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# Existence of mountain-pass solutions for $p(\cdot)$-biharmonic equations with Rellich-type term

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**Abstract.** This manuscript discusses the existence of nontrivial weak solution for the following nonlinear eigenvalue problem driven by the $p(\cdot)$-biharmonic operator with Rellich-type term

$$\begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda \frac{|u|^{q(x)-2}u}{\delta(x)^{2q(x)}}, & \text{for } x \in \Omega, \\ u = \Delta u \qquad = 0, & \text{for } x \in \partial\Omega. \end{cases}$$

Considering the case

$$1 < \min_{x\in\overline{\Omega}} p(x) \leq \max_{x\in\overline{\Omega}} p(x) < \min_{x\in\overline{\Omega}} q(x) \leq \max_{x\in\overline{\Omega}} q(x) < \min\left\{\frac{N}{2}, \frac{Np(x)}{N-2p(x)}\right\},$$

we extend the corresponding result of the reference [8], for the case

$$1 < \min_{x\in\overline{\Omega}} q(x) \leq \max_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p(x) \leq \max_{x\in\overline{\Omega}} .p(x) < \frac{N}{2}.$$

The proofs of the main results are based on the mountain pass theorem.

## 1. Introduction

In this note we assume that $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain with smooth boundary $\partial\Omega$. We consider the nonlinear eigenvalue problem

$$\begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda \frac{|u|^{q(x)-2}u}{\delta(x)^{2q(x)}}, & \text{for } x \in \Omega, \\ u = \Delta u = 0, & \text{for } x \in \partial\Omega. \end{cases} \tag{1}$$

where $\lambda$ is a positive real number, we denote by $\delta(x) := dist(x, \partial\Omega)$ the distance between a given $x \in \Omega$ and the boundary of $\Omega$, the exponent functions $p, q$ are continuous on $\overline{\Omega}$. The operator $\Delta^2_{p(\cdot)}u := \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is said to be the $p(\cdot)$-biharmonic, and becomes the $p$-biharmonic when $p(x) = p$.

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The study of the nonlinear problems involving the $p(\cdot)$-biharmonic operator occurs in interesting areas such as electrorheological fluids(see[18]), and elastic mechanics (see [20]). Problems with variable exponent growth conditions also appear in the modelling of stationary thermo-rheological viscous flows of non-Newtonian fluids, image processing(see [5]) and the mathematical description of the processes filtration of an ideal barotropic gas through a porous medium(see [1]).

In recent years the Hardy inequality and the related problems for nonlinear elliptic equation have attracted considerable interest. We refer to papers [3, 4, 7, 11, 16], where further bibliographical references can be found.

In the case when $\max_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p(x)$, problem of the form

$$\begin{cases}\Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda \frac{|u|^{q(x)-2}u}{\delta(x)^{2q(x)}} \ \text{ in } \ \Omega,\\ u \in W_0^{2,p(\cdot)}(\Omega),\end{cases} \tag{2}$$

have been studied by authors in [7]. They proved the existence of at least one non-decreasing sequence of positive eigenvalues, using an argument based on the Ljusternik-Schnirelman theory on $C^1$-manifold. Denoting by $\Lambda$ the set of all nonnegative eigenvalues, they showed that $\sup\Lambda = +\infty$ and they proved that $\inf\Lambda = 0$ if and only if the domain $\Omega$ satisfies the following $q(\cdot)$-Hardy inequality :

$$\int_\Omega \frac{|u|^{q(x)}}{\delta(x)^{2q(x)}}dx \leq \frac{1}{H}\int_\Omega |\Delta u|^{p(x)}\,dx,$$

for all $u \in W_0^{2,p(\cdot)}(\Omega)$.

In this work, our goal is to investigate problem (1) under the basic assumption

$$\max_{x\in\overline{\Omega}} p(x) < \min_{x\in\overline{\Omega}} q(x) < \max_{x\in\overline{\Omega}} q(x) < \min\Big\{\frac{N}{2}, \frac{Np(x)}{N-2p(x)}\Big\}.$$

In this new situation we will show that the problem (1) admits a nontrivial weak solution. The proofs of our main results are based on mountain pass theorem.

This paper is divided into four sections. In the second section, we present some necessary preliminaries on variable exponent Lebesgue-Sobolev spaces and we recall some basic properties of $p(\cdot)$-biharmonic operator. In the third section, we give a necessary and sufficient condition for Rellich-type inequality to holds true. In the fourth section, by using mountain pass theorem, we establish existence result of weak solution for problem (1).

