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L^p extremal polynomials $(0 < p < \infty)$ in the presence of a denumerable set of mass points

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Abstract. We study, for all $p > 0$ the asymptotic behavior of L^p extremal polynomials with respect to the measure $\alpha = \beta + \gamma, \alpha$ denotes a positive measure whose support is the unit circle Γ plus a denumerable set of mass points, which accumulate at Γ and satisfy Blaschke's condition and $\beta = \beta_a + \beta_s$, β_s the absolutely continuous part of the measure satisfies Szegö condition and β_s the singular part. Our main result is the explicit strong asymptotic formulas for the L^p extremal polynomials.

1. Introduction

Let α be a finite positive Borel measure with an infinite compact support in the complex plane. We denote $T_{n,p,\alpha}(z) = z^n + a_{n-1}z^{n-1} + ... + a_0, a_{n-1}, ..., a_0 \in \mathbb{C}$, the monic polynomial of degree n with respect to measure α. Then the extremal or general Chebyshev polynomial $T_{n,p,\alpha}$ is a monic polynomial that minimize the $L^p(\alpha)$ norm in the set of monic polynomials of degree n

$$m_{n,p}(\alpha) = \left\|T_{n,p,\alpha}\right\|_{L^p(\alpha)} = \min\left\{\|Q_n\|_{L^p(\alpha)} : Q_n\left(z\right) = z^n + a_{n-1}z^{n-1} + ... + a_0\right\}.$$

For $p = 2$, we have the special case of orthogonal polynomials with respect to the measure α. A large number of works have been done on this subject; see, for example [2], [3].

A series of results concerning the asymptotics of the L^p extremal polynomials was established. In [1] Geronimus has given such asymptotics in the case where the support of the measure α is a rectifiable Jordan curve with some smoothness condition. An extension of the Geronimus's result has been given by Kaliaguine [5], where the measure is supported by a rectifiable Jordan curve plus a finit number of mass points. In [8] Laskri, Benzine have obtained the asymptotics of L^p extremal polynomials on a complete curve plus an infinite number of mass points with some conditions of smoothness. Recently, X. Li and K. Pan [4] investigated the zero distributions of L^p extremal polynomials on the unit circle $(1 < p < \infty)$.

In this note we shall study the power asymptotic of the $L^p(\alpha)$ extremal polynomials $T_{n,p,\alpha}$ outside the unit circle Γ. We are also inspired by the work of Peherstorfer and Yuditskii [6] and Bello Hernandez,

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Marcellan and Minguez [7] to reach the desired asymptotic formula for L^p extremal polynomials $T_{n,p,\alpha}$. However, we can use a new technic due to Peherstorfer and Yuditskii [6]. In which the authors showed the asymptotic formula for the orthogonal polynomials, their method based on the use of a measure concentred on a segment plus an infinite points. we try to carry over some of the main ideas of [6] for $L^p(\alpha)$ extremal polynomials with the necessary modification imposed by the nature of our problem, we can show the asymptotic formula for the case of $L^p(\alpha)$ extremal polynomials. First of all, we set some notations.

Let $\Gamma = \{z \in \mathbb{C} : |z| = 1\}, G = \{z \in \mathbb{C} : |z| > 1\}$ the exterior of the unit circle and let α be a measure which has a decomposition of the form

$$\alpha = \beta + \gamma = \beta_a + \beta_s + \gamma, \tag{1}$$

where β_a is the absolutely continuous part of β with $supp(\beta_a) = \Gamma$, respect to the Lebesgue measure $|d\zeta|$ on $[-\pi, +\pi]$, that is

$$d\beta_a(\zeta) = \rho(\zeta)\,|d\zeta|\,, \rho \geq 0; \rho \in L^1([-\pi, +\pi], |d\zeta|), \tag{2}$$

and $supp(\beta_s) \subset \Gamma$ (β_s the singular part of β) and γ is a point measure supported on $\{z_k\}_{k=1}^{\infty}$, $(|z_k| > 1)$, that is

$$\gamma = \sum_{k=1}^{\infty} A_k \delta_{z_k}, A_k > 0 \text{ and } \sum_{k=1}^{\infty} A_k < \infty, \tag{3}$$

where each δ_{z_k} is the Dirac measure at the point z_k. Suppose that the absolutely continuous part β_a of β, satisfies the following Szegö condition :

$$\int_{\Gamma} \log\left(\rho\left(\zeta\right)\right) |d\zeta| > -\infty, \tag{4}$$

