Coefficient bounds and Fekete-Szegő inequality for a certain family of holomorphic and bi-univalent functions defined by (M,N)-Lucas polynomials

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Abstract. In the current work, we use the (M,N)-Lucas Polynomials to introduce a new family of holomorphic and bi-univalent functions which involve a linear combination between Bazilevič functions and \(\beta\)-pseudo-starlike function defined in the unit disk \(D\) and establish upper bounds for the second and third coefficients of functions belongs to this new family. Also, we discuss Fekete-Szegő problem in this new family.

1. Introduction

The Lucas Polynomials play an important role in a diversity of disciplines as the mathematical, statistical, physical and engineering sciences (see, for example [10, 14, 38]).

Let \(A\) stands for the collection of functions \(f\) that are holomorphic in the unit disk \(D = \{z \in \mathbb{C} : |z| < 1\}\) that have the form:

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n.
\] (1)

Further, let \(S\) indicate the sub-collection of the set \(A\) containing functions from \(D\) satisfying (1) which are univalent in \(D\). According to the Koebe one-quarter theorem (see [9]), every function \(f \in S\) has an inverse \(f^{-1}\) defined by \(f^{-1}(f(z)) = z, (z \in \mathbb{D})\) and \(f(f^{-1}(w)) = w, (|w| < r_0(f), r_0(f) \geq \frac{1}{4})\), where

\[
f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right)w^4 + \cdots.
\] (2)

A function \(f \in A\) is said to be bi-univalent in \(D\) if both \(f\) and \(f^{-1}\) are univalent in \(D\), let we name by the notation \(\Sigma\) the set of bi-univalent functions in \(D\) satisfying (1). In fact, Srivastava et al. [28] have

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actually revived the study of holomorphic and bi-univalent functions in recent years, recalling the following examples of functions in the class $\Sigma$:

\[
\frac{z}{1-z}, \quad -\log(1-z), \quad \frac{1}{2} \log \left( \frac{1+z}{1-z} \right).
\]

The Koebe function is not a member of the bi-univalent function class $\Sigma$, same as other common examples of functions in $S$ such as:

\[
z - \frac{z^2}{2}, \quad \frac{z}{1-z^2}.
\]

Their work was followed by such articles as those by Frasin and Aouf [11], Altinkaya and Yalçın [2], Srivastava and Wanas [29], Srivastava et al. [26] and others (see, for example [8, 17, 18, 20–25, 30, 32–37]).

More pioneering work was made by Srivastava et al. in [27] where they studied coefficients of meromorphic bi-univalent functions.

Lewin [13] was the first to investigate the class of bi-univalent functions, showing that the first coefficient of the Taylor series expansion of a bi-univalent function satisfies $|a_2| < 1.51$.

Brannan and Clunie [6] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$ and Netanyahu [16] showed that $\max |a_2| = \frac{1}{2}$. The coefficient estimate problem for each of the coefficients $|a_n| (n \in \mathbb{N} \setminus \{1, 2\})$ is still an open problem.

A function $f \in \mathcal{A}$ is called Bazilevič function in $\mathbb{D}$ if (see [19])

\[
\Re \left\{ \frac{z^{1-a} f'(z)}{(f(z))^{1-a}} \right\} > 0, \quad (z \in \mathbb{D}, a \geq 0).
\]

A function $f \in \mathcal{A}$ is called $\beta$-pseudo-starlike function in $\mathbb{D}$ if (see [5])

\[
\Re \left\{ \frac{zf'(z)^\beta}{f(z)} \right\} > 0, \quad (z \in \mathbb{D}, \beta \geq 1).
\]

We use the definition of subordination between holomorphic functions: let the functions $f$ and $g$ be holomorphic in $\mathbb{D}$, we say that the function $f$ is subordinate to $g$, if there exists a Schwarz function $\omega$ holomorphic in $\mathbb{D}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ ($z \in \mathbb{D}$) such that $f(z) = g(\omega(z))$. This subordination is indicated by $f \prec g$ or $f(z) \prec g(z)$ ($z \in \mathbb{D}$) (see [15]).

