Filomat 37:4 (2023), 1037–1044
https://doi.org/10.2298/FIL2304037W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Coefficient bounds and Fekete-Szegő inequality for a certain family of holomorphic and bi-univalent functions defined by (M,N)-Lucas polynomials

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Abstract. In the current work, we use the (M,N)-Lucas Polynomials to introduce a new family of holomorphic and bi-univalent functions which involve a linear combination between Bazilevič functions and β-pseudo-starlike function defined in the unit disk $\mathbb{D}$ and establish upper bounds for the second and third coefficients of functions belongs to this new family. Also, we discuss Fekete-Szegő problem in this new family.

1. Introduction

The Lucas Polynomials plays an important role in a diversity of disciplines as the mathematical, statistical, physical and engineering sciences (see, for example [10, 14, 38]).

Let $\mathcal{A}$ stands for the collection of functions f that are holomorphic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ that have the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Further, let S indicate the sub-collection of the set $\mathcal{A}$ containing functions from $\mathbb{D}$ satisfying (1) which are univalent in $\mathbb{D}$. According to the Koebe one-quarter theorem (see [9]), every function $f \in S$ has an inverse f^{-1} defined by $f^{-1}(f(z)) = z$, $(z \in \mathbb{D})$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \geq \frac{1}{4})$, where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$, let we name by the notation Σ the set of bi-univalent functions in $\mathbb{D}$ satisfying (1). In fact, Srivastava et al. [28] have

2020 *Mathematics Subject Classification*. Primary 30C45; Secondary 30C50
Keywords. Bi-Univalent function, (M,N)-Lucas Polynomials, Coefficient bounds, Fekete-Szegő problem, Subordination
Received: 26 January 2022; Revised: 17 June 2022; Accepted: 16 July 2022
Communicated by Hari M. Srivastava
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actually revived the study of holomorphic and bi-univalent functions in recent years, recalling the following examples of functions in the class Σ:

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

The Koebe function is not a member of the bi-univalent function class Σ, same as other common examples of functions in S such as:

$$z-\frac{z^2}{2}, \quad \frac{z}{1-z^2}.$$

Their work was followed by such articles as those by Frasin and Aouf [11], Altinkaya and Yalçin [2], Srivastava and Wanas [29], Srivastava et al. [26] and others (see, for example [8, 17, 18, 20–25, 30, 32–37]).

More pioneering work was made by Srivastava et al. in [27] where they studied coefficients of meromorphic bi-univalent functions.

Lewin [13] was the first to investigate the class of bi-univalent functions, showing that the first coefficient of the Taylor series expansion of a bi-univalent function satisfies $|a_2| < 1.51$.

Brannan and Clunie [6] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$ and Netanyahu [16] showed that $max\,|a_2| = \frac{4}{3}$. The coefficient estimate problem for each of the coefficients $|a_n|$ $(n \in \mathbb{N} \setminus \{1,2\})$ is still an open problem.

A function $f \in \mathcal{A}$ is called Bazilevič function in $\mathbb{D}$ if (see [19])

$$Re\left\{\frac{z^{1-\alpha} f'(z)}{(f(z))^{1-\alpha}}\right\} > 0, \quad (z \in \mathbb{D}, \alpha \geq 0).$$

A function $f \in \mathcal{A}$ is called β-pseudo-starlike function in $\mathbb{D}$ if (see [5])

$$Re\left\{\frac{z\,(f'(z))^{\beta}}{f(z)}\right\} > 0, \quad (z \in \mathbb{D}, \beta \geq 1).$$

We use the definition of subordination between holomorphic functions: let the functions f and g be holomorphic in $\mathbb{D}$, we say that the function f is subordinate to g, if there exists a Schwarz function ω holomorphic in $\mathbb{D}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{D})$ such that $f(z) = g(\omega(z))$. This subordination is indicated by $f \prec g$ or $f(z) \prec g(z)$ $(z \in \mathbb{D})$ (see [15]).

