Filomat 37:5 (2023), 1507–1522
https://doi.org/10.2298/FIL2305507A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Fredholm linear relation and some results of demicompact for multivalued matrix linear operator

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Abstract. The main purpose of this article is to give a relationship between Fredholm multivalued linear operators and the demicompact linear relation; we provide some sufficient conditions on the inputs of a closable block multivalued linear operator matrix to ensure the generalized demicompactness of its closure. Our results generalize many known ones in the literature.

1. Introduction

Recently, Spectral theory has witnessed a huge development. There are different types of spectra, both for one or serval commuting operators, with important applications, for example the approximate points spectrum, Taylor spectrum, essential spectrum, etc. Spectral theory is important when it comes to functional analysis, having numerous applications in different fields of physics and mathematics that include the matrix theory, complex analysis, differential and integral equations, control theory and quantum physics.

More precisely, we are interested in the study of the properties of demicompactness of a linear relation. In fact, the passage from the Fredholm relation and the multivalued demicompact operator is motivated by the close connection existing between them. However, we will be able to use the Fredholm theory for the linear relation matrices.

W. V. Petryshyn [22] in 1966, using the concept of demicompactness as a generalization of the class of demicompact operator. The notion of demicompactness has been used to discuss fixed points and has been studied in a large number of papers (see, for example, [10, 20, 21]). In 1984 W. Y. Akashi [2] generalized some known results in the classical theory of linear Fredholm operators. A demicompact linear operator $T : \mathcal{D}(T) \subset X \to X$ is defined as follows: for every bounded sequence x_n in $\mathcal{D}(T)$ such that $x_n - Tx_n \to x \in X$, there is a convergent subsequence of $\{x_n\}$. This description contained the use of some notions that were first developed for demicompactness in linear spaces in [15] and systematically treated in the context of the multivalued linear operator by A. Ammar, A. Jeribi and B. Saadaoui in [4, 5]. Recently, A. Jeribi [18] continued this study to investigate the essential spectra of densely defined linear operators. The purpose of this work is to pursue the analysis started in [12] and to extend it to more general classes by introducing the concept of relative demicompact linear relations. Lately, in [8] A. Ammar, H. Daoud and A. Jeribi defined the

2020 *Mathematics Subject Classification*. 47A06
Keywords. Demicompact, Linear relation, Fredholm, Matrix.
Received: 16 January 2021; Revised: 07 June 2021; Accepted: 06 August 2021
Communicated by Snežana Č. Živković Zlatanović
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demicompactness of a linear relation by $T : \mathcal{D}(T) \subseteq X \to X$ is said to be demicompact if for every bounded sequence $\{x_n\}$ in $\mathcal{D}(T)$ such that $Q_{I-T}(I-T)x_n \to y \in X/\overline{(I-T)(0)}$, there is a convergent subsequence of $\{Q_T x_n\}$. Afterwards, results given in have been generalized into multivalued linear operators .

Remark 1.1. *It is clear that the sum, the product of demicompact linear relation, and the product of a complex number by a demicompact linear relation are not necessarily demicompact.*

The theory of multivalued linear operators or linear relations is a branch of mathematics that has been developed intensively in the last years and lies at the junction of topology, the theory of functions of a real variable and nonlinear functional analysis. For some decades ago, the concept of linear relation has occurred in the literature for two main reasons. The first one is the need of considering adjoint (conjugates) of non-densely defined linear differential operators (see [13]). The second reason is the need of considering the inverses of certain operators.

The theory of block operator matrices arises in various areas of mathematics and its applications: in systems theory as Hamiltonians (see [16]), in the discretization of partial differential equations as large partitioned matrices due to sparsity patterns, in saddle point problems in non-linear analysis (see [11]), in evolution problems as linearization of second-order Cauchy problems and as linear operators describing coupled systems of partial differential equations. Such systems occur widely in mathematical physics, e.g., in fluid mechanics (see [17]), magnetohydrodynamics (see [19]). In these different application fields, the spectral properties of the block operator matrices are fundamental since they govern, for instance, the time evolution thus the stability of the underlying physical systems. The spectral theory of block linear relation matrix is of major interest since it describes coupled systems of partial differential equations of mixed order and type. From the most important works on the spectral theory of block linear relation matrix, we mention [1, 3, 6, 7] in which the author was proposed for developed the essential spectra of a 2×2 block linear relation matrix. We also mention [7] about the Weyl essential spectrum of the closure, $\mathcal{L}$, of an multivalued linear operator $\mathcal{L}_0$ represented by 2×2 block linear relation matrix acting on the product Banach spaces $X \times Y$ taking the form:

$$\mathcal{L}_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Finally, using the concept of the multivalued linear operator demicompact with more general hypotheses who it applies on the components of the matrix $\mathcal{L}_0$ under which $\lambda - \mathcal{L}$ is Fredholm linear relation for all $\lambda \in \mathbb{C}$.

We organize our paper in the following way: In the next Section, we introduce the class of Fredholm linear relations in Banach spaces and we prove several results which will be used to prove the main results. In Section 3, we give some sufficient conditions for the linear relation demicompact to become Fredholm linear relation. In Section 4, we prove in Theorem 5.4 that under some conditions, $\mu - \mathcal{L}$ is demicompact for each $\mu \in \rho(A)$ and we give a necessary condition for which $\mu - \mathcal{L}$ is Fredholm linear relations on a Banach space. In the last section, we apply the results of section 3 to describe that under some conditions, $\lambda\mathcal{L}$ is demicompact for each $\mu \in \rho(A)$.

2. Generalities and preliminaries

The goal of this section consists in establishing some preliminary results which will be needed in the sequel. In this paper, the symbols X, Y stand for infinite dimensional Banach spaces over the same field $\mathbb{K}$ ($\mathbb{K}$ being $\mathbb{R}$ or $\mathbb{C}$). A multivalued linear operator or linear relation is a mapping $T \subset X \times Y$ which goes from a subspace $\mathcal{D}(T) \subset X$ called the domain of T, into the collection of nonempty subsets of Y such that $T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$ for all nonzero scalars α_1, α_2 with $x_1,\ x_2 \in \mathcal{D}(T)$. For $x \in X \backslash \mathcal{D}(T)$ we define $Tx = \emptyset$. With this convention, we have

$$\mathcal{D}(T) := \Big\{x \in X : Tx \neq \emptyset\Big\}.$$

The collection of linear relations as defined above will be denoted by $L\mathcal{R}(X,Y)$. A linear relation $T \in L\mathcal{R}(X,Y)$ is uniquely determined by and identified with its graph, $G(T)$, which is defined by

$$G(T) := \Big\{(x,y) \in X \times Y : x \in \mathcal{D}(T),\ y \in Tx\Big\}.$$

Let $T \in L\mathcal{R}(X,Y)$. The range $R(T)$ of T is defined by $R(T) := \{y :\ (x,y) \in G(T)\}$ and T is called surjective if $R(T) = Y$. The inverse of $T \in L\mathcal{R}(X,Y)$ is the linear relation T^{-1} defined by

$$G(T^{-1}) := \Big\{(y,x) \in Y \times X :\ (x,y) \in G(T)\Big\}.$$

The subspace $T^{-1}(0)$ is denoted by $N(T)$ and T is called injective if $N(T) = \{0\}$. If T is both injective and surjective, then we say that T is bijective. Observe that $Tx = y + T(0)$, for any $y \in Tx$. For $T \in L\mathcal{R}(X,Y)$, we write $\alpha(T) := \dim N(T)$, $\beta(T) := \dim Y/R(T)$, $\overline{\beta}(T) := \dim Y/\overline{R(T)}$ and the index of T is the quantity $i(T) := \alpha(T) - \beta(T)$ provided that $\alpha(T)$ and $\beta(T)$ are not both infinite. Let $T,\ S \in L\mathcal{R}(X,Y)$, then their algebraic sum $T + S$ is also a linear relation defined by

$$G(T+S) := \Big\{(x, u+v) :\ (x,u) \in G(T),\ (x,v) \in G(S)\Big\}.$$

Similarly, if $T \in L\mathcal{R}(X,Y)$ and $S \in L\mathcal{R}(Y,Z)$, then their product ST is also a linear relation defined by

$$G(ST) := \Big\{(x,z) \in X \times Z :\ (x,y) \in G(T) \text{ and } (y,z) \in G(S) \text{ for some } y \in Y\Big\}.$$