Condition (4) allows us to construct the so-colled Szegö function $\mathcal{D}_{G,\rho}$ associated with G and the weight function $\rho(\zeta)$ with the following properties [5]:

(*i*) $\mathcal{D}_{G,\rho}(z)$ is analytic in G, $\mathcal{D}_{G,\rho}(z) \neq 0$, $\mathcal{D}_{G,\rho}(\infty) > 0$,

(*ii*) $\mathcal{D}_{G,\rho}(z)$ has boundary values almost everywhere on Γ such that $\rho(\zeta) = \left|\mathcal{D}_{G,\rho}(\zeta)\right|^p, \zeta \in \Gamma$.

The following function is the Szegö function for the domain G :

$$\mathcal{D}_{G,\rho}(z) = \exp\left\{-\frac{1}{2p\pi}\int_0^{2\pi} \frac{z + e^{i\theta}}{z - e^{i\theta}} \log\left(\rho\left(\theta\right)\right) d\theta\right\}.$$

One can find in the literature several technics to solve the problem of the asymptotic behavior of L^p extremal polynomials.

The technic that we use consists to generate and to study some sequences of extremal problems in Hardy spaces.

2. Asymptotic behavior

Let Γ be a unit circle, and the support of the measure α is Γ plus an infinite discrete set of mass points which accumulate on Γ. We associated to the measures β_a and α the extremal constants $m_{n,p}(\beta_a), m_{n,p}(\alpha)$ and the L^p extremal polynomial $T_{n,p,\beta_a}, T_{n,p,\alpha}$, as follows :

$$m_{n,p}(\beta_a) = \left\|T_{n,p,\beta_a}\right\|_{L^p(\Gamma)} = \min_{Q_n(z)=z^n+\dots} \left\{\|Q_n\|_{L^p(\Gamma)}\right\}, \tag{5}$$

$$m_{n,p}(\alpha) = \left\|T_{n,p,\alpha}\right\|_{L^p(\alpha)} = \min\left\{\|Q_n\|_{L^p(\alpha)}, Q_n(z) = z^n + ...\right\}, \tag{6}$$

with

$$\|f\|^p_{L^p(\alpha)} = \int_\Gamma \left|f(\zeta)\right|^p \rho(\zeta)\,|d\zeta| + \int_\Gamma \left|f(\zeta)\right|^p d\beta_s + \sum_{k=1}^{\infty} A_k \left|f(z_k)\right|^p, \tag{7}$$

$$\|f\|^p_{L^p(\Gamma)} = \int_\Gamma \left|f(\zeta)\right|^p \rho(\zeta)\,|d\zeta|. \tag{8}$$

We pose $0 < p < 1$. The optimal solution φ^* of the following extremal problem :

$$\inf\left\{\|\varphi\|^p_{H^p(G,\rho)}, \varphi \in H^p(G,\rho), \varphi(\infty) = 1\right\}, \tag{9}$$

is given by

$$\varphi^*(z) = \frac{\mathcal{D}_{G,\rho}(\infty)}{\mathcal{D}_{G,\rho}(z)}, \tag{10}$$

i.e., the infinimum (9) denote $\mu(\beta_a)$ is reached for (9):

$$\mu(\beta_a) = \left\|\varphi^*\right\|^p_{H^p(G,\rho)} = \mathcal{D}^p_{G,\rho}(\infty) = \mathcal{D}^p_{\Gamma,\rho}(0), \tag{11}$$

where $\mathcal{D}_{\Gamma,\rho}$, Szegö function associated with the unit disk and the weight function ρ and the analytic function φ^* belongs to $H^p(G,\rho)$ if and only if $\varphi^*(z)D_{G,\rho}(z) \in H^p(G)$, where $H^p(G)$ is the usual Hardy space associated with the exterior G of the unit circle.