For the polynomials $M(x)$ and $N(x)$ with real coefficients, the (M,N)-Lucas Polynomials $L_{M,N,k}(x)$ are defined by the following recurrence relation (see [12]):

$$L_{M,N,k}(x) = M(x)L_{M,N,k-1}(x) + N(x)L_{M,N,k-2}(x) \quad (k \geq 2),$$

with

$$L_{M,N,0}(x) = 2, \quad L_{M,N,1}(x) = M(x),$$

$$L_{M,N,2}(x) = M^2(x) + 2N(x), \quad L_{M,N,3}(x) = M^3(x) + 3M(x)N(x). \tag{3}$$

The generating function of the (M,N)-Lucas Polynomial $L_{M,N,k}(x)$ (see [14]) is given by

$$T_L\,(M,N;x,z) = \sum_{k=2}^{\infty} L_{M,N,k}(x)z^k = \frac{2-M(x)z}{1-M(x)z-N(x)z^2}.$$

Note that for particular values of M and N, the (M, N)-polynomial $L_n(x)$ leads to various polynomials, among those, we list few cases here (see, for more details [3]):

(i) For $M(x) = x$ and $N(x) = 1$, we obtain the Lucas polynomials $L_n(x)$.
(ii) For $M(x) = 2x$ and $N(x) = 1$, we obtain the Pell-Lucas polynomials $Q_n(x)$.
(iii) For $M(x) = 1$ and $N(x) = 2x$, we obtain the Jacobsthal-Lucas polynomials $j_n(x)$.
(iv) For $M(x) = 3x$ and $N(x) = -2$, we obtain the Fermat-Lucas polynomials $f_n(x)$.
(v) For $M(x) = 2x$ and $N(x) = -1$, we have the Chebyshev polynomials $T_n(x)$ of the first kind.

2. Main Results

We begin this section by defining the family $\mathscr{L}_{MN}(\lambda,\alpha,\beta;x)$ as follows:

Definition 2.1. *For $0 \le \lambda \le 1$; $\alpha \ge 0$; $\beta \ge 1$ let $\mathscr{L}_{MN}(\lambda,\alpha,\beta;x)$ denote the subclass of Σ such that*

$$(1-\lambda)\frac{z^{1-\alpha}f'(z)}{(f(z))^{1-\alpha}}+\lambda\frac{z\left(f'(z)\right)^{\beta}}{f(z)} \prec T_L(M,N;x,z)-1$$

and

$$(1-\lambda)\frac{w^{1-\alpha}\left(f^{-1}(w)\right)'}{\left(f^{-1}(w)\right)^{1-\alpha}}+\lambda\frac{w\left(\left(f^{-1}(w)\right)'\right)^{\beta}}{f^{-1}(w)} \prec T_L(M,N;x,w)-1,$$

where f^{-1} is given by (2).

In particular, if we choose $\alpha=\lambda=0$ or $\lambda=\beta=1$ in Definition 2.1, we have $\mathscr{L}_{MN}(0,0,\beta;x) = \mathscr{L}_{MN}(1,\alpha,1;x) := P_\sigma(0;x)$ for the family of functions $f \in \Sigma$ given by (1) and satisfying the following subordinations:

$$\frac{zf'(z)}{f(z)} \prec T_L(M,N;x,z)-1$$

and

$$\frac{w\left(f^{-1}(w)\right)'}{f^{-1}(w)} \prec T_L(M,N;x,w)-1.$$

If $M(x)=1, N(x)=0$ then $\dfrac{zf'(z)}{f(z)} \prec T_L(1,0;x,z)-1=\dfrac{1}{1-z}$.

If $M(x)=2x, N(x)=-1$ then $\dfrac{zf'(z)}{f(z)} \prec T_L(2x,-1;x,z)-1=\dfrac{1}{1-2xz+z^2}$.

