for all $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$. It follows that every $\rho_f \in \Omega$ is bounded on E. Hence E is τ_{so}-bounded by (b) of Theorem 1.37 in [11].

Finally, suppose E is τ_{so}-bounded. Fix $f \in \mathcal{A}^*$. Since

$$\{a \in \mathcal{A}; \|T_a(f)\| < 1\}$$

is a neighborhood of 0, $E \subseteq n\{a \in \mathcal{A}; \|T_a(f)\| < 1\}$ for some $n \in \mathbb{N}$. Hence $\|T_a(f)\| \leq n$ for every $a \in E$. On the other hand $\mathcal{A}$ has a bounded approximate identity bounded by 1. Thus $|\langle f, a\rangle| \leq n$ for every $a \in E$. Put $K = \{f \in \mathcal{A}^*; \|f\| \leq 1\}$. Since K is convex and weak*-compact and since the functions $f \mapsto \langle f, a\rangle$ are weak*-continuous, we can apply Theorem 2.9 in [11] to conclude that there is a constant $M < \infty$ such that $|\langle f, a\rangle| \leq M$ for every $a \in E$ and $f \in K$. It follows that $\|a\| \leq M$ for every $a \in E$. This finishes the proof of the proposition. □

Suppose X and Y are Banach spaces. It is shown that in [8], there is a 'right topology' for X such that a linear map from X into Y is weakly compact precisely when it is a continuous map from X, equipped with the right topology, into Y, equipped with the norm topology. Quasi completely continuous multilinear operators have been studied by Peralta et al. in [9]. They have obtained a number of interesting and nice results. The following Proposition shows that a linear operator L from $\mathcal{A}$ to itself is τ_{so}-continuous if and only if L is τ_{wo}-continuous.

Proposition 2.4. *Let $\mathcal{A}$ have a bounded approximate identity bounded by 1. A linear operator L on $\mathcal{A}$ is continuous with respect to the τ_{so}-topology if and only if it is continuous with respect to the τ_{wo}-topology.*

Proof. Let U be the open unit ball in $\mathcal{A}$. By Proposition 2.3 (ii), U is τ_{so}-bounded. We claim that $L(U)$ is τ_{so}-bounded. Every τ_{so}-neighborhood of 0 in $\mathcal{A}$ contains a set of form

$$\{a \in \mathcal{A}; \|T_a(f_i)\| < \epsilon \text{ for } 1 \leq i \leq n\},$$

where $f_i \in \mathcal{A}^*$ and $\epsilon > 0$. By the hypothesis, there exist $f_1', \cdots, f_m' \in \mathcal{A}^*$ and $\delta > 0$ such that

$$L\Big(\bigcap_{i=1}^{m}\{a \in \mathcal{A}; \|T_a(f_i')\| < \delta\}\Big) \subseteq \bigcap_{i=1}^{n}\{a \in \mathcal{A}; \|T_a(f_i)\| < \epsilon\}.$$

Since U is τ_{so}-bounded,

$$U \subseteq k\bigcap_{i=1}^{m}\{a \in \mathcal{A}; \|T_a(f_i')\| < \delta\}$$

for some $k \in \mathbb{N}$. Thus

$$L(U) \subseteq k\bigcap_{i=1}^{n}\{a \in \mathcal{A}; \|T_a(f_i)\| < \epsilon\}.$$

This shows that $L(U)$ is τ_{so}-bounded and, by Proposition 2.3 (ii), is also norm bounded. We conclude that L is norm continuous. Now, let $\{a_\alpha\}$ be a net in $\mathcal{A}$ such that $a_\alpha \to a$ in the τ_{wo}-topology. For $f \in \mathcal{A}^*$, $T_{a_\alpha}(f) \to T_a(f)$ in the weak topology. Since L is norm bounded, $foT \in \mathcal{A}^*$, and so $T_{a_\alpha}(foL) \to T_a(foL)$ in the weak topology. This shows that L is τ_{wo}-continuous.