We denote by $\mu(\alpha)$ the extremal value of the problem:

$$\mu(\alpha) = \inf\left\{\|\varphi\|^p_{H^p(G,\rho)}, \varphi \in H^p(G,\rho), \varphi(\infty) = 1, \varphi(z_k) = 0, k = 1, 2, ..\right\}. \tag{12}$$

We denote by

$$B(z) = \prod_{k=1}^{\infty} \frac{z - z_k}{z\bar{z}_k - 1} \frac{|z_k|^2}{z_k},$$

the Blaschke product and we denote by $\psi^*(z) = \varphi^*(z)B(z)$ is an extremal function of problem (12). The optimal values of the problems (9) and (12) are connected by:

$$\mu(\beta_a) = \left(\prod_{k=1}^{\infty} |z_k|\right)^{-p} \mu(\alpha). \tag{13}$$

Lemma 2.1. *[5] If $f(z) \in H^p(G,\rho)$, then for every compact set $K \subset G$ there is a constant $C(K)$ ($C(K)$ depending only on K) such that :*

$$\sup\left\{\left|f(z)\right| : z \in K\right\} \le C(K)\left\|f\right\|_{H^p(G,\rho)}.$$

Lemma 2.2. *[5] Let $\{f_n\}$ be a sequence of functions in $H^p(G,\rho)$ and*

(i) *$f_n \to f$ uniformly on the compact sets of G,*

(ii) $\left\|f_n\right\|^p_{H^p(G,\rho)} \le M$ *(constant),* *then*

$$f \in H^p(G,\rho) \text{ and } \|f\|^p_{H^p(G,\rho)} \le \lim_{n\to\infty} \inf \|f_n\|^p_{H^p(G,\rho)}.$$

Definition 2.3. *If the measure $\alpha = \beta_a + \beta_s + \gamma$ is such that a verifies the condition (4) and its discrete part verifies*

$$\sum_{k=1}^{\infty} (1 - |z_k|) < \infty,$$

then we say that $\alpha \in \mathcal{E}$.

Theorem 2.4. *Let Γ be the unit circle and $\alpha = \beta_a + \beta_s + \gamma$, such that $\alpha \in \mathcal{E}$, and $\mathcal{D}_{G,\rho}$, the Szegö function associated with G, then*

(*i*) $\lim_{n\to\infty} m_{n,p}(\alpha) = \{\mu(\alpha)\}^{\frac{1}{p}}$,

(*ii*) $\lim_{n\to\infty} \left\| \frac{T_{n,p,\alpha}(\alpha)}{z^n} - \frac{\mathcal{D}_{G,\rho}(\infty)}{\mathcal{D}_{G,\rho}(z)} B(z) \right\|_{H^p(G,\rho)} = 0$

(*iii*) $T_{n,p,\alpha}(z) = z^n \frac{\mathcal{D}_{G,\rho}(\infty)}{\mathcal{D}_{G,\rho}(z)} \prod_{k=1}^{\infty} \frac{z - z_k}{z\bar{z}_k - 1} \frac{|z_k|^2}{z_k} [1 + \epsilon_n(z)]$,

$\epsilon_n(z) \to 0$ *uniformly on the compact sets of G.*

Proof. First, we will prove the inequality follow

$$\lim_{n\to\infty} \sup m_{n,p}(\alpha) \le \{\mu(\alpha)\}^{\frac{1}{p}}.$$

Let $\mathcal{D}_{\Gamma,\rho,\epsilon}$ be a function analytic belong $C^\infty(\Gamma)$ such that $\min_\Gamma \left|\mathcal{D}_{\Gamma,\rho,\epsilon}(\zeta)\right| \ge 1$ and by hypothesis $\inf_\Gamma \left|\mathcal{D}_{\Gamma,\rho,\epsilon}(\zeta)\right| > 0$ and

$$\int_\Gamma \left| \left|\mathcal{D}_{\Gamma,\rho,\epsilon}(\zeta)\right|^p - \left|\mathcal{D}_{\Gamma,\rho}(\zeta)\right|^p \right| |d\zeta| < \epsilon^p, (\epsilon > 0), \tag{14}$$

we put $\frac{1}{\eta(\epsilon)} = \max_\Gamma \left|\mathcal{D}_{\Gamma,\rho,\epsilon}(\zeta)\right|$.