A linear relation T is said to be closed if its graph is closed. Similarly, T is called closable if $\overline{T}$ is an extension of T where the closure of T, $\overline{T}$, is defined by $G(\overline{T}) := \overline{G(T)}$. We denote the class of all bounded linear relations from X to Y by $B\mathcal{R}(X,Y)$. The collection of all closed linear relations from X to Y is denoted by $C\mathcal{R}(X,Y)$. The quotient map from Y onto $Y/\overline{T(0)}$ is denoted by Q_T. A linear relation T is said to be compact if $Q_T T(B_{\mathcal{D}(T)})$ is compact in Y ($B_{\mathcal{D}(T)} := \{x \in \mathcal{D}(T) : \|x\| \leq 1\}$). We denote the class of compact linear relations from X to Y by $K\mathcal{R}(X,Y)$ and by $\mathcal{K}(X,Y)$ the subspace of all compact operators. It is easy to see that $Q_T T$ is single valued so that we can define

$$\|Tx\| := \|Q_T Tx\| \text{ for all } x \in\ \mathcal{D}(T) \text{ and } \|T\| := \|Q_T T\|.$$

We say that $T \in L\mathcal{R}(X,Y)$ is continuous if $\|T\| < \infty$; bounded if it is continuous and $\mathcal{D}(T) = X$; open if T^{-1} is continuous; equivalent if its minimum modulus $\gamma(T)$ is a positive number, where

$$\gamma(T) := \sup\Big\{\lambda \geq 0 : \lambda d(x, N(T)) \leq \|Tx\|,\ \ x \in \mathcal{D}(T)\Big\},$$

where $d(x, N(T))$ the distance between x and $N(T)$.

We say that a closed linear relation T is a upper semi Fredholm linear relation if it has finite dimensional null space and closed range is defined by:

$$\Phi_+(X,Y) = \Big\{T \in C\mathcal{R}(X,Y) : \alpha(T) < \infty \text{ and } R(T) \text{ is closed in } Y\Big\},$$

T is lower semi Fredholm linear relation if its range is closed and has a finite codimensional is defined by:

$$\Phi_-(X,Y) = \Big\{T \in C\mathcal{R}(X,Y) : \beta(T) < \infty \text{ and } R(T) \text{ is closed in } Y\Big\}.$$

T is Fredholm linear relation if it is both upper and lower semi Fredholm linear relation is defined by $\Phi(X,Y) := \Phi_+(X,Y) \cap \Phi_-(X,Y)$ denotes the set of Fredholm relations from X into Y. If $X = Y$ then $\Phi_+(X,Y)$, $\Phi_-(X,Y)$, $\Phi_\pm(X,Y)$ and $\Phi(X,Y)$ are replaced by $\Phi_+(X)$, $\Phi_-(X)$, $\Phi_\pm(X)$ and $\Phi(X)$, respectively.

Lemma 2.1. [13] *Let $T \in L\mathcal{R}(X,Y)$. Then,*

(i) $\mathcal{D}(T^{-1}) = R(T)$ *and* $\mathcal{D}(T) = R(T^{-1})$.

(iv) $TT^{-1}y = y + T(0)$ *and* $T^{-1}Tx = x + T^{-1}(0)$.

(iii) T is single valued if, and only if, $T(0) = \{0\}$.

(ii) T injective if, and only if, $T^{-1}T = I_{\mathcal{D}(T)}$.

Lemma 2.2. [13, Lemma V.2.9] *If $T \in L\mathcal{R}(X,Y)$ and $S \in L\mathcal{R}(Y,Z)$ such that $\overline{T(0)} \subset \mathcal{D}(S)$ and S is a continuous, then $Q_{ST}ST = Q_{ST}SQ_T^{-1}Q_TT$.*

Proposition 2.3. [13, Proposition 11.3.4] *we have:*

(i) *$N(T) \subset N(Q_TT)$ with equality if $T(0)$ is relatively closed in $R(T)$.*

(ii) *$\gamma(T) < \gamma(Q_TT)$ with equality if $T(0)$ is relatively closed in $R(T)$.*

Proposition 2.4. [13, Proposition I.4.2] *Let $R, S, T \in L\mathcal{R}(X)$. Then,*

(i) *$(R + S)T \subset RT + ST$ with equality if T is single valued.*

(ii) *$T(R + S)$ is an extension of $TR + TS$ and $TR + TS = T(R + S)$ if $\mathcal{D}(T)$ is the whole space.*

(iii) *Let $T \in L\mathcal{R}(X,Y)$ and $S, R \in L\mathcal{R}(Y,Z)$. If $T(0) \subset N(S)$ or $T(0) \subset N(R)$, then $(R + S)T = RT + ST$.*

Lemma 2.5. [13, Proposition II.5.3] *Let X and Y be two vector spaces.*

T is closed if, and only if, Q_TT is closed single valued and $T(0)$ is a closed space.

Theorem 2.6. [7, Theorem 3.1] *Let $T \in B\mathcal{R}(X,Y)$ be a single valued bijective and assume that $R \in B\mathcal{R}(Z,W)$ is bijective with $R(0)$ closed.*

(i) *If S is closable, then RST is closable and $\overline{RST} = R\overline{S}T$.*

(ii) *If R is bounded single valued bijective, then S is closable if, and only if, RST is closable and $\overline{RST} = R\overline{S}T$.*

Corollary 2.7. [4, Corollary 2.4.] *Let D a linear subspace of a space X with $dim(D) < \infty$. Let $\{x_n\}$ in X be a sequence such that $\{Q_Dx_n\}$ is a convergent sequence, then $\{x_n\}$ has a convergent subsequence.*

Theorem 2.8. *Let X be Banach space, then $B_X(0,1)$ is compact if, and only if, X has a finite dimension.*

Theorem 2.9. [1, Theorem 2.2] *Let $S, T \in C\mathcal{R}(X)$. Then,*

(i) *$T \in \Phi_+(X)$ if, and only if, $Q_TT \in \Phi_+(X)$. In such case $i(T) = i(Q_TT)$.*

(ii) *If $S, T \in \Phi_+(X)$, then $ST \in \Phi_+(X)$ and $TS \in \Phi_+(X)$.*

(iii) *If S and T are everywhere defined and $TS \in \Phi_+(X)$, then $S \in \Phi_+(X)$.*

Lemma 2.10. [13, Corollary V.15.7] *Let $T \in C\mathcal{R}(X,Y)$ closed. Then, for any linear operator S satisfying $\mathcal{D}(S) \supset \mathcal{D}(T)$ and $\|S\| < \gamma(T)$ we have*

$$i(T + S) = i(T).$$

Lemma 2.11. (i) ([13, Lemma V.7.8]) *Let $T \in L\mathcal{R}(X,Y)$ if $dimT(0) < \infty$, then $S + T - T \in \Phi_+(X,Y)$ if, and only if, $S \in \Phi_+(X)$.* (ii) ([13, Lemma VII.1.4]) *Let $S \in L\mathcal{R}(X,Y)$ satisfy $\mathcal{D}(F) \supset \mathcal{D}(T)$ and $dimT(0) < \infty$. Then, $T + S \in \Phi(X,Y)$ if, and only if, $T \in \Phi(X,Y)$.*

Lemma 2.12. [14, Corollary 3.2] *Let $S, T \in L\mathcal{R}(X)$. Suppose that $\mathcal{D}(S) = X$ and T, S have finite indices. Then, ST has a finite index and $i(ST) = i(S) + i(T) - \dim(T(0) \cap S^{-1}(0))$.*

Theorem 2.13. [13, Theorem V.10.3] *Let $T \in L\mathcal{R}(X)$. Then, the following are equivalent:*

(i) *$T \in \Phi_+(X)$.*

(ii) *There exists $A \in B\mathcal{R}(X)$ and a finite rank projection K such that $AT = I - K$.*

Below, we introduce some definitions on Fredholm perturbations:

Definition 2.14. [7, Definition 2.1] *Let $S \in L\mathcal{R}(X,Y)$ be continuous.*

(i) *S is called a Fredholm perturbation if $T+S \in \Phi(X,Y)$ whenever $T \in \Phi(X,Y)$ with $\dim(S(0)) < \infty$ and $S(0) \subset T(0)$.*

(ii) *S is called an upper semi-Fredholm perturbation if $T+S \in \Phi_+(X,Y)$ whenever $T \in \Phi_+(X,Y)$ with $\dim(S(0)) < \infty$ and $S(0) \subset T(0)$.*

We denote by $\mathcal{P}(X)$ the set of Fredholm perturbations, $\mathcal{P}_+(X)$ the set of upper semi-Fredholm perturbations and $\mathcal{P}_-(X)$ the set of lower semi-Fredholm perturbations.