Conversely, assume that L is not τ_{so}-continuous. Then there is a neighborhood

$$W = \bigcap_{i=1}^{n}\{a \in \mathcal{A}; \|T_a(f_i)\| < \epsilon\}$$

of 0 in $\mathcal{A}$ such that $L^{-1}(W)$ contains no neighborhood of 0 in $\mathcal{A}$. An argument similar to the proof of Proposition 2.3 shows that there is a net $\{a_\alpha\}$ in $\mathcal{A}$ such that $T_{a_\alpha}(f) \to 0$ for all $f \in \mathcal{A}^*$ (in the weak topology) and $\max\{\|T_{a_\alpha}(f_i)\|; 1 \leq i \leq n\} \geq \epsilon$ for all α, which is a contradiction. □

3. Some more results on group algebras

Let G be a locally compact group. The left Haar measure on the locally compact group G is λ. Let $M(G)$ be the Banach algebra of regular Borel measures on G. Let $L^p(G)$ $(1 \leq p \leq \infty)$ have the usual meanings. Convolution of functions φ and f is defined by $\varphi * f(x) = \int \varphi(y)f(y^{-1}x)dy$ whenever the integral makes sense. Usually, φ, ψ will be elements of the $L^1(G)$, f, g elements of $L^\infty(G)$, and E, F elements of $L^1(G)^{**}$. For any subset A of G, 1_A denotes the characteristic function of A. For every $\varphi : G \to \mathbb{C}$, we define $\tilde{\varphi}(x) = \varphi(x^{-1})$. The second dual $L^1(G)^{**}$ of $L^1(G)$ is a Banach algebra with the first Arens product. For each $f \in L^\infty(G)$ and $\varphi \in L^1(G)$, $f\varphi = \frac{1}{\Delta}\tilde{\varphi} * f$ and $\varphi f = f * \tilde{\varphi}$, here Δ is the modular function of G. Duality between Banach spaces is denoted by $\langle\ \rangle$; thus for $f \in L^\infty(G)$ and $\varphi \in L^1(G)$, we have $\langle f, \varphi\rangle = \int f(x)\varphi(x)dx$.

Example 3.1. *A locally compact group G is finite if and only if τ_{so}-topology is compatible with τ_{wo}-topology on $L^1(G)$. Indeed, if G is a finite group, then the dimension of $L^1(G)$ is finite. Let $\dim L^1(G) = n$. Then every basis of $L^1(G)$ induces an isomorphism of $L^1(G)$ onto $\mathbb{C}^n$. Theorem 1.21 in [11] shows that this isomorphism must be a homeomorphism. Consequently $\tau_{so} = \tau_{wo}$.*

Conversely, let $\tau_{so} = \tau_{wo}$. Consider a fixed element φ in $L^1(G)$. For every $\psi \in L^1(G)$,

$$\langle 1_G\varphi, \psi\rangle = \langle 1_G, \varphi * \psi\rangle = \int\int \psi(y^{-1}x)\varphi(y)dydx = \langle 1_G, \psi\rangle\langle 1_G, \varphi\rangle.$$

This shows that $1_G\varphi = \langle 1_G, \varphi\rangle 1_G$. By hypothesis, there exist $F_1, \cdots, F_n \in L^\infty(G)^*$, $f_1, \cdots, f_m \in L^\infty(G)$ and $\epsilon > 0$ such that

$$W = \bigcap_{i=1}^{n}\bigcap_{j=1}^{m}\{\phi; |\langle F_i, f_j\phi\rangle| < \epsilon\} \subseteq \{\phi; |\langle 1_G, \varphi\rangle| = \|1_G\phi\| < 1\}.$$

Since $\varphi \mapsto (\langle F_1, f_1\varphi\rangle, \cdots, \langle F_n, f_m\varphi\rangle)$ maps $L^1(G)$ into $\mathbb{C}^{mn}$ with null space N, we see that $\dim L^1(G) \leq mn + \dim N$. If G is an infinite group, then $L^1(G)$ is an infinity dimensional space. If $\varphi \in N \subseteq W$ and $\langle 1, \varphi\rangle \neq 0$, $k\varphi \in N \subseteq W$ for all $k \in \mathbb{N}$. It follows that $k|\langle 1, \varphi\rangle| < 1$ for all $k \in \mathbb{N}$, which is a contradiction.

A further consequence is one expressing a relationship between convergence and τ_{wo} convergence in $L^1(G)$.