In the following, let η where $\eta < 1$ and $0 < \eta < \eta(\epsilon)$. Let as define a smooth function $\chi_{\epsilon,\eta}$

$$\chi_{\epsilon,\eta}(\zeta) = \begin{cases} \frac{1}{\left|\mathcal{D}_{\Gamma,\rho,\epsilon}(\zeta)\right|} & \zeta \notin \Gamma_s \cup \tilde{\Gamma}_+ \cup \tilde{\Gamma}_- \\ \eta & \zeta \in \Gamma_s \setminus \tilde{\Gamma}_+ \cup \tilde{\Gamma}_- \\ |\zeta \pm 1|^2 & \zeta \in \tilde{\Gamma}_\pm, \end{cases}$$

and

$$|\zeta \pm 1|^2 \le \chi_{\epsilon,\eta}(\zeta) \le \frac{1}{\left|\mathcal{D}_{\Gamma,\rho,\epsilon}(\zeta)\right|} \qquad \forall \zeta \in \tilde{\Gamma}_\pm \setminus \Gamma_\pm,$$

where

$$\Gamma_\pm = \left\{ \zeta \in \Gamma : |\zeta \pm 1|^2 \le \frac{\eta}{2} \right\}, \Gamma_\pm \subsetneq \tilde{\Gamma}_\pm = \left\{ \zeta \in \Gamma : |\zeta \pm 1|^2 \le \eta \right\},$$

and

$$\int_{\Gamma \setminus \Gamma_s} d\beta_s \le \eta, \quad |\Gamma_s| \le \eta. \tag{15}$$

It's clear that $\chi_{\epsilon,\eta}(\zeta)$ is differentiable and

$$\left|\mathcal{D}_{\Gamma,\rho,\epsilon}(\zeta)\,\chi_{\epsilon,\eta}(\zeta)\right| \leq 1, \tag{16}$$

also

$$\left|\mathcal{D}_{\Gamma,\rho,\epsilon}(\zeta)\right| \leq \frac{1}{\eta}. \tag{17}$$

Hence, by the above settings

$$0 \leq \log\frac{1}{\chi_{\epsilon,\eta}(0)} - \log\mathcal{D}_{\Gamma,\rho,\epsilon}(0) \leq \int_{\tilde{\Gamma}_+\cup\tilde{\Gamma}_-\cup\Gamma_s} \log\frac{1}{\chi_{\epsilon,\eta}(\zeta)\,\mathcal{D}_{\Gamma,\rho,\epsilon}(\zeta)}\,|d\zeta|.$$

$$0 \leq \int_{\tilde{\Gamma}_+} \log\frac{1}{|\zeta-1|}\,|d\zeta| + \int_{\tilde{\Gamma}_-} \log\frac{1}{|\zeta-1|}\,|d\zeta| + \log\frac{1}{\eta}\int_{\Gamma_s}|d\zeta| = \circ(1), \quad \eta \to 0,$$

implies $\chi_{\epsilon,\eta}(0)\,\mathcal{D}_{\Gamma,\rho,\epsilon}(0) \simeq 1$ except in the neighbors of the two points $\pm 1, \eta \to 0, \epsilon \to 0$.

We see that the function $B\chi_{\epsilon,\eta}$, is regular and $(B\chi_{\epsilon,\eta})' \in L^\infty(\Gamma)$, implies the uniform convergence of the Fourie's serie on Γ, also

$$\max_{\Gamma}\left|\left(B\chi_{\epsilon,\eta}\right)(\zeta)\right| \leq \max\left|\frac{1}{\mathcal{D}_{\Gamma,\rho,\epsilon}(\zeta)}\right| \leq 1, \tag{18}$$