Proposition 2.15. [7, Proposition 2.2] *Let $T \in L\mathcal{R}(X,Y)$ be closed and $S \in L\mathcal{R}(X,Y)$ be continuous. We have*

(i) *If $T \in \Phi_+(X,Y)$ and $S \in \mathcal{P}_+(X,Y)$, then $T+S \in \Phi_+(X,Y)$ and $i(T+S) = i(T)$.*

(ii) *If $T \in \Phi_-(X,Y)$ and $S \in \mathcal{P}_-(X,Y)$, then $T+S \in \Phi_-(X,Y)$ and $i(T+S) = i(T)$.*

(iii) *If $T \in \Phi(X,Y)$ and $S \in \mathcal{P}(X,Y)$, then $T+S \in \Phi(X,Y)$ and $i(T+S) = i(T)$.*

3. Demicompact and Fredholm linear relation

In the first part of this section, we introduce a Theorem which shows the connection between a linear relation demicompact and its selection. Just below, we recall the following definition.

Definition 3.1. [13, Definition I.5.1] *Let $T \in L\mathcal{R}(X)$. A linear operator S is called a selection of T if*

$$T = S + T - T \ \text{ and } \ \mathcal{D}(T) = \mathcal{D}(S).$$

If S is a selection of T, then we have $\forall x \in \mathcal{D}(T)$

$$Tx = Sx + T(0).$$

Theorem 3.2. *Let S be selection of $T \in L\mathcal{R}(X)$ and $dim(T(0)) < \infty$. Then, T is a demicompact linear relation if, and only if, S is a demicompact operator.*

Proof. Let T be a demicompact linear relation by using Theorem 3.3, we get $I - T \in \Phi_+(X)$. Since $I - S$ is selection of $I - T$ with $dim(T(0)) < \infty$, then by Lemma 2.11 we obtain $I - S \in \Phi_+(X)$. From Theorem 2.13, it follows that there exists $A \in B\mathcal{R}(X)$ and a finite rank projection K such that $A(I - S) = I - K$. Let $\{x_n\}$ be a bounded sequence of $\mathcal{D}(S)$ such as $(I - S)x_n \longrightarrow y$, then

$$A(I - S)x_n \longrightarrow Ay.$$

We conclude that

$$(I - K)x_n \longrightarrow Ay.$$

Since K is compact, then $(Kx_n)_n$ has a convergent subsequence and so $\{x_n\}$ has also a convergent subsequence. Conversely, let S be a demicompact operator, then by using Theorem 3.3, we get $I - S \in \Phi_+(X)$. Since $I - S$ is a selection of $I - T$ with $dim(T(0)) < \infty$, then by Lemma 2.11 we obtain $I - T \in \Phi_+(X)$. From Theorem 2.13, it follows that there exists $A \in B\mathcal{R}(X)$ and a finite rank projection K such as $A(I - T) = I - K$. Let $\{x_n\}$ be a bounded sequence of $\mathcal{D}(T)$ such as $Q_T(I - T)x_n \longrightarrow y$. Since $Q_{A(I-T)}AQ_T^{-1}$ is a bounded operator, then

$$Q_{A(I-T)}AQ_T^{-1}Q_T(I - T)x_n \longrightarrow Q_{A(I-T)}AQ_T^{-1}y.$$

Since $\overline{T(0)} \subset \mathcal{D}(A)$ and A is continuous, then by using Lemma 2.2, we obtain $Q_{I-K}A(I - T)x_n \longrightarrow Q_{I-K}AQ_T^{-1}y$. Equivalently to $Q_K(I - K)x_n \longrightarrow Q_K AQ_T^{-1}y$. Since K is a compact operator, then $\{x_n\}$ has also a convergent subsequence. □

Theorem 3.3. *Let $T \in C\mathcal{R}(X)$ and $\mu \in \mathbb{C}^*$. If $\frac{1}{\mu}T$ is a demicompact, then $\mu - T \in \Phi_+(X)$.*

Proof. Since $I - \frac{1}{\mu}T \in C\mathcal{R}(X)$, then $N\left(I - \frac{1}{\mu}T\right) = N\left(Q_T\left(I - \frac{1}{\mu}T\right)\right)$. Notice that $\alpha\left(I - \frac{1}{\mu}T\right) < \infty$. Indeed, let

$$B_{N\left(Q_T\left(I-\frac{1}{\mu}T\right)\right)}(0,1) = \left\{x \in \mathcal{D}(T) : Q_T\left(I - \frac{1}{\mu}T\right)x = 0 \text{ and } \|x\| = 1\right\}$$

and let $\{x_n\}$ be a bounded sequence of $B_{N\left(Q_T\left(I-\frac{1}{\mu}T\right)\right)}(0,1)$, then $Q_T\left(I - \frac{1}{\mu}T\right)x_n = 0$ and $\|x_n\| = 1$. Since μT is demicompact, there exists a subsequence $\{x_{\varphi}(n)\}$ of $\{x_n\}$ which converges to x. Since $I - \frac{1}{\mu}T \in C\mathcal{R}(X)$, then by Lemma 2.5 we get

$Q_T\left(I - \frac{1}{\mu}T\right)$ is closed. Thus, $x \in \mathcal{D}\left(Q_T\left(I - \frac{1}{\mu}T\right)\right)$ and $Q_T\left(I - \frac{1}{\mu}T\right)x = 0$.

It follows from the continuity of the norm that $\|x\| = 1$. We obtain that

$$x \in B_{N\left(Q_T\left(I-\frac{1}{\mu}T\right)\right)}(0,1)$$

implies that $B_{N\left(Q_T\left(I-\frac{1}{\mu}T\right)\right)}(0,1)$ is compact. Therefore, by using Theorem 2.8 we get $\alpha\left(Q_T\left(I - \frac{1}{\mu}T\right)\right) < \infty$. By using Proposition 2.3 (*i*), we conclude that

$$\alpha\left(I - \frac{1}{\mu}T\right) < \infty.$$

We will check that $R\left(I - \frac{1}{\mu}T\right)$ is closed. Indeed, applying [23, Proposition II.5.3], there exists a closed subspace X_0 of X such that $X = N\left(Q_T\left(I - \frac{1}{\mu}T\right)\right) \oplus X_0$. Then,

$$\mathcal{D}(T) = N\left(Q_T\left(I - \frac{1}{\mu}T\right)\right) \oplus W \text{ with } W = \mathcal{D}(T) \cap X_0.$$

Since $Q_T T$ is closed, then $(\mathcal{D}(T), \|\cdot\|_{Q_T T})$ is a Banach space. Since $\|x\|_{Q_T T} = \|x\| + \|Q_T Tx\| = \|x\| + \|Tx\| = \|x\|_T$, then $(\mathcal{D}(T), \|\cdot\|_T)$ is a Banach space. Since W is a closed subspace of $\mathcal{D}(T)$, then $(W, \|\cdot\|_T)$ is a Banach space. We suppose that $R\left(Q_T\left(I - \frac{1}{\mu}T\right)\right)$ is not closed. By [23, Theorem 3.12.], it suffices to prove that there is a constant $\lambda > 0$ such as for every $x \in W$, $\|Tx\| \geq \lambda\|x\|_T$. If not, there exists a sequence $\{x_n\}$ of W such as $\|x_n\|_T = 1$ and $\left\|Q_T\left(I - \frac{1}{\mu}T\right)x_n\right\| \longrightarrow 0$. Since $\frac{1}{\mu}T$ is demicompact, there exists a subsequence $\{x_{\varphi}(n)\}$ which converges to x. Moreover, $Q_T\left(I - \frac{1}{\mu}T\right)$ is closed, then $Q_T\left(I - \frac{1}{\mu}T\right)x_{\varphi}(n) \longrightarrow Q_T\left(I - \frac{1}{\mu}T\right)x$, thus $Q_T\left(I - \frac{1}{\mu}T\right)x = 0$.

Accordingly, we have $x \in N\left(Q_T\left(I - \frac{1}{\mu}T\right)\right) \cap W = \left\{0\right\}$. Hence $x = 0$ which contradicts the continuity of the norm. Therefore, $R\left(Q_T\left(I - \frac{1}{\mu}T\right)\right)$ is closed. Since $I - \frac{1}{\mu}T$ is closed, then by Proposition 2.3 (*ii*), we get $R\left(I - \frac{1}{\mu}T\right)$ is closed. □

Theorem 3.4. *Let $T \in C\mathcal{R}(X)$ and $dimT(0) < \infty$. If $\frac{1}{\mu}T$ is demicompact for each $\mu \in [1, +\infty[$, then $\mu - T \in \Phi(X)$.*

Proof. Let S be a selection of T. Since $\frac{1}{\mu}T$ is demicompact for each $\mu \in [1, +\infty[$, then by Theorem 3.3 we get $\mu - T \in \Phi_+(X)$. By using Lemma 2.11 (*i*) and Theorem 3.2, we get $\mu - S \in \Phi(X)$.