Theorem 2.2. *For $0 \le \lambda \le 1$, $\alpha \ge 0$ and $\beta \ge 1$, let f belongs to the family $\mathscr{L}_{MN}(\lambda,\alpha,\beta;x)$ and $N(x) \ne 0$; let denote*

$$\Omega(\lambda,\alpha,\beta)=(1-\lambda)(\alpha+1)+\lambda(2\beta-1), \tag{4}$$

$$E(\lambda,\alpha,\beta,M(x),N(x))=\frac{\sqrt{2}\,|M(x)|\,\sqrt{|M(x)|}}{\sqrt{\left|\left[(1-\lambda)(\alpha+2)(\alpha+1)+2\lambda\beta(2\beta-1)-2\Omega^2(\lambda,\alpha,\beta)\right]M^2(x)-4\Omega^2(\lambda,\alpha,\beta)N(x)\right|}}$$

and

$$F(\lambda,\alpha,\beta,M(x))=\frac{|M(x)|}{\Omega(\lambda,\alpha,\beta)};$$

then

$$|a_2| \le min\left\{E(\lambda,\alpha,\beta,M(x),N(x)),F(\lambda,\alpha,\beta,M(x))\right\}$$

and

$$|a_3| \le \frac{M^2(x)}{\Omega^2(\lambda,\alpha,\beta)}+\frac{|M(x)|}{(1-\lambda)(\alpha+2)+\lambda(3\beta-1)}.$$

Proof. Suppose that $f \in \mathscr{L}_{MN}(\lambda, \alpha, \beta; x)$. Then there exists two holomorphic functions $\phi, \psi : \mathbb{D} \longrightarrow \mathbb{D}$ given by

$$\phi(z) = r_1 z + r_2 z^2 + r_3 z^3 + \cdots \quad (z \in \mathbb{D}) \tag{5}$$

and

$$\psi(w) = s_1 w + s_2 w^2 + s_3 w^3 + \cdots \quad (w \in \mathbb{D}), \tag{6}$$

with $\phi(0) = \psi(0) = 0$, $|\phi(z)| < 1$, $|\psi(w)| < 1$, $z, w \in \mathbb{D}$ such that

$$(1-\lambda)\frac{z^{1-\alpha} f'(z)}{(f(z))^{1-\alpha}} + \lambda \frac{z\,(f'(z))^{\beta}}{f(z)} = -1 + L_{M,N,0}(x) + L_{M,N,1}(x)\phi(z) + L_{M,N,2}(x)\phi^2(z) + \cdots \tag{7}$$

and

$$(1-\lambda)\frac{w^{1-\alpha}\left(f^{-1}(w)\right)'}{(f^{-1}(w))^{1-\alpha}} + \lambda \frac{w\left(\left(f^{-1}(w)\right)'\right)^{\beta}}{f^{-1}(w)}$$
$$= -1 + L_{M,N,0}(x) + L_{M,N,1}(x)\psi(w) + L_{M,N,2}(x)\psi^2(w) + \cdots. \tag{8}$$

Combining (5), (6), (7) and (8), yield

$$(1-\lambda)\frac{z^{1-\alpha} f'(z)}{(f(z))^{1-\alpha}} + \lambda \frac{z\,(f'(z))^{\beta}}{f(z)} = 1 + L_{M,N,1}(x) r_1 z + \left[L_{M,N,1}(x) r_2 + L_{M,N,2}(x) r_1^2\right] z^2 + \cdots \tag{9}$$

and

$$(1-\lambda)\frac{w^{1-\alpha}\left(f^{-1}(w)\right)'}{(f^{-1}(w))^{1-\alpha}} + \lambda \frac{w\left(\left(f^{-1}(w)\right)'\right)^{\beta}}{f^{-1}(w)}$$
$$= 1 + L_{M,N,1}(x) s_1 w + \left[L_{M,N,1}(x) s_2 + L_{M,N,2}(x) s_1^2\right] w^2 + \cdots. \tag{10}$$

It is quite well-known that if $|\phi(z)| < 1$ and $|\psi(w)| < 1$, $z, w \in \mathbb{D}$, we get

$$|r_j| \le 1 \quad \textit{and} \quad |s_j| \le 1 \ (j \in \mathbb{N}). \tag{11}$$