For the polynomials $M(x)$ and $N(x)$ with real coefficients, the $(M,N)$-Lucas Polynomials $L_{MN,k}(x)$ are defined by the following recurrence relation (see [12]):

\[
L_{MN,k}(x) = M(x)L_{MN,k-1}(x) + N(x)L_{MN,k-2}(x) \quad (k \geq 2),
\]

with

\[
L_{MN,0}(x) = 2, \quad L_{MN,1}(x) = M(x),
\]

\[
L_{MN,2}(x) = M^2(x) + 2N(x), \quad L_{MN,3}(x) = M^3(x) + 3M(x)N(x).
\]

The generating function of the $(M,N)$-Lucas Polynomial $L_{MN,k}(x)$ (see [14]) is given by

\[
T_L(M,N;x,z) = \sum_{k=2}^{\infty} L_{MN,k}(x)z^k = \frac{2 - M(x)z}{1 - M(x)z - N(x)z^2}.
\]

Note that for particular values of $M$ and $N$, the $(M, N)$-polynomial $L_m(x)$ leads to various polynomials, among those, we list few cases here (see, for more details [3]):

(i) For $M(x) = x$ and $N(x) = 1$, we obtain the Lucas polynomials $L_n(x)$.
(ii) For $M(x) = 2x$ and $N(x) = 1$, we obtain the Pell-Lucas polynomials $Q_n(x)$.
(iii) For $M(x) = 1$ and $N(x) = 2x$, we obtain the Jacobsthal-Lucas polynomials $j_n(x)$.
(iv) For $M(x) = 3x$ and $N(x) = -2$, we obtain the Fermat-Lucas polynomials $f_n(x)$.
(v) For $M(x) = 2x$ and $N(x) = -1$, we have the Chebyshev polynomials $T_n(x)$ of the first kind.
2. Main Results

We begin this section by defining the family $\mathcal{L}_{MN}(\lambda, \alpha, \beta; x)$ as follows:

**Definition 2.1.** For $0 \leq \lambda \leq 1; \alpha \geq 0; \beta \geq 1$ let $\mathcal{L}_{MN}(\lambda, \alpha, \beta; x)$ denote the subclass of $\Sigma$ such that

\[
(1 - \lambda) \frac{z^{1 - \alpha} f'(z)}{(f(z))^{1 - \alpha}} + \lambda \frac{z (f'(z))^{\beta}}{f(z)} < T_L (M, N; x, z) - 1
\]

and

\[
(1 - \lambda) \frac{w^{1 - \alpha} \left(f^{-1}(w)\right)'}{(f^{-1}(w))^{1 - \alpha}} + \lambda \frac{w \left(f^{-1}(w)\right)'^{\beta}}{f^{-1}(w)} < T_L (M, N; x, w) - 1,
\]

where $f^{-1}$ is given by (2).

In particular, if we choose $\alpha = \lambda = 0$ or $\alpha = \beta = 1$ in Definition 2.1, we have $\mathcal{L}_{MN}(0, 0, \beta; x) = \mathcal{L}_{MN}(1, \alpha, 1; x) = P_{\sigma}(0; x)$ for the family of functions $f \in \Sigma$ given by (1) and satisfying the following subordinations:

\[
\frac{zf'(z)}{f(z)} < T_L (M, N; x, z) - 1
\]

and

\[
\frac{w (f^{-1}(w))'}{f^{-1}(w)} < T_L (M, N; x, w) - 1.
\]

If $M(x) = 1, N(x) = 0$ then $\frac{zf'(z)}{f(z)} < T_L (1, 0; x, z) - 1 = \frac{1}{1 - z}$.

If $M(x) = 2x, N(x) = -1$ then $\frac{zf'(z)}{f(z)} < T_L (2x, -1; x, z) - 1 = \frac{1}{1 - 2xz + z^2}$.

**Theorem 2.2.** For $0 \leq \lambda \leq 1, \alpha \geq 0$ and $\beta \geq 1$, let $f$ belongs to the family $\mathcal{L}_{MN}(\lambda, \alpha, \beta; x)$ and $N(x) \neq 0$; let denote

\[
\Omega(\lambda, \alpha, \beta) = (1 - \lambda)(\alpha + 1) + \lambda(2\beta - 1),
\]

\[
E(\lambda, \alpha, \beta, M(x), N(x)) = \sqrt{\frac{\sqrt{2 M(x)} \sqrt{|M(x)|}}{\sqrt{\left|(1 - \lambda)(\alpha + 2)(\alpha + 1) + 2\lambda\beta(2\beta - 1) - 2\Omega^2(\lambda, \alpha, \beta)\right|m^2(x) - 4\Omega^2(\lambda, \alpha, \beta)N(x)}}}
\]

and

\[
F(\lambda, \alpha, \beta, M(x)) = \frac{|M(x)|}{\Omega(\lambda, \alpha, \beta)};
\]