We shall prove that the map

$$\begin{array}{rccc} \varphi : & [1, +\infty[& \longrightarrow & \mathbb{Z} \\ & \mu & \longrightarrow & i(\mu - S) \end{array}$$

is continuous in μ. Let $\mu, \mu_0 \in [1, +\infty[$ arbitrary but fixed such as $|\mu - \mu_0| < \gamma(\mu - S)$. By using Lemma 2.10, we have

$$i(\mu - S) = i(\mu - S - \mu + \mu_0) = i(\mu_0 - S).$$

Let $\varepsilon > 0$ there exists $\delta := \gamma(\mu - S)$ such as, if $\mu, \mu_0 \in [1, +\infty[$ with $|\mu - \mu_0| < \delta$, then $|i(\mu - S) - i(\mu_0 - S)| = |0| = 0 < \varepsilon$. So, that $\varphi(\mu)$ is continuous. Now, we know that every continuous mapping of a connected in $\mathbb{Z}$ is constant.

If $\mu \longrightarrow +\infty$, then $i\left(\left(I - \frac{1}{\mu}S\right)\mu\right) = i\left(I - \frac{1}{\mu}S\right) = i(I) = 0$. Showing that

$i(\mu - S) = 0$ for each $\mu \in [1, +\infty[$. We conclude that $\alpha(\mu - S) = \beta(\mu - S) < \infty$, then $\mu - S$ is a Fredholm linear operator. Now, by Lemma 2.11 we conclude that $\mu - T \in \Phi(X)$.
□

Theorem 3.5. *Let $T \in B\mathcal{R}(X)$, $\lambda_0 \in [1, +\infty[$ and $\lambda \in \mathbb{C}$ such that $\lambda \neq \lambda_0$ with $T(0) = N(\lambda_0 - T)$ and $dim(T(0)) < \infty$. If $\frac{1}{\lambda_0}T$ is a demicompact linear relation, then $\lambda - T$ is Fredholm linear relation.*

Proof. If $\frac{1}{\lambda_0}T$ is a demicompact linear relation, then for every bounded sequence $\{x_n\}$ in $\mathcal{D}(T)$ such that $Q_T\left(I - \frac{1}{\lambda_0}T\right)x_n \to y \in X/\overline{\left(I - \frac{1}{\lambda_0}T\right)}$, there is a convergent subsequence of $\{Q_T x_n\}$. We claim that $-\frac{1}{\lambda_0}\left(\lambda - \lambda_0\right)\left(I - \frac{1}{\lambda_0}T\right)^{-1}$ is demicompact. Indeed, let $\{x_n\}$ be a bounded sequence of $\mathcal{D}(T)$ such that

$$Q_{\frac{1}{\lambda_0}\left(\lambda-\lambda_0\right)\left(I-\frac{1}{\lambda_0}T\right)^{-1}}\left(I + \frac{1}{\lambda_0}\left(\lambda - \lambda_0\right)\left(I - \frac{1}{\lambda_0}T\right)^{-1}\right)x_n \to y_0.$$

Since $\frac{1}{\lambda_0}\left(\lambda - \lambda_0\right)\left(I - \frac{1}{\lambda_0}T\right)^{-1}(0) = \left(\lambda - \lambda_0\right)(\lambda_0 - T)^{-1}(0) = \left(\lambda - \lambda_0\right)T(0) = T(0)$, then

$$Q_T\left(I + \frac{1}{\lambda_0}\left(\lambda - \lambda_0\right)\left(I - \frac{1}{\lambda_0}T\right)^{-1}\right)x_n \to y_0.$$

In fact,

$$Q_T\left(I + \frac{1}{\lambda_0}\left(\lambda - \lambda_0\right)\left(I - \frac{1}{\lambda_0}T\right)^{-1}\right)x_n = Q_T x_n + Q_T\frac{1}{\lambda_0}\left(\lambda - \lambda_0\right)\left(I - \frac{1}{\lambda_0}T\right)^{-1}x_n.$$

Since $Q_T\left(I - \frac{1}{\lambda_0}T\right)Q_T^{-1}$ is bounded linear relation, then

$$Q_T\left(I - \frac{1}{\lambda_0}T\right)Q_T^{-1}\left(Q_T x_n + Q_T\frac{1}{\lambda_0}\left(\lambda - \lambda_0\right)\left(I - \frac{1}{\lambda_0}T\right)^{-1}x_n\right) \to Q_T\left(I - \frac{1}{\lambda_0}T\right)Q_T^{-1}y_0.$$

By using Lemma 2.1 (*iv*) and Proposition 2.4 (*ii*), we obtain

$$\begin{aligned}
&Q_T\left(I - \tfrac{1}{\lambda_0}T\right)Q_T^{-1}\left(Q_T x_n + Q_T\tfrac{1}{\lambda_0}\left(\lambda - \lambda_0\right)\left(I - \tfrac{1}{\lambda_0}T\right)^{-1}x_n\right)\\
&= Q_T\left(I - \frac{1}{\lambda_0}T\right)Q_T^{-1}Q_T x_n + Q_T\left(I - \frac{1}{\lambda_0}T\right)Q_T^{-1}\frac{1}{\lambda_0}\left(\lambda - \lambda_0\right)Q_T\left(I - \frac{1}{\lambda_0}T\right)^{-1}x_n,\\
&= Q_T\left[\left(I - \frac{1}{\lambda_0}T\right)\left(x_n + Q_T^{-1}(0)\right)\right]\\
&\quad + \frac{1}{\lambda_0}\left(\lambda - \lambda_0\right)Q_T\left(I - \frac{1}{\lambda_0}T\right)Q_T^{-1}Q_T\left(I - \frac{1}{\lambda_0}T\right)^{-1}x_n,\\
&= Q_T\left[\left(I - \frac{1}{\lambda_0}T\right)\left(x_n + T(0)\right)\right]\\
&\quad + \frac{1}{\lambda_0}\left(\lambda - \lambda_0\right)Q_T\left[\left(I - \frac{1}{\lambda_0}T\right)\left(I - \frac{1}{\lambda_0}T\right)^{-1}\left(x_n + Q_T^{-1}(0)\right)\right],\\
&= Q_T\left(I - \frac{1}{\lambda_0}T\right)x_n + \frac{1}{\lambda_0}\left(\lambda - \lambda_0\right)Q_T\left(x_n + \left(I - \frac{1}{\lambda_0}T\right)(0)\right),\\
&= Q_T\left(I - \frac{1}{\lambda_0}T\right)x_n + \frac{1}{\lambda_0}\left(\lambda - \lambda_0\right)Q_T x_n.
\end{aligned}$$

Since $Q_T\left(I-\frac{1}{\lambda_0}T\right)x_n \to y$, then $\frac{1}{\lambda_0}\left(\lambda-\lambda_0\right)Q_Tx_n \longrightarrow Q_T\left(I-\frac{1}{\lambda_0}T\right)Q_T^{-1}y_0 - y$. We obtain that, $Q_Tx_n \longrightarrow \lambda_0\left(\lambda-\lambda_0\right)^{-1}\left(Q_T\left(I-\frac{1}{\lambda_0}T\right)Q_T^{-1}y_0-y\right)$. It follows, from Corollary 2.7, that $\{x_n\}$ has a convergent subsequence, then $-\frac{1}{\lambda_0}\left(\lambda-\lambda_0\right)\left(I-\frac{1}{\lambda_0}T\right)^{-1}$ is demicompact. Hence,

$$\lambda - T = \lambda_0\left(I-\frac{1}{\lambda_0}T\right)\left(I+\frac{1}{\lambda_0}\left(\lambda-\lambda_0\right)\left(I-\frac{1}{\lambda_0}T\right)^{-1}\right).$$

Since $\frac{1}{\lambda_0}T$ and $-\frac{1}{\lambda_0}\left(\lambda-\lambda_0\right)\left(I-\frac{1}{\lambda_0}T\right)^{-1}$ are demicompact, then by Theorem 3.3 we get $I-\frac{1}{\lambda_0}T \in \Phi(X)$ and $\left(I+\frac{1}{\lambda_0}\left(\lambda-\lambda_0\right)\left(I-\frac{1}{\lambda_0}T\right)^{-1}\right) \in \Phi(X)$, then by Theorem 2.9 we have $\lambda - T \in \Phi(X)$. $\square$

4. Polynomially demicompact linear relation

It was shown in [1] that a polynomially compact linear relation T, element of

$$\mathcal{P}(X) := \left\{ T \in L\mathcal{R}(X) \ : \ \begin{array}{c} \text{there exists a nonzero complex polynomial} \\ P(z) = \sum_{r=0}^{p} a_r z^r \ \text{ satisfying } P(Q_T) \neq 0, \\ P(Q_T) - a_0 \neq 0 \text{ and } P(Q_T T) \in \mathcal{K}(X) \end{array} \right\},$$

is demicompact. In this section, we show that this result remains valid for a broader class of polynomially demicompact operators on X. To this end we let $\mathcal{PDC}(X)$ be the set defined by

$$\mathcal{PDC}(X) := \left\{ T \in L\mathcal{R}(X) : \begin{array}{c} \text{there exists a nonzero complex polynomial} \\ P(z) = \sum_{r=0}^{p} a_r z^r \ \text{ satisfying } \ P(Q_T) \neq 0, \\ P(Q_T) - a_0 \neq 0 \text{ and } \frac{Q_T T}{P(Q_T)} \ \text{ is demicompact} \end{array} \right\}.$$

We note that $\mathcal{PDC}(X)$ contains the set $\mathcal{P}(X)$.