In the light of (9) and (10), after simplifying, we find that

$$[(1-\lambda)(\alpha+1) + \lambda(2\beta-1)]\, a_2 = L_{M,N,1}(x) r_1, \tag{12}$$

$$[(1-\lambda)(\alpha+2) + \lambda(3\beta-1)]\, a_3 + \left[\frac{1}{2}(1-\lambda)(\alpha+2)(\alpha-1) + \lambda\,(2\beta(\beta-2)+1)\right] a_2^2$$
$$= L_{M,N,1}(x) r_2 + L_{M,N,2}(x) r_1^2, \tag{13}$$

$$-\left[(1-\lambda)(\alpha+1) + \lambda(2\beta-1)\right] a_2 = L_{M,N,1}(x) s_1 \tag{14}$$

and

$$[(1-\lambda)(\alpha+2) + \lambda(3\beta-1)]\left(2a_2^2 - a_3\right) + \left[\frac{1}{2}(1-\lambda)(\alpha+2)(\alpha-1) + \lambda\,(2\beta(\beta-2)+1)\right] a_2^2$$
$$= L_{M,N,1}(x) s_2 + L_{M,N,2}(x) s_1^2. \tag{15}$$

It follows from (12) and (14) that

$$r_1 = -s_1 \tag{16}$$

and

$$2\left[(1-\lambda)(\alpha+1)+\lambda(2\beta-1)\right]^2 a_2^2 = L_{M,N,1}^2(x)(r_1^2+s_1^2). \tag{17}$$

If we add (13) to (15), we obtain

$$\left[(1-\lambda)(\alpha+2)(\alpha+1)+2\lambda\beta\,(2\beta-1)\right] a_2^2 = L_{M,N,1}(x)(r_2+s_2)+L_{M,N,2}(x)(r_1^2+s_1^2). \tag{18}$$

Substituting the value of $r_1^2+s_1^2$ from (17) in the right hand side of (18), we deduce that

$$\left[(1-\lambda)(\alpha+2)(\alpha+1)+2\lambda\beta\,(2\beta-1)-\frac{2L_{M,N,2}(x)}{L_{M,N,1}^2(x)}\Omega^2(\lambda,\alpha,\beta)\right]a_2^2 = L_{M,N,1}(x)(r_2+s_2), \tag{19}$$

where $\Omega(\lambda,\alpha,\beta)$ is given by (4).
Moreover computations using (3), (11) and (19), we find that

$$|a_2| \le \frac{\sqrt{2}\,|M(x)|\,\sqrt{|M(x)|}}{\sqrt{\left|\left[(1-\lambda)(\alpha+2)(\alpha+1)+2\lambda\beta(2\beta-1)-2\Omega^2(\lambda,\alpha,\beta)\right]M^2(x)-4\Omega^2(\lambda,\alpha,\beta)N(x)\right|}}.$$

From (12) and (14) we can also obtain

$$|a_2| \le \frac{\left|L_{M,N,1}(x)\right|}{(1-\lambda)(\alpha+1)+\lambda(2\beta-1)} \le \frac{|M(x)|}{(1-\lambda)(\alpha+1)+\lambda(2\beta-1)}. \tag{20}$$

Next, if we subtract (15) from (13), we can easily see that

$$2\left[(1-\lambda)(\alpha+2)+\lambda(3\beta-1)\right](a_3-a_2^2) = L_{M,N,1}(x)(r_2-s_2)+L_{M,N,2}(x)(r_1^2-s_1^2). \tag{21}$$

In view of (16) and (17), we get from (21)

$$a_3 = \frac{L_{M,N,1}^2(x)}{2\Omega^2(\lambda,\alpha,\beta)}(r_1^2+s_1^2)+\frac{L_{M,N,1}(x)}{2\left[(1-\lambda)(\alpha+2)+\lambda(3\beta-1)\right]}(r_2-s_2).$$

Thus applying (3), we conclude that

$$|a_3| \le \frac{M^2(x)}{\Omega^2(\lambda,\alpha,\beta)}+\frac{|M(x)|}{(1-\lambda)(\alpha+2)+\lambda(3\beta-1)}.$$

□

Putting $\lambda=\beta=1$ in Theorem 2.2, we conclude the following result:

Corollary 2.3. *If f belongs to the family $P_\sigma(0;x)$, then*

$$|a_2| \le |M(x)|\sqrt{\left|\frac{M(x)}{2N(x)}\right|}$$

and

$$|a_3| \le M^2(x)+\frac{|M(x)|}{2}.$$

The previous result was obtained in Corollary 1 from [3].