then

\[
|\varphi_2| \leq \min \{E(\lambda, \alpha, \beta, M(x), N(x)), F(\lambda, \alpha, \beta, M(x))\}
\]

and

\[
|\varphi_3| \leq \frac{M^2(x)}{\Omega^2(\lambda, \alpha, \beta)} + \frac{|M(x)|}{(1 - \lambda)(\alpha + 2) + \lambda(3\beta - 1)}.
\]
Proof. Suppose that \( f \in \mathcal{L}_{MN}(\lambda, \alpha, \beta; x) \). Then there exist two holomorphic functions \( \phi, \psi : \mathbb{D} \rightarrow \mathbb{D} \) given by
\[
\phi(z) = r_1z + r_2z^2 + r_3z^3 + \cdots \quad (z \in \mathbb{D})
\]
and
\[
\psi(w) = s_1w + s_2w^2 + s_3w^3 + \cdots \quad (w \in \mathbb{D}),
\]
with \( \phi(0) = \psi(0) = 0, |\phi(z)| < 1, |\psi(w)| < 1, z, w \in \mathbb{D} \) such that
\[
(1 - \lambda) \frac{z^{1-a}f'(z)}{(f(z))^{1-a}} + \lambda \frac{z}{f(z)} = -1 + L_{MN,0}(x) + L_{MN,1}(x)\phi(z) + L_{MN,2}(x)\psi(z) + \cdots
\]
Combining (5), (6), (7) and (8), yield
\[
(1 - \lambda) \frac{z^{1-a}f'(z)}{(f(z))^{1-a}} + \lambda \frac{z}{f(z)} = 1 + L_{MN,1}(x)r_1z + \left[ L_{MN,1}(x)r_2 + L_{MN,2}(x)r_2^2 \right] z^2 + \cdots
\]
and
\[
(1 - \lambda) \frac{w^{1-a}(f^{-1}(w))'}{(f^{-1}(w))^{1-a}} + \lambda \frac{w}{f^{-1}(w)} = 1 + L_{MN,1}(x)s_1w + \left[ L_{MN,1}(x)s_2 + L_{MN,2}(x)s_2^2 \right] w^2 + \cdots.
\]
It is quite well-known that if \( |\phi(z)| < 1 \) and \( |\psi(w)| < 1, z, w \in \mathbb{D} \), we get
\[
|r_j| \leq 1 \quad \text{and} \quad |s_j| \leq 1 \quad (j \in \mathbb{N}).
\]
In the light of (9) and (10), after simplifying, we find that
\[
[(1 - \lambda)(\alpha + 1) + \lambda(2\beta - 1)]a_2 = L_{MN,1}(x)r_1,
\]
\[
[(1 - \lambda)(\alpha + 2) + \lambda(3\beta - 1)]a_3 + \left[ \frac{1}{2}(1 - \lambda)(\alpha + 2)(\alpha - 1) + \lambda(2\beta(\beta - 2) + 1) \right] a_2^2 = L_{MN,1}(x)r_2 + L_{MN,2}(x)r_1^2,
\]
\[
- [(1 - \lambda)(\alpha + 1) + \lambda(2\beta - 1)] a_2 = L_{MN,1}(x)s_1
\]
and
\[
[(1 - \lambda)(\alpha + 2) + \lambda(3\beta - 1)](2a_2^2 - a_3) + \left[ \frac{1}{2}(1 - \lambda)(\alpha + 2)(\alpha - 1) + \lambda(2\beta(\beta - 2) + 1) \right] a_2^2 = L_{MN,1}(x)s_2 + L_{MN,2}(x)s_1^2.
\]
It follows from (12) and (14) that
\[ r_1 = -s_1 \] (16)
and
\[ 2[(1 - \lambda)(\alpha + 1) + \lambda(2\beta - 1)]^2 a_2^2 = L_{MN,1}^2(x)(r_1^2 + s_1^2). \] (17)

If we add (13) to (15), we obtain
\[ [(1 - \lambda)(\alpha + 2)(\alpha + 1) + 2\lambda\beta(2\beta - 1)]a_2^2 = L_{MN,1}(x)(r_2 + s_2) + L_{MN,2}(x)(r_2^2 + s_2^2). \] (18)

Substituting the value of \( r_1^2 + s_1^2 \) from (17) in the right hand side of (18), we deduce that
\[ \left[ (1 - \lambda)(\alpha + 2)(\alpha + 1) + 2\lambda\beta(2\beta - 1) - \frac{2L_{MN,2}(x)}{L_{MN,1}(x)}\Omega^2(\lambda, \alpha, \beta) \right] a_2^2 = L_{MN,1}(x)(r_2 + s_2), \] (19)
where \( \Omega(\lambda, \alpha, \beta) \) is given by (4).