Theorem 3.2. *A sequence $\{\varphi_n\}$ converges to φ in $L^1(G)$ if and only if the following conditions hold:*

(i) $\{\varphi_n\}$ converges to φ in the τ_{wo}-topology;

(ii) $\{\varphi_n\}$ converges to φ locally in measure, that is, for each compact set $K \subseteq G$ and each number $\epsilon > 0$ one has

$$\lim_{n\to\infty} \lambda(K \cap \{x \in G; |\varphi_n(x) - \varphi(x)| \geq \epsilon\}) = 0.$$

Proof. Since norm topology is stronger than τ_{wo}-topology, it is clear that norm convergence implies τ_{wo} convergence. If $\{\varphi_n\}$ converges to φ in $L^1(G)$, then a simple verification shows that $\{\varphi_n\}$ converges to φ locally in measure.

To prove the converse, let $\{e_\alpha\}$ be an approximate identity for $L^1(G)$ of bound 1. Without loss of generality, we may assume that $e_\alpha \to E$ $(E \in L^\infty(G)^*)$ in the weak*-topology. Let $A \subseteq G$ be a measurable set. It is easy to see that $\langle 1_A, \varphi_n\rangle \to \langle 1_A, \varphi\rangle$. Indeed,

$$\begin{aligned}\lim_{n\to\infty}\langle 1_A, \varphi_n\rangle &= \lim_{n\to\infty}\langle E1_A, \varphi_n\rangle = \lim_{n\to\infty}\langle E, 1_A\varphi_n\rangle \\ &= \langle E, 1_A\varphi\rangle = \langle 1_A, \varphi\rangle.\end{aligned}$$

We first show that for any given $\epsilon > 0$ there exists a compact subset K in G such that

$$\limsup\left\{\langle 1_{G\setminus K}, |\varphi_n - \varphi|\rangle, n \in \mathbb{N}\right\} < \epsilon.$$

Since $\varphi_n \in L^1(G)$, the function φ_n vanishes outside of a σ-finite set. Therefore $\bigcup_{n=1}^{\infty}\{x;\ \varphi_n(x) \neq 0\} \cup \{x; \varphi(x) \neq 0\}$ can be written as $S = (\bigcup_{m=1}^{\infty} K_m) \cup N$ where $\lambda(N) = 0$ and $\{K_m\}$ is an increasing sequence of compact sets. Let M be the subset of $L^\infty(G)$ formed of all functions equal almost everywhere to characteristic functions of measurable subsets of G. We define $\|s - s'\|_m = \langle 1_{K_m}, |s - s'|\rangle$. These norms $\|.\|_m$ $(m = 1, 2, \cdots)$ make M into a Banach space. Applying Baire's theorem, we infer the existence of a measurable $A \subseteq S$ and natural numbers m and l and a number $\delta > 0$ such that

$$\left|\langle 1_B, \varphi_n - \varphi\rangle\right| \leq \frac{\epsilon}{16} \text{ for all } n \geq l$$

whenever $B \subseteq S$ is measurable, and $\|1_B - 1_A\|_m < \delta$. For arbitrary $n \in \mathbb{N}$, we write

$$\varphi_n - \varphi = ({\varphi_n}^1 - \varphi^1) - ({\varphi_n}^2 - \varphi^2) + i[({\varphi_n}^3 - \varphi^3) - ({\varphi_n}^4 - \varphi^4)],$$

where ${\varphi_n}^i - \varphi^i$'s are functions in $L^1(G)^+$ and

$$\min\{{\varphi_n}^1 - \varphi^1, {\varphi_n}^2 - \varphi^2\} = \min\{{\varphi_n}^3 - \varphi^3, {\varphi_n}^4 - \varphi^4\} = 0.$$

If $B \subseteq S$ is measurable, $\|1_B - 1_A\|_m < \delta$ and $n \geq l$, then

$$\left|\langle 1_B, {\varphi_n}^i - \varphi^i\rangle\right| \leq \frac{\epsilon}{16} \text{ for all } 1 \leq i \leq 4.$$