Theorem 4.1. *Let $T \in \mathcal{PDC}(X)$. If $Q_T(I-T)$ commutes with Q_T, then $T \in \Phi_+(X)$.*

Proof. Since $Q_T(I-T)$ commutes with Q_T, Newton's binomial formula allows us to write the following relation

$$(Q_T T)^j = Q_T^j + \sum_{i=1}^{j} (-1)^i C_j^i Q_T^{j-i}\left(Q_T(I-T)\right)^i.$$

We may write $P(Q_T T)$ in another manner

$$P(Q_T T) = P(Q_T) + \sum_{j=1}^{p} a_j \left(\sum_{i=1}^{j} (-1)^i C_j^i Q_T^{j-i}\left(Q_T(I-T)\right)^i \right). \tag{1}$$

To this end we let $T \in \mathcal{PDC}(X)$, we shall prove that $I-T$ is an upper semi-Fredholm multivalued linear operator. Hence, $N(Q_T(I-T)) \subset N\left(I - \frac{P(Q_T T)}{P(Q_T)}\right)$.

Since $\frac{P(Q_T T)}{P(Q_T)}$ is demicompact, we deduce that $\alpha\left(I-\frac{P(Q_T T)}{P(Q_T)}\right) < \infty$ and as consequence, $\alpha(Q_T(I-T)) < \infty$. We will check that $R\,(I-T)$. Indeed, applying [23, Proposition II.5.3], there exist a closed subspace X_0 of X such that $X = N\left(Q_T\left(I-\frac{1}{\mu}T\right)\right) \oplus X_0$. Then,

$$\mathcal{D}(T) = N\left(Q_T\left(I-\frac{1}{\mu}T\right)\right) \oplus W \text{ with } W = \mathcal{D}(T) \cap X_0.$$

Since $Q_T T$ is closed, then $(\mathcal{D}(Q_T T), \|\cdot\|_{Q_T T})$ is a Banach space. Since W is a closed subspace of $\mathcal{D}(T)$, then $(W, \|\cdot\|_{Q_T T})$ is a Banach space. We suppose that$R(Q_T(I-T))$ is not closed. By [23, Theorem 3.12.], it suffices to prove that there is a constant $\lambda > 0$ such that for every $x \in W$, $\|Tx\| \geq \lambda \|x\|_T$. If not, there exists a sequence $\{x_n\}$ of W such that $\|x_n\|_T = 1$ and $Q_T(I-T)x_n \longrightarrow 0$. This together with (1) leads

to $\left(P(Q_T T)x_n - P(Q_T)x_n\right)_n \to 0$. Since $P(Q_T) \neq 0$, then $\left(I - \frac{P(Q_T T)}{P(Q_T)}\right)x_n \to 0$.

Using the demicompactness of $\frac{P(Q_T T)}{P(Q_T)}$, we deduce that $(x_n)_n$ has a converging subsequence to an element x in X_0, verifying $\|x\| = 1$. Moreover, $Q_T(I-T)$ is closed, then $Q_T(I-T)x_{\varphi(n)} \longrightarrow Q_T(I-T)x$, thus $Q_T(I-T)x = 0$. Therefore,

$$x \in N(Q_T(I-T)) \cap W = \left\{0\right\}.$$

Hence $x = 0$ which contradicts the continuity of the norm. Which implies that $R(Q_T(I-T))$ is closed. Since $I-T$ is closed, then by Proposition 2.3 (*ii*), we get $R(I-T)$ is closed.
□

5. Demicompactness results for multivalued 2 × 2 matrices linear operator $\mathcal{L}$

In this section, we consider the following 2×2 block matrices multivalued linear operator of the form

$$\mathcal{L} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where the linear relation A acts on X and has the domain $\mathcal{D}(A)$, D is defined on $\mathcal{D}(D)$ and acts on the Banach space X, and the intertwining B (resp. C) is defined on the domain $\mathcal{D}(B)$ (resp. $\mathcal{D}(C)$) and acts on X (resp. on X).

Lemma 5.1. *Let A and B are two demicompact linear relations, then*

(*i*) *The matrix linear relation $\mathcal{T} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is demicompact.*

(*ii*) *If $C(0) \subset A(0)$ and $Q_A C$ is bounded, then the matrix linear relation $M_C =$*
$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is demicompact.

Proof. (*i*) Let $\left\{\left(x_n, y_n\right)\right\}$ bounded sequence in $\mathcal{D}(\mathcal{T}) = \mathcal{D}(A) \times \mathcal{D}(B)$ such that:

$$Q_{I-\mathcal{T}}(I-\mathcal{T})\begin{pmatrix} x_n \\ y_n \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ y \end{pmatrix}.$$

The latter is equivalent to $\begin{pmatrix} Q_A(I-A) & 0 \\ 0 & Q_B(I-B) \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ y \end{pmatrix}$. It

follows that $\begin{pmatrix} Q_A(I-A)x_n \\ Q_B(I-B)y_n \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ y \end{pmatrix}$. Since A is demicompact, then

$Q_A x_n$ has a convergent subsequence we denote by $Q_A x_{\varrho(n)}$. On the other hand,

$Q_B(I-B)y_n \longrightarrow y$, then $Q_B(I-B)y_{\varrho(n)} \longrightarrow y$. Hence B is demicompact, then $Q_B x_{\varrho(n)}$ has a convergent subsequence we denote by $Q_B x_{\psi(\varrho(n))}$. We obtain that, $Q_{\mathcal{T}}\begin{pmatrix} x_n \\ y_n \end{pmatrix}$ has a convergent subsequence.

(ii) Let $\left\{\left(x_n, y_n\right)\right\}$ bounded sequence in $\mathcal{D}\left(M_C\right)$ such that:

$$Q_{I-M_C}(I-M_C)\begin{pmatrix} x_n \\ y_n \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ y \end{pmatrix}.$$

Since $M_C\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} A(0) \\ B(0) \end{pmatrix}$, then $\begin{pmatrix} Q_A(I-A)x_n - Q_ACy_n \\ Q_B(I-B)y_n \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ y \end{pmatrix}$.

We conclude that $Q_B(I-B)y_n$ is convergent. This together with the demicompact of B show that $(y_n)_n$ has a convergent subsequence. Since Q_AC is a bounded operator and A is demicompact, we infer that $(x_n)_n$ has a convergent subsequence, which proves the demicompact of M_C. $\square$

5.1. *Some results for multivalued* 2×2 *matrices linear operator*

This subsection based on the Frobenuis-Schur factorization associated to this kind of matrix multivalued linear operator $\mathcal{L}$ on the product of Banach spaces $X \times Y$. We will first collect some hypotheses, introduced in [1], which we will need in the sequel.

$(H1)$ The linear relation A is continuous, $\mathcal{D}(A)$ is closed, and $\dim(A(0)) < \infty$.

$(H2)$ The linear relation C is closable which satisfies $\mathcal{D}(A) \subset \mathcal{D}(C)$. Then, for $\mu \in \rho(A)$ the linear relation $F(\mu) = C(A-\mu I)^{-1}$ is bounded.

$(H3)$ The linear relation B is closable and $\overline{\mathcal{D}(B)} = Y$. Then, for $\mu \in \rho(A)$ the the linear relation $(A-\mu I)^{-1}B$ is closable. Then, for $\mu \in \rho(A)$ the linear relation $E(\mu) = \overline{(A-\mu I)^{-1}B}$ is bounded.

$(H4)$ The linear relation D is bounded and $D(0) \subset C(A-\mu I)^{-1}B(0) \subset C(0)$. Then, for $\mu \in \rho(A)$, the linear relation $M(\mu) = D - C(A-\mu I)^{-1}B$ is closable.

It is always assumed that the entries of this matrix satisfy the following conditions, introduced in [7].

$(H5)$ $Z(\mu)$ is a continuous selection of $F(\mu)$.

$(H6)$ $W(\mu)$ is a continuous selection of $E(\mu)$.