Moreover computations using (3), (11) and (19), we find that
\[ |a_2| \leq \frac{\sqrt{2}|M(x)| \sqrt{|M(x)|}}{\sqrt{\left[ (1 - \lambda)(\alpha + 2)(\alpha + 1) + 2\lambda\beta(2\beta - 1) - 2\Omega^2(\lambda, \alpha, \beta) \right] M^2(x) - 4\Omega^2(\lambda, \alpha, \beta)N(x)}}. \]

From (12) and (14) we can also obtain
\[ |a_2| \leq \frac{|L_{MN,1}(x)|}{(1 - \lambda)(\alpha + 1) + \lambda(2\beta - 1)} \leq \frac{|M(x)|}{(1 - \lambda)(\alpha + 1) + \lambda(2\beta - 1)}. \] (20)

Next, if we subtract (15) from (13), we can easily see that
\[ 2[(1 - \lambda)(\alpha + 2) + \lambda(3\beta - 1)](a_3 - a_2^2) = L_{MN,1}(x)(r_2 - s_2) + L_{MN,2}(x)(r_2^2 - s_2^2). \] (21)

In view of (16) and (17), we get from (21)
\[ a_3 = \frac{L_{MN,2}(x)}{2\Omega^2(\lambda, \alpha, \beta)}(r_1^2 + s_1^2) + \frac{L_{MN,1}(x)}{2[(1 - \lambda)(\alpha + 2) + \lambda(3\beta - 1)]}(r_2 - s_2). \]

Thus applying (3), we conclude that
\[ |a_3| \leq \frac{M^2(x)}{\Omega^2(\lambda, \alpha, \beta)} + \frac{|M(x)|}{(1 - \lambda)(\alpha + 2) + \lambda(3\beta - 1)}. \]

□

Putting \( \lambda = \beta = 1 \) in Theorem 2.2, we conclude the following result:

**Corollary 2.3.** If \( f \) belongs to the family \( P_a(0; x) \), then
\[ |a_2| \leq |M(x)| \sqrt{\frac{|M(x)|}{2N(x)}} \]
and
\[ |a_3| \leq M^2(x) + \frac{|M(x)|}{2}. \]
Theorem 2.6. The class $\mathcal{LMN}(\lambda, \alpha; \beta)$ is a generalization of many classes considered earlier:

(i) If $a = 0, \lambda = 0$ then $\mathcal{LMN}(0, 0, \beta; x) = P_a(0; x)$ from [3].

(ii) If $\lambda = 0$ then $\mathcal{LMN}(0, \alpha, \beta; x) = \mathcal{S}_\alpha^\beta(1, 0)$ from [4].

(iii) If $a = 0, \lambda = 0, M(x) = x, N(x) = 1$ and from article [1] $a = 2, b = 1, p = 1, q = 1, \lambda = 0$ then $\mathcal{LMN}(0, 0, \beta; x) = S_0^1(0, x)$. 

(iv) If $a = 0, \lambda = 1$ then $\mathcal{LMN}(1, \alpha, \beta; x) = G_\alpha^\beta(\Phi(0); x)$ from [31].

(v) If $a = 0, \lambda = 0, M(x) = 2x, N(x) = -1$ or $\beta = 1, \lambda = 1, M(x) = 2x, N(x) = -1$ then $\mathcal{LMN}(0, 0, \beta; x) = \mathcal{LMN}(1, \alpha, 1; x) = \mathcal{S}_0^\beta(1, 1)$ from [7].

(vi) If $f \in \mathcal{LMN}(\alpha, \beta; x)$ then $f \in \mathcal{T}_\alpha^\beta(\lambda, \alpha, \beta; 1)$ from [30].

From Theorem 2.2, in particular cases, one can reobtain the same type of results for the classes mentioned above.

Remark 2.5. In the estimation of $|a_2|$, the minimum depends on $M(x)$ and $N(x)$.

In the case $M(x) = 1, N(x) = 0, \alpha = \lambda = 0$ or $(\lambda = \beta = 1)$ we obtain for $f(z) = \frac{z}{1 - z}$ then $f(z) = \frac{1}{1 - z}$ so $f \in \mathcal{LMN}(0, 0, \beta; x) = \mathcal{LMN}(1, \alpha, 1; x)$, but $f(z) = z \cdot (1 + z + z^2 + \ldots) = z + z^2 + z^3 + \ldots$ is Koebe’s convex function which $E(0, 0, 1, 0) = \infty, F(0, 0, \beta, 1) = 1$, hence $|a_2| \leq 1$ and this is the best estimation.