It follows that $\langle 1_B, |\varphi_n - \varphi|\rangle < \frac{\epsilon}{4}$ for all $n \geq l$. If $B \subseteq G \setminus K_m$, then

$$\begin{aligned} \langle 1_B, |\varphi_n - \varphi|\rangle &= \langle 1_{B\cap S}, |\varphi_n - \varphi|\rangle \\ &= \langle 1_{(B\cap S)\cup(A\cap K_m)}, |\varphi_n - \varphi|\rangle - \langle 1_{A\cap K_m}, |\varphi_n - \varphi|\rangle < \frac{\epsilon}{2} \end{aligned}$$

for all $n \geq l$. This being true for each measurable $B \subseteq G \setminus K_m$, it follows that $\langle 1_{G\setminus K_m}, |\varphi_n - \varphi|\rangle < \frac{\epsilon}{2}$. By hypothesis $\{\varphi_n\}$ converges to φ locally in measure. Therefore $\langle \chi_{K_m}, |\varphi_n - \varphi|\rangle$ converges to 0. Consequently

$$\begin{aligned} \limsup \|\varphi_n - \varphi\|_1 &\leq \limsup\{\langle 1_{K_m}, |\varphi_n - \varphi|\rangle; n \in \mathbb{N}\} \\ &+ \limsup\left\{\langle 1_{G\setminus K_m}, |\varphi_n - \varphi|\rangle, n \in \mathbb{N}\right\} < \epsilon. \end{aligned}$$

Therefore $\{\varphi_n\}$ converges to φ with respect to norm topology. $\square$

Recall that a left multiplier of a Banach algebra $\mathcal{A}$ is a bounded linear operator T which maps $\mathcal{A}$ into $\mathcal{A}$ satisfing $T(ab) = T(a)b$ for any a and b in $\mathcal{A}$. Wendel's Theorem tells us that the left multiplier algebra of $L^1(G)$ is the measure algebra $M(G)$.

Proposition 3.3. *Let G be a locally compact group. Then the absolutely convex hulls of τ_{so}-relatively compact sets in $L^1(G)$ are again τ_{so}-relatively compact.*

Proof. For each $f \in L^\infty(G)$, let $L^\infty(G)_f$ be a copy of $L^\infty(G)$. Let

$$\mathcal{X} = \prod\left\{L^\infty(G)_f; f \in L^\infty(G)\right\}.$$

If $\{x_\alpha\}$ is a net in $\mathcal{X}$, then $x_\alpha \to x$ if and only if $x_\alpha(f) \to x(f)$ for all $f \in L^\infty(G)$. Therefore $\varphi \mapsto \{f\varphi\}_{f\in L^\infty(G)}$ is a continuous one-to-one linear function from $L^1(G)$ with respect to τ_{so}-topology into the product of the norm topology on $L^\infty(G)$. Let K be a compact subset of $L^1(G)$. Define the projection function $\pi_f : \mathcal{X} \to L^\infty(G)$ by $\pi_f(x) = x(f)$. π_f is continuous by definition of the product topology; hence $\pi_f(K)$ is a compact subset of $L^\infty(G)$ for all $f \in L^\infty(G)$. Obviously $K \subseteq \prod\{\pi_f(K); f \in L^\infty(G)\}$. Hence

$$\left\{\sum_{i=1}^{n} c_i k_i;\ k_i \in K, c_i \in \mathbb{C}, n \in \mathbb{N}, \sum_{i=1}^{n} |c_i| \leq 1\right\} \subseteq \prod\{S_f; f \in L^\infty(G)\}$$

where S_f is absolutely convex hull of $\pi_f(K)$ in $L^\infty(G)$. On the other hand, the absolutely convex hull of a compact subset of a Banach space is compact. Hence $\prod\{S_f; f \in L^\infty(G)\}$ is compact by the Tychonoff theorem, since S_f is compact for each $f \in L^\infty(G)$. We will show that the absolutely convex hull of K is actually a closed subset of $\mathcal{X}$. Indeed, let $\{\varphi_\alpha\}$ be a net in absolutely convex hull of K that converges to some $T \in \mathcal{X}$ in the product topology. As K is compact, K is τ_{so}-bounded. By Proposition 2.3 (*ii*), K is norm bounded. Finally, the absolutely convex hull of K is again norm bounded. It is easy to see that T is a bounded linear operator on $L^\infty(G)$. Let $T^* : L^\infty(G)^* \to L^\infty(G)^*$ be adjoint to T. Then T^* is a left multiplier on $L^\infty(G)^*$. In fact, for $E, F \in L^\infty(G)^*$, $f \in L^\infty(G)$, we have

$$\begin{aligned}\langle T^*(EF), f\rangle &= \langle EF, T(f)\rangle = \lim_\alpha \langle EF, f\varphi_\alpha\rangle \\ &= \lim_\alpha \langle E, Ff\varphi_\alpha\rangle = \langle E, T(Ff)\rangle \\ &= \langle T^*(E), Ff\rangle = \langle T^*(E)F, f\rangle.\end{aligned}$$