Remark 2.4. *The class $\mathscr{L}_{MN}(\lambda,\alpha,\beta;x)$ is a generalization of many classes considered earlier:*

(i) If $\alpha=0,\lambda=0$ then $\mathscr{L}_{MN}(0,0,\beta;x)=P_\sigma(0;x)$ from [3].

(ii) If $\lambda=0$ then $\mathscr{L}_{MN}(0,\alpha,\beta;x)=\mathfrak{B}^{\alpha}_{\Sigma}(1,0)$ from [4].

(iii) If $\alpha=0,\lambda=0,M(x)=x,N(x)=1$ and from article [1] $a=2,b=1,p=1,q=1,\lambda=0$ then $\mathscr{L}_{MN}(0,0,\beta;x)=\mathcal{S}^*_\sigma(0,x)$.

(iv) If $\alpha=0,\lambda=1$ then $\mathscr{L}_{MN}(1,\alpha,\beta;x)=\mathcal{G}_\Sigma(\beta,\Phi(0);x)$ from [31].

(v) If $\alpha=0,\lambda=0,M(x)=2x,N(x)=-1$ or $\beta=1,\lambda=1,M(x)=2x,N(x)=-1$ then $\mathscr{L}_{MN}(0,0,\beta;x)=\mathscr{L}_{MN}(1,\alpha,1;x)=\mathfrak{B}^{0}_{\Sigma}(1,t)$ from [7].

(vi) If $f\in\mathscr{L}_{1,0}(\lambda,\alpha,\beta;x)$ then $f\in\mathcal{T}_\Sigma(\lambda,\alpha,\beta;1)$ from [30].

From Theorem 2.2, in particular cases, one can reobtain the same type of results for the classes mentioned above.

Remark 2.5. *In the estimation of $|a_2|$, the minimum depends on $M(x)$ and $N(x)$.*

In the case $M(x)=1,N(x)=0,\ \alpha=\lambda=0$ or $(\lambda=\beta=1)$ we obtain for $f(z)=\dfrac{z}{1-z}$ then $\dfrac{z\cdot f'(z)}{f(z)}=\dfrac{1}{1-z}$ so $f\in\mathscr{L}_{MN}(0,0,\beta;x)=\mathscr{L}_{MN}(1,\alpha,1;x)$, but $f(z)=z\cdot(1+z+z^2+\ldots)=z+z^2+z^3+\ldots$ is Koebe's convex function which $E(0,0,\beta,1,0)=\infty$, $F(0,0,\beta,1)=1$, hence $|a_2|\le 1$ and this is the best estimation.

In the next theorem, we discuss "the Fekete-Szegő Problem" for the family $\mathscr{L}_{MN}(\lambda,\alpha,\beta;x)$.

Theorem 2.6. *For $0\le\lambda\le1$, $\alpha\ge0$, $\beta\ge1$ and $\delta\in\mathbb{R}$, let $f\in\mathcal{A}$ belongs to the family $\mathscr{L}_{MN}(\lambda,\alpha,\beta;x)$. Then*