In the next theorem, we discuss “the Fekete-Szego Problem” for the family $\mathcal{LMN}(\lambda, \alpha; \beta; x)$.

Theorem 2.6. For $0 \leq \lambda \leq 1, \alpha \geq 0, \beta \geq 1$ and $\delta \in \mathbb{R}$, let $f \in \mathcal{A}$ belongs to the family $\mathcal{LMN}(\lambda, \alpha; \beta; x)$. Then

$$|a_3 - \delta a_2^2| \leq \begin{cases} \frac{|f(z)|}{(1 - \lambda)(\alpha + 2) + \lambda(3\beta - 1)} \times \\ \times (1 - \lambda)(\alpha + 2)(\alpha + 1) + 2\lambda\beta(2\beta - 1) - 2\Omega^2(\lambda, \alpha, \beta) - \frac{4\Omega(\lambda, \alpha, \beta)N(x)}{M^2(x)} \\ for \ |\delta - 1| \leq \frac{1}{2[(1 - \lambda)(\alpha + 2) + \lambda(3\beta - 1)]} \\ \times (1 - \lambda)(\alpha + 2)(\alpha + 1) + 2\lambda\beta(2\beta - 1) - 2\Omega^2(\lambda, \alpha, \beta) - \frac{4\Omega(\lambda, \alpha, \beta)N(x)}{M^2(x)} \\ for \ |\delta - 1| \geq \frac{1}{2[(1 - \lambda)(\alpha + 2) + \lambda(3\beta - 1)]} \\ \times (1 - \lambda)(\alpha + 2)(\alpha + 1) + 2\lambda\beta(2\beta - 1) - 2\Omega^2(\lambda, \alpha, \beta) - \frac{4\Omega(\lambda, \alpha, \beta)N(x)}{M^2(x)} \end{cases}$$

where $\Omega(\lambda, \alpha, \beta)$ is given by (4).

Proof. By making use of (19) and (21), we conclude that

$$a_3 - \delta a_2^2 = (1 - \delta) \frac{L_{\mathcal{LMN}, 1}^3(x)(r_2 + s_2)}{L_{\mathcal{LMN}, 1}(r_2 - s_2)}$$

$$+ \left( \frac{\phi(\delta; x)}{2[(1 - \lambda)(\alpha + 2) + \lambda(3\beta - 1)]} \right) r_2$$

$$= L_{\mathcal{LMN}, 1}(x) \left( \frac{\phi(\delta; x)}{2[(1 - \lambda)(\alpha + 2) + \lambda(3\beta - 1)]} \right) r_2$$

$$+ \left( \phi(\delta; x) - \frac{1}{2[(1 - \lambda)(\alpha + 2) + \lambda(3\beta - 1)]} \right) s_2,$$
According to (3), we find that
\[ |a_3 - \delta a_2^2| \leq \frac{|M(x)|}{(1 - \lambda)(\alpha + 1) + \beta}, \quad 0 \leq |\varphi(\delta; x)| \leq \frac{1}{2[(1 - \lambda)(\alpha + 1) + \lambda(3\beta - 1)]}, \]
\[ 0 \leq \varphi(\delta; x) \leq \frac{1}{2[(1 - \lambda)(\alpha + 1) + \lambda(3\beta - 1)]}. \]

After some computations, we obtain the desired result. \( \square \)

**Corollary 2.7.** If \( f \) belongs to the family \( P_c(0; x) \), then
\[ |a_3 - \delta a_2^2| \leq \frac{|M(x)|}{2}; \quad \text{for } |\delta - 1| \leq \frac{|N(x)|}{M(\delta)}, \]
\[ |M(x)|/|\delta - 1|; \quad \text{for } |\delta - 1| \geq \frac{|N(x)|}{M(\delta)}. \]

Putting \( \delta = 1 \) in Theorem 2.6, we conclude the following result:

**Corollary 2.8.** If \( f \) belongs to the family \( \mathcal{L}_{MN}(\lambda, \alpha, \beta; x) \), then
\[ |a_3 - a_2^2| \leq \frac{|M(x)|}{(1 - \lambda)(\alpha + 2) + \lambda(3\beta - 1)}. \]

**References**