Hence $T^*(EF) = T^*(E)F$, showing that T^* is a left multiplier on $L^\infty(G)^*$. We next show that for each $\varphi \in L^1(G)$, $T^*(\varphi) \in L^1(G)$. Indeed, if $\{F_\beta\}$ is a net in $L^\infty(G)^*$ and $F_\beta \to F$ in the weak*-topology, then

$$\begin{aligned}\lim_\beta \langle T^*(\varphi)F_\beta, f\rangle &= \lim_\beta \langle \varphi, T(F_\beta f)\rangle = \lim_\beta \langle \varphi, F_\beta T(f)\rangle \\ &= \lim_\beta \langle \varphi F_\beta, T(f)\rangle = \lim_\beta \langle F_\beta, T(f)\varphi\rangle \\ &= \langle F, T(f)\varphi\rangle = \langle \varphi F, T(f)\rangle \\ &= \langle T^*(\varphi F), f\rangle = \langle T^*(\varphi)F, f\rangle\end{aligned}$$

for all $f \in L^\infty(G)$. Hence $T^*(\varphi)F_\beta \to T^*(\varphi)F$, showing that $T^*(\varphi)$ is in the topological center of $L^\infty(G)^*$. It is known that if G is a locally compact topological group, then the topological center of $L^1(G)^{**}$ is $L^1(G)$, so $T^*(\varphi) \in L^1(G)$ [7]. Therefore T^* restricted to $L^1(G)$ is a left multiplier from $L^1(G)$ into $L^1(G)$. Consequently there exist $\mu \in M(G)$ such that $T^*(\varphi) = \mu * \varphi$ for all $\varphi \in L^1(G)$. For every $f \in L^\infty(G)$, $f\varphi_\alpha$ converges to $f\mu$ in the norm topology. Since $L^1(G)$ is a closed two-sided ideal in the algebra $M(G)$, we have $\mu \in L^1(G)$ [4]. Consequently $\{\varphi_\alpha\}$ converges to $\mu \in L^1(G)$. It follows that the absolutely convex hull of K in $L^1(G)$ is compact, being a closed subset of the compact set $\prod\{S_f; f \in L^\infty(G)\}$.

□

Theorem 3.4. *Let G be a locally compact group. Then the following statements are equivalent.*

(i) G is a finite group.

(ii) ball $L^1(G)$ is compact with respect to the τ_{wo}-topology.

Proof. It is clear that (*i*) implies (*ii*), so it will be shown that (*ii*) implies (*i*). Let $F \in$ball $L^\infty(G)^*$. We claim that for every finite subset $A = \{f_1, ..., f_n\}$ of $L^\infty(G)$ and $\epsilon > 0$, there exists $\varphi_{A,\epsilon} \in$ ball $L^1(G)$ such that $\sum_{i=1}^n |\langle F - \hat{\varphi}_{A,\epsilon}, f_i\rangle|^2 < \epsilon$. Let $A = \{f_1, ..., f_n\}$, say, and let $\epsilon > 0$ be given. Consider the map

$$\Lambda : \text{ball } L^1(G) \to \mathbb{R}^n \quad \varphi \mapsto (\langle F - \hat{\varphi}, f_1\rangle, ..., \langle F - \hat{\varphi}, f_n\rangle).$$

We show that $\beta := \inf\{|\Lambda(\varphi)|^2;\ \varphi \in \text{ball } L^1(G)\} = 0$. For every $k \in \mathbb{N}$ there exists $\varphi_k \in$ ball $L^1(G)$ such that $\beta \le |\Lambda(\varphi_k)| < \beta + \frac{1}{k}$. Since $\{(\langle F - \hat{\varphi}_k, f_1\rangle, ..., \langle F - \hat{\varphi}_k, f_n\rangle);\ k \in \mathbb{N}\} \subseteq \mathbb{R}^n$ is bounded, withought loss of generality, we can assume that

$$(\langle F - \hat{\varphi}_k, f_1\rangle, ..., \langle F - \hat{\varphi}_k, f_n\rangle) \to (r_1, ..., r_n).$$