$(H7)$ For some $\mu \in \rho(A)$, $\mathcal{D}(K)$ contains the ranges of both

$$\begin{pmatrix} I & W(\mu) \\ 0 & I \end{pmatrix} \text{ and } \begin{pmatrix} 0 & E(\mu)-E(\mu) \\ 0 & 0 \end{pmatrix}.$$

We recall the following result established by [7] which describes the closure of the multivalued linear operator $\mathcal{L}_0$.

Theorem 5.2. [7, Theorem 4.5] *Let* $\mu \in \rho(A)$. *We suppose that the conditions* $(H1)$-$(H7)$ *are satisfied, then for* $\lambda \in \mathbb{C}$ *we have:*

$$\lambda I - \mathcal{L} = HV(\lambda)J - (\lambda-\mu)\mathcal{I}(\mu) + \mathcal{J}(\mu), \tag{2}$$

where H and J are the lower- and upper-triangular linear relations matrices defined by:

$$H = \begin{pmatrix} I & 0 \\ Z(\mu) & I \end{pmatrix},\ J = \begin{pmatrix} I & W(\mu) \\ 0 & I \end{pmatrix},$$

$V(\lambda)$ is the diagonal multivalued linear relation matrix

$$V(\lambda)=\begin{pmatrix}\lambda I-A & 0\\ 0 & \lambda I-\overline{M(\mu)}\end{pmatrix}$$

with

$$\mathcal{I}(\mu)=\begin{pmatrix}0 & W(\mu)\\ Z(\mu) & Z(\mu)W(\mu)\end{pmatrix},$$

$$\mathcal{J}(\mu)=\begin{pmatrix}A-A & B-B\\ (F(\mu)-\overline{F(\mu)})(\mu I-A)+Z(\mu)(A-A) & (F(\mu)-\overline{F(\mu)})B+Z(\mu)(B-B)\end{pmatrix}.$$

To find the demicompact of a matrix relation it is necessary to go through the Frobenius-Schur decomposition but in our work the definition of demicompact attached with the quotient operator. For this reason, we seek in the Theorem following the relationship of the factorization of Frobenius-Schur with its quotient operator.

Theorem 5.3. *Let $\mathcal{L}$ be the multivalued matrices linear operator defined in* (2) *satisfies* $(H1)-(H7)$. *Suppose that there is $\mu\neq 0$ such*

that $\frac{1}{\mu}\in\rho(A)$. If $B(0)\subset A(0)\subset N\left(F\left(\frac{1}{\mu}\right)\right)$, then

$$Q_{\mathcal{L}}(I-\mu\mathcal{L})=Q_{\mathcal{L}}\left(H_1V\left(\tfrac{1}{\mu}\right)J_1\right),$$

where H_1 and J_1 are the lower- and upper-triangular linear relations matrices defined by:

$$H_1=\begin{pmatrix}I & 0\\ Z\left(\frac{1}{\mu}\right) & I\end{pmatrix},\ J_1=\begin{pmatrix}I & W\left(\frac{1}{\mu}\right)\\ 0 & I\end{pmatrix},$$

$V(\lambda)$ is the diagonal multivalued linear relation matrix

$$V\left(\frac{1}{\mu}\right)=\begin{pmatrix}I-\mu A & 0\\ 0 & I-\mu\overline{M\left(\frac{1}{\mu}\right)}\end{pmatrix}$$

with

$$\mathcal{I}\left(\tfrac{1}{\mu}\right)=\begin{pmatrix}0 & W\left(\frac{1}{\mu}\right)\\ Z\left(\frac{1}{\mu}\right) & Z\left(\frac{1}{\mu}\right)W\left(\frac{1}{\mu}\right)\end{pmatrix}.$$

Proof. Recalling the factorization (2), one has $I-\mu\mathcal{L}=H_1V\left(\frac{1}{\mu}\right)J_1+\mathcal{J}\left(\frac{1}{\mu}\right)$ with

$$\mathcal{J}\left(\frac{1}{\mu}\right)=\begin{pmatrix}A-A & B-B\\ \mathcal{F}(\mu)\left(\frac{1}{\mu}-A\right)+Z\left(\frac{1}{\mu}\right)(A-A) & \mathcal{F}(\mu)B+Z\left(\frac{1}{\mu}\right)(B-B)\end{pmatrix}$$

where $\mathcal{F}(\mu)=F\left(\frac{1}{\mu}\right)-\overline{F\left(\frac{1}{\mu}\right)}$. Then,

$$\begin{aligned}Q_{\mathcal{L}}(I-\mu\mathcal{L}) &= Q_{\mathcal{L}}\left(H_1V\left(\frac{1}{\mu}\right)J_1+\mathcal{J}\left(\frac{1}{\mu}\right)\right)\\ &= Q_{\mathcal{L}}\left(H_1V\left(\frac{1}{\mu}\right)J_1\right)+Q_{\mathcal{L}}\left(\mathcal{J}\left(\frac{1}{\mu}\right)\right).\end{aligned}$$

Since $\mathcal{L}\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}A(0)\\C(0)\end{pmatrix}$, we can write the following relation

$Q_{\mathcal{L}}\left(\mathcal{J}\left(\frac{1}{\mu}\right)\right)=\begin{pmatrix}Q_A(A-A) & Q_A(B-B)\\ Q_C\left(\mathcal{F}(\mu)\left(\frac{1}{\mu}-A\right)+Z\left(\frac{1}{\mu}\right)(A-A)\right) & Q_C\left(\mathcal{F}(\mu)B+Z\left(\frac{1}{\mu}\right)(B-B)\right)\end{pmatrix}$. Let $(x,y)\in\mathcal{D}(\mathcal{L})$, then

$$Q_A(B-B)y=Q_AB(0)\subset Q_AA(0)=0 \tag{3}$$

Moreover, by making some simple calculations,

$$\begin{aligned} Q_C\Big(\mathcal{F}(\mu)\Big(\frac{1}{\mu}-A\Big)+Z\Big(\frac{1}{\mu}\Big)(A-A)\Big)x &= Q_C\Big(\mathcal{F}(\mu)\Big(\frac{1}{\mu}-A\Big)\Big)x \\ &\quad +Q_C\Big(Z\Big(\frac{1}{\mu}\Big)(A-A)\Big)x. \end{aligned}$$

Since $A(0) \subset N\Big(F\Big(\frac{1}{\mu}\Big)\Big)$, then by Proposition 2.4 (*iii*) we get

$$Q_C\Big(\mathcal{F}(\mu)\Big(\frac{1}{\mu}-A\Big)\Big)x = Q_C\Big(F\Big(\frac{1}{\mu}\Big)\Big(\frac{1}{\mu}-A\Big)-F\Big(\frac{1}{\mu}\Big)\Big(\frac{1}{\mu}-A\Big)\Big)x.$$

Let $\frac{1}{\mu} \in \rho(A)$, then $F\Big(\frac{1}{\mu}\Big)\Big(\frac{1}{\mu}-A\Big) = C$. We obtain that

$$Q_C\Big(\mathcal{F}(\mu)\Big(\frac{1}{\mu}-A\Big)\Big)x = Q_C\Big(\Big(F\Big(\frac{1}{\mu}\Big)\Big(\frac{1}{\mu}-A\Big)\Big)\Big(0\Big) = Q_C C\Big(0\Big) = 0. \tag{4}$$

Since $\mathcal{F}(\mu)$ is bounded, then by Proposition 2.4 (*ii*) we obtain

$$\begin{aligned} Q_C\Big(\mathcal{F}(\mu)(A-A)\Big)x &= Q_C\Big(\mathcal{F}(\mu)(\frac{1}{\mu}-A-\frac{1}{\mu}+A)\Big)x \\ &= Q_C\Big(\mathcal{F}(\mu)\Big(\frac{1}{\mu}-A\Big)-\mathcal{F}(\mu)\Big(\frac{1}{\mu}-A\Big)\Big)x \\ &= Q_C\Big(\mathcal{F}(\mu)\Big(\frac{1}{\mu}-A\Big)\Big)(0) = 0. \end{aligned}$$

We can prove that

$$Q_C\Big(Z\Big(\frac{1}{\mu}\Big)(A-A)\Big)x = Q_C\Big(Z\Big(\frac{1}{\mu}\Big)(A-A)\Big)x + Q_C\Big(\mathcal{F}(\mu)(A-A)\Big)x.$$

Since $Z\Big(\frac{1}{\mu}\Big)$ is selection of $F\Big(\frac{1}{\mu}\Big)$ and $A(0) \subset N\Big(F\Big(\frac{1}{\mu}\Big)\Big) \subset N(\mathcal{F}(\mu))$, then by Proposition 2.4 (*iii*) we get

$$Q_C\Big(Z\Big(\frac{1}{\mu}\Big)(A-A)\Big)x = Q_C\Big(F\Big(\frac{1}{\mu}\Big)(A-A)\Big)x = 0. \tag{5}$$