$Q_{n,\epsilon,\eta}$ the Fourie's some of n order of the function $B\chi_{\epsilon,\eta}$, so $\forall\epsilon > 0, \forall\eta > 0, \exists m(\epsilon,\eta)$:

$$\sup_{n\geq m(\epsilon,\eta)}\left|Q_{n,\epsilon,\eta}(\zeta)\right| \leq \left|\left(Q_{n,\epsilon,\eta} - \chi_{\epsilon,\eta}\right)(\zeta)\right| + \left|\chi_{\epsilon,\eta}(\zeta)\right| \leq 2, \tag{19}$$

and for all ϵ, η fixed

$$\left\|B\chi_{\epsilon,\eta} - Q_{n,\epsilon,\eta}\right\|_{L^\infty(\Gamma)} \to 0, n \to \infty. \tag{20}$$

So taking into account that (17), (18), (20):

$$\begin{aligned}
\left\|\mathcal{D}_{\Gamma,\rho,\epsilon}Q_{n,\epsilon,\eta}\right\|_{L^p(\Gamma)} &\leq \left\|\mathcal{D}_{\Gamma,\rho,\epsilon}BQ_{n,\epsilon,\eta}\right\|_{L^p(\Gamma)} + \left\|\mathcal{D}_{\Gamma,\rho,\epsilon}\left(B\chi_{\epsilon,\eta} - Q_{n,\epsilon,\eta}\right)\right\|_{L^p(\Gamma)} \\
&\leq \left\|\mathcal{D}_{\Gamma,\rho,\epsilon}BQ_{n,\epsilon,\eta}\right\|_{L^p(\Gamma)} + \left\|\mathcal{D}_{\Gamma,\rho,\epsilon}\right\|_{L^p(\Gamma)}\left\|\left(B\chi_{\epsilon,\eta} - Q_{n,\epsilon,\eta}\right)\right\|_{L^\infty(\Gamma)} \\
&\leq 1 + \epsilon_1, \forall n \geq s(\epsilon_1,\eta).
\end{aligned} \tag{21}$$

Consequently, by using the following formulas (14), (19), and (21):

$$\begin{aligned}
\left\|\mathcal{D}_{\Gamma,\rho}Q_{n,\epsilon,\eta}\right\|_{L^p(\Gamma)} &\leq \int_\Gamma\left(\left|\mathcal{D}_{\Gamma,\rho}\right|^p - \left|\mathcal{D}_{\Gamma,\rho,\epsilon}\right|^p + \left|\mathcal{D}_{\Gamma,\rho,\epsilon}\right|^p\right)\left|Q_{n,\epsilon,\eta}(\zeta)\right|^p|d\zeta| \\
&\leq (1+\epsilon_1)^p + \int_\Gamma\left(\left|\mathcal{D}_{\Gamma,\rho}\right|^p - \left|\mathcal{D}_{\Gamma,\rho,\epsilon}\right|^p\right)\left|Q_{n,\epsilon,\eta}(\zeta)\right|^p|d\zeta| \\
&\leq 1 + p\epsilon_1 + 2^p\epsilon^p, \forall n \geq \max(m,s),
\end{aligned} \tag{22}$$

for $n \to \infty, \eta \to 0, \epsilon_1 \to 0, \epsilon \to 0$.

For the singular measure β_s we have

$$\begin{aligned}
\left\|Q_{n,\epsilon,\eta}\right\|_{L^p(\beta_s)} &\leq \left\|\mathcal{D}_{\Gamma,\rho}B\chi_{\epsilon,\eta}\right\|_{L^p(\beta_s)} \\
&\leq \left[\int_{\Gamma_s}\left|\chi_{\epsilon,\eta}(\zeta)\right|^p d\beta_s + \int_{\Gamma\setminus\Gamma_s}\left|\chi_{\epsilon,\eta}(\zeta)\right|^p d\beta_s\right]^{\frac{1}{p}} + \circ(1),
\end{aligned}$$

due to (15), we get

$$\left\|Q_{n,\epsilon,\eta}\right\|_{L^p(\beta_s)} \leq C\eta^{\frac{1}{p}} + \circ(1), \quad n \to \infty. \tag{23}$$

At last, for the discrete measure γ we have, for fixed ϵ, η and with the Blaschke product equal zero in the points z_k then,

$$\left\|Q_{n,\epsilon,\eta}\right\|_{L^p(\gamma)} = \left\|B\chi_{\epsilon,\eta} - Q_{n,\epsilon,\eta}\right\|_{L^p(\gamma)} = \circ(1), \quad n \to \infty, \tag{24}$$