For every $0 < t < 1$, $\varphi \in$ ball $L^1(G)$ and $k \in \mathbb{N}$, we have

$$|\Lambda((1-t)\varphi_k + t\varphi)|^2 = \sum_{i=1}^n |\langle F, f_i\rangle - \langle f_i, \varphi_k\rangle + t\langle f_i, \varphi_k - \varphi\rangle|^2.$$

It is easy to see that

$$\begin{aligned}\beta &\leq \limsup |\Lambda((1-t)\varphi_k + t\varphi)|^2 \\ &\leq \beta + 2tRe\sum_{i=1}^{n}(\overline{\langle F, f_i\rangle - r_i})(r_i - \langle f_i, \varphi\rangle) + t^2\sum_{i=1}^{n}|r_i - \langle f_i, \varphi\rangle|^2.\end{aligned}$$

Therefore

$$2Re\sum_{i=1}^{n} s_i(r_i - \langle f_i, \varphi\rangle) \geq -t\sum_{i=1}^{n}|r_i - \langle f_i, \varphi\rangle|^2.$$

where $s_i = \overline{\langle F, f_i\rangle - r_i}$. As $0 < t < 1$ may be chosen arbitrary, we must have $Re\sum_{i=1}^{n} r_i s_i \geq Re\sum_{i=1}^{n} s_i\langle f_i, \varphi\rangle$. Let $f = \sum_{i=1}^{n} s_i f_i$. Therefore $Re\langle f, \varphi\rangle \leq Re\sum_{i=1}^{n} r_i s_i$. As $\varphi \in$ ball $L^1(G)$ may be chosen arbitrary, we must have $\|f\| \leq Re\sum_{i=1}^{n} r_i s_i$. We have

$$\begin{aligned}\beta &= \lim_k |\Lambda(\varphi_k)|^2 = \sum_{i=1}^{n}|\langle F, f_i\rangle - r_i|^2 = Re\langle F, f\rangle - Re\sum_{i=1}^{n} r_i s_i \\ &\leq \|F\|\|f\| - \|f\| \leq \|f\| - \|f\| = 0.\end{aligned}$$

Now, let ball $L^1(G)$ is compact in the τ_{wo}-topology. For every finite set $A = \{f_1, ..., f_n\}$ and for every $\epsilon > 0$, we can choose an $\varphi_{A,\epsilon} \in$ ball $L^1(G)$ such that $\sum_{i=1}^{n}|\langle F - \hat{\varphi}_{A,\epsilon}, f_i\rangle|^2 < \epsilon$. Order the pairs (A, ϵ) in the obvious manner. After passing to a subnet if necessary, we can assume that $\varphi_{A,\epsilon} \to \varphi$ in the τ_{wo}-topology. We show that $F = \hat{\varphi}$.
Let E be a right identity in $L^\infty(G)^*$. Suppose that $f_0 \in L^\infty(G)$ and $\epsilon > 0$ be given. There exists a finite subset A_0 in ball $L^1(G)$ and $\epsilon_0 > 0$ such that $|\langle E, f_0\varphi_{A',\epsilon'}\rangle - \langle E, f_0\varphi\rangle| < \frac{\epsilon}{2}$ for all $(A', \epsilon') \geq (A_0, \epsilon_0)$. Put $A' = A_0 \cup \{f_0\}$ and $\epsilon' = \frac{\min\{\epsilon_0, \epsilon\}}{2}$. We have

$$\begin{aligned}|\langle F - \hat{\varphi}, f_0\rangle &\leq |\langle F, f_0\rangle - \langle E, f_0\varphi_{A',\epsilon'}\rangle| + |\langle E, f_0\varphi_{A',\epsilon'} - \langle E, f_0\varphi\rangle| \\ &< \epsilon' + \frac{\epsilon}{2} \leq \epsilon.\end{aligned}$$

This shows that $F = \hat{\varphi} \in L^1(G)$. Consequently $L^1(G)$ is an ideal in $L^\infty(G)^*$. It is known that $L^1(G)$ is a two-sided ideal in $L^\infty(G)^*$ if and only if G is a compact group [6], and so G is compact. On the other hand $E \in L^\infty(G)^* = L^1(G)$ is a two-sided identity for $L^1(G)$, and so G is discrete [4]. Finally G is a finite group. □

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