Hence $B(0) \subset N\Big(F\Big(\frac{1}{\mu}\Big)\Big)$, then by Proposition 2.4 (*iii*) we get

$$Q_C\left(\mathcal{F}(\mu)B\right)y = Q_C\Big(F\Big(\frac{1}{\mu}\Big)B\Big)\Big(0\Big).$$

Using the fact that $F\Big(\frac{1}{\mu}\Big)B(0) \subset C(0)$, we conclude that

$$Q_C\left(\mathcal{F}(\mu)B\right)y = 0. \tag{6}$$

Now, notice that $Q_C\left(\mathcal{F}(\mu)(B-B)\right)y = Q_C\left(\mathcal{F}(\mu)B(0)\right)$. Therefore,

$$Q_C\left(\mathcal{F}(\mu)(B-B)\right)y = 0. \tag{7}$$

By using Eq. (7) we have

$$\begin{aligned} Q_C\Big(Z\Big(\frac{1}{\mu}\Big)(B-B)\Big)y &= Q_C\Big(Z\Big(\frac{1}{\mu}\Big)(B-B)\Big)y + Q_C\Big(\mathcal{F}(\mu)(B-B)\Big)y \\ &= Q_C\Big(\Big(Z\Big(\frac{1}{\mu}\Big)+\mathcal{F}(\mu)\Big)(B-B)\Big)y. \end{aligned}$$

Since $Z\left(\frac{1}{\mu}\right)$ is a selection of $F\left(\frac{1}{\mu}\right)$, then

$$Q_C\left(Z\Big(\frac{1}{\mu}\Big)(B-B)\right)y = Q_C\left(F\Big(\frac{1}{\mu}\Big)(B-B)\right)y = Q_C\left(F\Big(\frac{1}{\mu}\Big)B(0)\right).$$

We deduce that

$$Q_C\left(Z\Big(\frac{1}{\mu}\Big)(B-B)\right)y = 0. \tag{8}$$

By applying Eqs. (3) − (6) and (8), we get

$$Q_{\mathcal{L}}\left(\mathcal{J}\Big(\frac{1}{\mu}\Big)\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{9}$$

□

Theorem 5.4. *Let $\mathcal{L}$ be the multivalued matrices linear operator defined in (2) satisfies $(H1)-(H7)$. Suppose that there is $\mu \neq 0$ such*

that $\frac{1}{\mu} \in \rho(A)$ and $B(0) \subset A(0) \subset N\left(F\left(\frac{1}{\mu}\right)\right)$ with $Q_A\left((I-\mu A)W\left(\frac{1}{\mu}\right)\right)$ is

bounded. If the linear relations μA and $\mu\overline{M(\frac{1}{\mu})}$ are demicompact, then the linear relation $\mu\mathcal{L}$ is a demicompact.

Proof. Let $\begin{pmatrix} x_n \\ y_n \end{pmatrix}_n \in \mathcal{D}(\mathcal{L})$ be a bounded sequence such that

$$Q_{\mathcal{L}}\Big(I-\mu\mathcal{L}\Big)\begin{pmatrix} x_n \\ y_n \end{pmatrix} \to \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Recalling the factorization in Theaorem 5.3, one has

$$Q_{\mathcal{L}}\Big(I-\mu\mathcal{L}\Big)\begin{pmatrix} x_n \\ y_n \end{pmatrix} = Q_{\mathcal{L}}\left(H_1V\Big(\frac{1}{\mu}\Big)J_1\right)\begin{pmatrix} x_n \\ y_n \end{pmatrix} \to \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Since H_1 is bounded with $V\left(\frac{1}{\mu}\right)J_1\begin{pmatrix} 0 \\ 0 \end{pmatrix} = V\left(\frac{1}{\mu}\right)\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, then by using Lemma 2.2 we get

$$Q_{\mathcal{L}}\left(H_1V\Big(\frac{1}{\mu}\Big)J_1\right) = Q_{\mathcal{L}}H_1Q^{-1}_{V(\frac{1}{\mu})J_1}Q_{V(\frac{1}{\mu})J_1}V\Big(\frac{1}{\mu}\Big)J_1. \tag{10}$$

We claim that $\left(Q_{\mathcal{L}}H_1Q^{-1}_{V(\frac{1}{\mu})J_1}\right)^{-1}$ is operator. Indeed,

$$\left(Q_{\mathcal{L}}H_1Q^{-1}_{V(\frac{1}{\mu})J_1}\right)^{-1} = Q_{V(\frac{1}{\mu})J_1}H_1^{-1}Q_{\mathcal{L}}^{-1}.$$

Furthermore,

$$Q_{V(\frac{1}{\mu})J_1}H_1^{-1}Q_{\mathcal{L}}^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = Q_{V(\frac{1}{\mu})}H_1^{-1}\mathcal{L}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = Q_{V(\frac{1}{\mu})}\begin{pmatrix} A(0) \\ Z\big(\frac{1}{\mu}\big)A(0)+C(0) \end{pmatrix}.$$

Since,

$$\begin{aligned} Z\Big(\frac{1}{\mu}\Big)A(0)+C(0) &= Z\Big(\frac{1}{\mu}\Big)A(0)+\mathcal{F}(\mu)A(0) \\ &= \Big(Z\Big(\frac{1}{\mu}\Big)+\mathcal{F}(\mu)\Big)A(0) \\ &= \mathcal{F}(\mu)A(0)=C(0). \end{aligned}$$

Therefore, $Q_{V(\frac{1}{\mu})J_1}\begin{pmatrix} A(0) \\ Z\big(\frac{1}{\mu}\big)A(0)+C(0) \end{pmatrix}=\begin{pmatrix} Q_A A(0) \\ Q_C C(0) \end{pmatrix}$. It is obvious that,

$$Q_{V(\frac{1}{\mu})J_1}H_1^{-1}Q_{\mathcal{L}}^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Obviously, $Q_{V(\frac{1}{\mu})J_1}$, H_1^{-1} and $Q_{\mathcal{L}}^{-1}$ be three bounded, then $Q_{V(\frac{1}{\mu})J_1}H_1^{-1}Q_{\mathcal{L}}^{-1}$ is bounded operator. We claim that

$$\Big(Q_{\mathcal{L}}H_1Q^{-1}_{V(\frac{1}{\mu})J_1}\Big)^{-1}\Big(Q_{\mathcal{L}}H_1Q^{-1}_{V(\frac{1}{\mu})J_1}\Big)Q_{V(\frac{1}{\mu})J_1}V\Big(\frac{1}{\mu}\Big)J_1\begin{pmatrix} x_n \\ y_n \end{pmatrix}\to\Big(Q_{\mathcal{L}}H_1Q^{-1}_{V(\frac{1}{\mu})J_1}\Big)^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

It follows that

$$\begin{aligned} &\Big(Q_{\mathcal{L}}H_1Q^{-1}_{V(\frac{1}{\mu})J_1}\Big)^{-1}\Big(Q_{\mathcal{L}}H_1Q^{-1}_{V(\frac{1}{\mu})J_1}\Big)Q_{V(\frac{1}{\mu})J_1}V\Big(\frac{1}{\mu}\Big)J_1\begin{pmatrix} x_n \\ y_n \end{pmatrix} \\ = \;& Q_{V(\frac{1}{\mu})J_1}V\Big(\frac{1}{\mu}\Big)J_1\begin{pmatrix} x_n \\ y_n \end{pmatrix}+\Big(Q_{\mathcal{L}}H_1Q^{-1}_{V(\frac{1}{\mu})J_1}\Big)^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

This implies that

$$\Big(Q_{\mathcal{L}}H_1Q^{-1}_{V(\frac{1}{\mu})J_1}\Big)^{-1}Q_{\mathcal{L}}\Big(I-\mu\mathcal{L}\Big)\begin{pmatrix} x_n \\ y_n \end{pmatrix}=Q_{V(\frac{1}{\mu})J_1}V\Big(\frac{1}{\mu}\Big)J_1\begin{pmatrix} x_n \\ y_n \end{pmatrix}.$$

Moreover, by making some simple calculations, we may show that $V\big(\frac{1}{\mu}\big)\begin{pmatrix} 0 \\ 0 \end{pmatrix}=\begin{pmatrix} A(0) \\ C(0) \end{pmatrix}$. Hence, $V\big(\frac{1}{\mu}\big)J_1=$ $V\big(\frac{1}{\mu}\big)=\begin{pmatrix} I-\mu A & \overline{0} \\ 0 & I-\mu M\big(\frac{1}{\mu}\big) \end{pmatrix}\begin{pmatrix} I & W\big(\frac{1}{\mu}\big) \\ 0 & I \end{pmatrix}$.