$$\left|a_3-\delta a_2^2\right|\le\begin{cases}\frac{|M(x)|}{(1-\lambda)(\alpha+2)+\lambda(3\beta-1)}; & for\ \ |\delta-1|\le\frac{1}{2\left[(1-\lambda)(\alpha+2)+\lambda(3\beta-1)\right]}\times\\ & \times\left|(1-\lambda)(\alpha+2)(\alpha+1)+2\lambda\beta(2\beta-1)-2\Omega^2(\lambda,\alpha,\beta)-\frac{4\Omega^2(\lambda,\alpha,\beta)N(x)}{M^2(x)}\right|,\\ \frac{2|M(x)|^3|\delta-1|}{\left|\left[(1-\lambda)(\alpha+2)(\alpha+1)+2\lambda\beta(2\beta-1)-2\Omega^2(\lambda,\alpha,\beta)\right]M^2(x)-4\Omega^2(\lambda,\alpha,\beta)N(x)\right|}; & \\ & for\ \ |\delta-1|\ge\frac{1}{2\left[(1-\lambda)(\alpha+2)+\lambda(3\beta-1)\right]}\times\\ & \times\left|(1-\lambda)(\alpha+2)(\alpha+1)+2\lambda\beta(2\beta-1)-2\Omega^2(\lambda,\alpha,\beta)-\frac{4\Omega^2(\lambda,\alpha,\beta)N(x)}{M^2(x)}\right|,\end{cases}$$

where $\Omega(\lambda,\alpha,\beta)$ is given by (4).

Proof. By making use of (19) and (21), we conclude that

$$\begin{aligned}a_3-\delta a_2^2&=(1-\delta)\frac{L^3_{M,N,1}(x)(r_2+s_2)}{\left[(1-\lambda)(\alpha+2)(\alpha+1)+2\lambda\beta(2\beta-1)\right]L^2_{M,N,1}(x)-2\Omega^2(\lambda,\alpha,\beta)L_{M,N,2}(x)}\\&+\frac{L_{M,N,1}(x)(r_2-s_2)}{2\left[(1-\lambda)(\alpha+2)+\lambda(3\beta-1)\right]}\\&=L_{M,N,1}(x)\left[\left(\varphi(\delta;x)+\frac{1}{2\left[(1-\lambda)(\alpha+2)+\lambda(3\beta-1)\right]}\right)r_2\right.\\&\left.+\left(\varphi(\delta;x)-\frac{1}{2\left[(1-\lambda)(\alpha+2)+\lambda(3\beta-1)\right]}\right)s_2\right],\end{aligned}$$

where

$$\varphi(\delta;x)=\frac{L^2_{M,N,1}(x)(1-\delta)}{\left[(1-\lambda)(\alpha+2)(\alpha+1)+2\lambda\beta(2\beta-1)\right]L^2_{M,N,1}(x)-2\Omega^2(\lambda,\alpha,\beta)L_{M,N,2}(x)}.$$

According to (3), we find that

$$\left|a_3 - \delta a_2^2\right| \leq \begin{cases} \frac{|M(x)|}{(1-\lambda)(\alpha+2)+\lambda(3\beta-1)}, & 0 \leq \left|\varphi(\delta;x)\right| \leq \frac{1}{2\left[(1-\lambda)(\alpha+2)+\lambda(3\beta-1)\right]}, \\ 2\,|M(x)|\left|\varphi(\delta;x)\right|, & \left|\varphi(\delta;x)\right| \geq \frac{1}{2\left[(1-\lambda)(\alpha+2)+\lambda(3\beta-1)\right]}. \end{cases}$$

After some computations, we obtain the desired result. □

Putting $\lambda = \beta = 1$ in Theorem 2.6, we conclude the following result:

Corollary 2.7. *If f belongs to the family $P_\sigma(0;x)$, then*

$$\left|a_3 - \delta a_2^2\right| \leq \begin{cases} \frac{|M(x)|}{2}; & for\ |\delta - 1| \leq \frac{|N(x)|}{M^2(x)}, \\ \frac{|M(x)|^3|\delta-1|}{2|N(x)|}; & for\ |\delta - 1| \geq \frac{|N(x)|}{M^2(x)}. \end{cases}$$

Putting $\delta = 1$ in Theorem 2.6, we conclude the following result:

Corollary 2.8. *If f belongs to the family $\mathscr{L}_{MN}(\lambda, \alpha, \beta; x)$, then*

$$\left|a_3 - a_2^2\right| \leq \frac{|M(x)|}{(1-\lambda)(\alpha+2)+\lambda(3\beta-1)}.$$

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