Indeed, if we notice that

$$Q_{n,\epsilon,\eta}(z) = B\chi_{\epsilon,\eta}(0) + ...,$$

by using (6) we get, with the help of (22), (23) and (24)

$$\begin{aligned} m_{n,p}(\alpha) &\leq \left\|\frac{1}{\left(B\chi_{\epsilon,\eta}\right)(0)} Q_{n,\epsilon,\eta}\right\|_{L^p(\alpha)} = \frac{1}{\left(B\chi_{\epsilon,\eta}\right)(0)} \left\|Q_{n,\epsilon,\eta}\right\|_{L^p(\alpha)} \\ &\leq \frac{1 + p\epsilon_1 + 2^p\epsilon^p}{\left(B\chi_{\epsilon,\eta}\right)(0)}. \end{aligned}$$

Finally, by using (11), (13) we obtain

$$\lim_{n\to\infty} \sup m_{n,p}(\alpha) \leq \mathcal{D}_{G,\rho}(\infty) \prod_{k=1}^{\infty} |z_k| = \left(\mu(\beta_a)\right)^{\frac{1}{p}} \prod_{k=1}^{\infty} |z_k| = \left(\mu(\alpha)\right)^{\frac{1}{p}}. \tag{25}$$

To get rid of the assumption that $\left|\mathcal{D}_{\Gamma,\rho}\right|$ is bounded from below we use the following standard trick. We define

$$\left|\mathcal{D}_{\Gamma,\rho,\epsilon}(\zeta)\right|^p = \left|\mathcal{D}_{\Gamma,\rho}(\zeta)\right|^p + \epsilon^p, \zeta \in \Gamma \quad (\epsilon > 0). \tag{26}$$

Note that $\left|\mathcal{D}_{\Gamma,\rho,\epsilon}\right|$ is bounded from below. The extremal properties of $T_{n,\rho,\alpha}$ and $T_{n,\rho,\alpha_\epsilon}$ imply

$$m_{n,p}(\alpha) \geq m_{n,p}(\alpha_\epsilon), \tag{27}$$

with

$$\begin{aligned} m_{n,p}(\alpha_\epsilon) &= \left\|T_{n,\rho,\alpha_\epsilon}\right\|_{L^p(\alpha_\epsilon)} \\ \|f\|^p_{L^p(\alpha_\epsilon)} &= \int_\Gamma \left|f(\zeta)\right|^p \left|\mathcal{D}_{\Gamma,\rho,\epsilon}\right|^p |d\zeta| + \int_\Gamma \left|f(\zeta)\right|^p d\beta_s + \sum_{k=1}^{\infty} A_k \left|f(z_k)\right|^p. \end{aligned}$$

We also have (see [7, Theorem 2])

$$\lim_{n\to\infty} m_{n,p}(\alpha_\epsilon) = \left|\mathcal{D}_{\Gamma,\rho,\epsilon}(0)\right| \prod_{k=1}^{\infty} |z_k|. \tag{28}$$

Since $\mathcal{D}_{\Gamma,\rho,\epsilon}(0) \to \mathcal{D}_{\Gamma,\rho}(0), \epsilon \to 0$ using (25), (27), (28) we get

$$\mu(\alpha)^{\frac{1}{p}} \leq \lim_{n\to\infty} \inf m_{n,p}(\alpha) \leq \lim_{n\to\infty} \sup m_{n,p}(\alpha) \leq \{\mu(\alpha)\}^{\frac{1}{p}},$$

and (i) of theorem is proved.

The function

$$\Phi_n^+(z) = \frac{1}{2}\left(\frac{T_{n,p,\alpha}(z)}{z^n} + \psi^*(z)\right) \text{ and } \Phi_n^-(z) = \frac{1}{2}\left(\frac{T_{n,p,\alpha}(z)}{z^n} - \psi^*(z)\right),$$

where $\left\|\psi^*\right\|^p_{H^p(G,\rho)}$, tends to the following limits

$$\Phi_n^+(\infty) = 1 \text{ and } \lim_{n\to\infty} \Phi_n^+(z) = 0.$$

As in (i) of theorem we have

$$\lim_{n\to\infty} \inf \left\|\Phi_n^+\right\|^p_{H^p(G,\rho)} \geq \mu(\alpha).$$

Finally, (ii) of theorem follows from an extension of the Keldysh lemma and the Clarkson inequality. □

For $0 \leq p \leq \infty$, we use the extension of the Keldysh lemma due to Bello Hernandez, Marcellan and Minguez [7]. If we adapted this result to our case, we obtain the following version of the Keldysh lemma.