By [7, Lemma 3.1], we get $V\big(\frac{1}{\mu}\big)J_1=\begin{pmatrix} I-\mu A & \overline{(I-\mu A)W\big(\frac{1}{\mu}\big)} \\ 0 & I-\mu M\big(\frac{1}{\mu}\big) \end{pmatrix}$. Then, the use of Lemma [9, Lemma 2.3] allows us to conclude that

$$Q_{V(\frac{1}{\mu})}V\Big(\frac{1}{\mu}\Big)J_1=\begin{pmatrix} Q_A(I-\mu A) & Q_A\Big((I-\mu A)W\big(\frac{1}{\mu}\big)\Big) \\ 0 & Q_C\Big(I-\mu\overline{M\big(\frac{1}{\mu}\big)}\Big) \end{pmatrix}.$$

Therefore,

$$\begin{aligned} Q_{V(\frac{1}{\mu})}V\Big(\frac{1}{\mu}\Big)J_1\begin{pmatrix} x_n \\ y_n \end{pmatrix} &= \begin{pmatrix} Q_A(I-\mu A) & Q_A\Big((I-\mu A)W\big(\frac{1}{\mu}\big)\Big) \\ 0 & Q_C\Big(I-\mu\overline{M\big(\frac{1}{\mu}\big)}\Big) \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix} \\ &= \begin{pmatrix} Q_A(I-\mu A)x_n+Q_A\Big((I-\mu A)W\big(\frac{1}{\mu}\big)\Big)y_n \\ Q_C\Big(I-\mu\overline{M\big(\frac{1}{\mu}\big)}\Big)y_n \end{pmatrix}. \end{aligned}$$

We conclude that $Q_C\left(I-\mu\overline{M\left(\frac{1}{\mu}\right)}\right)y_n$ is convergent. This together with the demicompact of $\mu\overline{M\left(\frac{1}{\mu}\right)}$ show that $(y_n)_n$ has a convergent subsequence. Since

$Q_A\left((I-\mu A)W\left(\frac{1}{\mu}\right)\right)$ is bounded linear relation and μA is demicompact, we infer that $(x_n)_n$ has a convergent subsequence, which proves the demicompact of $\mu\mathcal{L}$.

□

Theorem 5.5. *Let $\mathcal{L}_0$ be the multivalued matrices linear operator defined in* (2) *and let $\mathcal{L}$ be its closure. Suppose that for a certain $\mu\in\rho(A)$, there is $\lambda\in\mathbb{C}\backslash\{0\}$*

such that $Q_CZ(\mu)\in\mathcal{K}(X)$ and $B(0)\subset A(0)\subset N\left(F\left(\frac{1}{\mu}\right)\right)$ with $Q_A\left((I-\mu A)W\left(\frac{1}{\mu}\right)\right)$

is bounded. If λA and $\lambda S(\mu)$ are demicompact, then $\lambda\mathcal{L}\in\mathcal{DC}(X\times X)$.

Proof. Take the following bounded sequence $\begin{pmatrix} x_n \\ y_n \end{pmatrix}_n \in\mathcal{D}(\mathcal{L})$ such that

$$Q_{(I-\lambda\mathcal{L})}(I-\lambda\mathcal{L})\begin{pmatrix} x_n \\ y_n \end{pmatrix}\to\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Recalling the factorization in Theorem (5.2), one has

$$\frac{1}{\lambda}I-\mathcal{L} \;=\; HV\left(\frac{1}{\lambda}\right)J-\left(\frac{1}{\lambda}-\mu\right)\mathcal{I}(\mu)+\mathcal{J}(\mu),$$

From Eq. (9), it follows that $Q_{\mathcal{L}}(I-\lambda\mathcal{L})=Q_{\mathcal{L}}(HV(\lambda)J)-(1-\lambda\mu)Q_{\mathcal{L}}\mathcal{I}(\mu)$. Since $Q_{\mathcal{L}}HQ^{-1}_{V(\lambda)}$ is bounded operator inverse and by Eq. (10), we get

$$\begin{aligned}
\left(Q_{\mathcal{L}}HQ^{-1}_{V(\lambda)}\right)^{-1}Q_{\mathcal{L}}\left(I-\mu\mathcal{L}\right) &= Q_{V(\lambda)}V(\lambda)J-(1-\lambda\mu)\left(Q_{\mathcal{L}}HQ^{-1}_{V(\lambda)}\right)^{-1}Q_{\mathcal{L}}\mathcal{I}(\mu)\\
&= Q_{V(\lambda)}V(\lambda)J-(1-\lambda\mu)Q_{V(\lambda)}H^{-1}Q^{-1}_{\mathcal{L}}Q_{\mathcal{L}}\mathcal{I}(\mu)\\
&= Q_{V(\lambda)}V(\lambda)J-(1-\lambda\mu)Q_{V(\lambda)}H^{-1}\left[\mathcal{I}(\mu)+Q^{-1}_{\mathcal{L}}(0)\right]\\
&= Q_{V(\lambda)}V(\lambda)J-(1-\lambda\mu)Q_{V(\lambda)}H^{-1}\mathcal{I}(\mu)+(1-\lambda\mu)Q_{V(\lambda)}H^{-1}\mathcal{L}(0)
\end{aligned}$$

Since, $H^{-1}\mathcal{L}(0)=\begin{pmatrix} A(0) \\ Z(\mu)A(0)+C(0) \end{pmatrix}=\begin{pmatrix} A(0) \\ C(0) \end{pmatrix}=V(\lambda)(0)$, then

$$(1-\lambda\mu)Q_{V(\lambda)}H^{-1}\mathcal{L}(0)=0.$$

Consequently,

$$\left(Q_{\mathcal{L}}HQ^{-1}_{V(\lambda)}\right)^{-1}Q_{\mathcal{L}}\left(I-\mu\mathcal{L}\right)=Q_{V(\lambda)}\left[V(\lambda)J-(1-\lambda\mu)H^{-1}\mathcal{I}(\mu)\right].$$

By using [7, Lemma 3.1], we get

$$H^{-1}\mathcal{I}(\mu)=\begin{pmatrix} 0 & W(\mu) \\ Z(\mu) & 0 \end{pmatrix} \text{ and } V(\lambda)J=\begin{pmatrix} I-\lambda A & (I-\lambda)AW(\mu) \\ 0 & I-\lambda\overline{M(\mu)} \end{pmatrix}.$$

It is easy to conclude that

$$\left(Q_{\mathcal{L}}HQ^{-1}_{V(\lambda)}\right)^{-1}Q_{\mathcal{L}}\left(I-\mu\mathcal{L}\right)=\begin{pmatrix} Q_A(I-\lambda A) & (Q_A(I-\lambda A)-(1-\lambda\mu)Q_A)W(\mu) \\ (1-\lambda\mu)Q_CZ(\mu) & Q_C(I-\lambda\overline{M(\mu)}) \end{pmatrix}.$$

Moreover, by making some simple calculations, we may show that

$$\begin{pmatrix} Q_A(I-\lambda A)x_n + (Q_A(I-\lambda A) - (1-\lambda\mu)Q_A)W(\mu)y_n \\ (1-\lambda\mu)Q_C Z(\mu)x_n + Q_C(I-\lambda\overline{M(\mu)})y_n \end{pmatrix} \longrightarrow Q_{V(\lambda)}H^{-1}Q_{\mathcal{L}}^{-1}\begin{pmatrix} x_n \\ y_n \end{pmatrix}. \tag{11}$$

Since $Q_C Z(\mu) \in \mathcal{K}(X)$ and $(x_n)_n$ is bounded, $(I-\lambda\mu)Q_C Z(\mu)x_n$ has a convergent subsequence. Hence, from the second equation of system (11), we infer that $Q_C(I-\lambda\overline{M(\mu)})y_n$ has a convergent subsequence. Using the demicompactness of $\lambda\overline{M(\mu)}$, we deduce that there exists a convergent subsequence of $(y_n)_n$. Now, since $Q_A(I-\lambda A)W(\mu) - (1-\lambda\mu)Q_A W(\mu)$ is bounded, we conclude from the first equation of system (11) that $(I-\lambda A)x_n$ has a convergent subsequence. This together with the fact that λA is demicompact allows us to conclude that $(x_n)_n$ has a convergent subsequence. Therefore, there exists a subsequence of $\begin{pmatrix} x_n \\ y_n \end{pmatrix}_n$ which converges on $\mathcal{D}(\mathcal{L})$. Thus, $\lambda\mathcal{L}$ is demicompact. □

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