Lemma 2.5. *Let $\{z_k\}_{k=1}^\infty$ be a set of points in G, $\alpha = \beta + \gamma = \beta_a + \beta_s + \gamma$ where $\alpha \in \mathcal{E}$ and $\{f_n\} \subset H^p(G,\rho)$, $0 < p < \infty$. Let*

$$g_n = \frac{f_n}{\varphi^*}, \text{ where } \varphi^*(z) = \frac{\mathcal{D}_{G,\rho}(\infty)}{\mathcal{D}_{G,\rho}(z)}.$$

If

(a) $\lim_{n\to\infty} g_n(\infty) = 1$,

(b) $\lim_{n\to\infty} g_n(z_k) = 0, k = 1, 2, ...,$

(c) $\sum_{k=1}^\infty (|z_k| - 1) < \infty$,

(d) $\lim_{n\to\infty} \left\|f_n\right\|^p_{H^p(G,\rho)} = \mathcal{D}_{G,\rho}(\infty) \prod_{k=1}^\infty |z_k|$.

Then

$$\lim_{n\to\infty} \left\|f_n - \varphi^* B\right\|^p_{H^p(G,\rho)} = 0.$$

We get (ii) of theorem in the case $0 < p < 1$, by applying Lemma 2.5 to the sequence $\left\{f_n = \frac{T_{n,p,\alpha}}{z^n}\right\} \subset H^p(G,\rho)$, We have

$$f_n(\infty) = 1 \text{ and } \varphi^*(\infty) = 1.$$

Hence (a) follows. On the other hand, (b) is a consequence of the fact that
$\varphi^*(z_k) \neq 0$ and $\lim_{n\to\infty} f_n(z_k) = 0, k = 1, 2, ...$
(c) is exactly the condition of convergence of the Blaschke product. We obtain (d) by (i) of theorem.
for $1 \leq p \leq 2$:

$$\left[\int_\Gamma \left|\Phi_n^+(\zeta)\right|^p \rho(\zeta)|d\zeta|\right]^{\frac{1}{p-1}} + \left[\int_\Gamma \left|\Phi_n^-(\zeta)\right|^p \rho(\zeta)|d\zeta|\right]^{\frac{1}{p-1}}$$
$$\leq \left[\frac{1}{2}\int_\Gamma \left|T_{n,p,\alpha}(\zeta)\right|^p \rho(\zeta)|d\zeta| + \frac{1}{2}\int_\Gamma \left|\psi^*(\zeta)\right|^p \rho(\zeta)|d\zeta|\right]^{\frac{1}{p-1}}.$$

For $0 < p < 1$:

$$\begin{aligned} &\int_{\Gamma} \left|\Phi_n^+ (\zeta)\right|^p \rho(\zeta)\, |d\zeta| + \int_{\Gamma} \left|\Phi_n^- (\zeta)\right|^p \rho(\zeta)\, |d\zeta| \\ \leq\ & \frac{1}{2} \int_{\Gamma} \left|T_{n,p,\alpha} (\zeta)\right|^p \rho(\zeta)\, |d\zeta| + \frac{1}{2} \int_{\Gamma} \left|\psi^* (\zeta)\right|^p \rho(\zeta)\, |d\zeta| . \end{aligned}$$

To prove (*iii*)of theorem we consider the function

$$\epsilon_n = T_{n,p,\alpha} - \psi^*,$$

which belongs to the space $H^p(G,\rho)$. Then by applying lemma 2.1, we obtain

$$\sup \{|\epsilon_n (z)| : z \in K\} \leq C(K)\, \|\epsilon_n\|_{H^p(G,\rho)} \to 0,$$

for all compact subsets K of G. This achieves the proof of the theorem.

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