## 2. Abstract setting

To study the problem (1), we introduce some theories of the spaces $L^{p(\cdot)}(\Omega)$ and $W^{m,p(\cdot)}(\Omega)$, respectively, which will be used later. For a deeper treatment, we refer the reader to [9, 10, 12] and the references therein. Suppose that $\Omega$, is a bounded domain of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$.

The Lebesgue space with variable exponent is defined by

$$L^{p(\cdot)}(\Omega) := \Big\{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)}dx < \infty\Big\},$$

where $p \in C_1^+(\overline{\Omega})$ and

$$C_1^+(\overline{\Omega}) := \Big\{p \in C(\overline{\Omega}) \Big| p(x) > 1, \text{ for any } x \in \overline{\Omega}\Big\}.$$

Denote

$$p^+ := \max_{x\in\overline{\Omega}} p(x), \quad p^- := \min_{x\in\overline{\Omega}} p(x).$$

One introduces in $L^{p(\cdot)}(\Omega)$ the so-called Luxemburg norm

$$|u|_{L^{p(\cdot)}(\Omega)} := |u|_{p(\cdot)} := \inf\Big\{\alpha > 0\Big| \int_\Omega |\frac{u(x)}{\alpha}|^{p(x)}dx \leq 1\Big\}.$$

**Proposition 2.1.** *(See. [10]) The space $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ is separable, uniformly convex, reflexive and its conjugate space is $L^{q(\cdot)}(\Omega)$ where $q(x)$ is the conjugate function of $p(x)$; i.e.,*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1,$$

*for all $x \in \Omega$. For $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$ we have*

$$\Big|\int_\Omega u(x)v(x)dx\Big| \leq \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u(x)|_{p(\cdot)}|v(x)|_{q(\cdot)}. \tag{3}$$

The mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$, called the $p(\cdot)$-modular of the $L^{p(\cdot)}(\Omega)$ space, is defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}dx.$$

Then we have the following relations connecting this application to the Luxemburg norm:

**Proposition 2.2.** *(See. [10]) For all $u_n, u \in L^{p(\cdot)}(\Omega)$, we have*

1 $|u|_{p(\cdot)} = a \Leftrightarrow \rho_{p(\cdot)}(\frac{u}{a}) = 1$, *for $u \neq 0$ and $a > 0$.*

2 $|u|_{p(\cdot)} > 1$ $(= 1; < 1)$ $\Leftrightarrow \rho_{p(\cdot)}(u) > 1$ $(= 1; < 1)$.

3 $|u|_{p(\cdot)} \to 0$ *(resp.* $\to +\infty$*)* $\Leftrightarrow \rho_{p(\cdot)}(u) \to 0$ *(resp.* $\to +\infty$*).*

4 *The following statements are equivalent to each other:*

*4.i* $\lim_{n\to\infty} |u_n - u|_{p(\cdot)} = 0$,
*4.ii* $\lim_{n\to\infty} \rho_{p(\cdot)}(u_n - u) = 0$,
*4.iii* $u_n \to u$ *in measure in* $\Omega$ *and* $\lim_{n\to\infty} \rho_{p(\cdot)}(u_n) = \rho_{p(\cdot)}(u)$.

The Sobolev space with variable exponent $W^{k,p(\cdot)}(\Omega)$ is defined as

$$W^{k,p(\cdot)}(\Omega) := \Big\{u \in L^{p(\cdot)}(\Omega)\Big| D^\alpha u \in L^{p(\cdot)}(\Omega), |\alpha| \leq k\Big\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_N^{\alpha_N}}u$, with $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$. The space $W^{k,p(\cdot)}(\Omega)$, equipped with the norm

$$||u||_{k,p(\cdot)} := \sum_{|\alpha|\leq k} |D^\alpha u|_{p(\cdot)},$$

also becomes a Banach, separable and reflexive space. For more details, we refer the reader to [6, 9, 10]. We denote by $W_0^{k,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(\cdot)}(\Omega)$.

Note that the weak solutions of problem (1) are considered in the generalized Sobolev space

$$X := W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega).$$

Generally, we know that if $(E, \|.\|_E)$ and $(F, \|.\|_F)$ are Banach spaces, we define the norm on the space $X := E \cap F$ as $\|u\|_X = \|u\|_E + \|u\|_F$ . In our case, we have, for any $u \in X$, $\|u\|_X = \|u\|_{1,p(\cdot)} + \|u\|_{2,p(\cdot)}$, thus $\|u\|_X = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)} + \sum_{|\alpha|=2} |D^\alpha u|_{p(\cdot)}$.

In [19], the equivalence of the norms was proved, and it was even proved that the norm $|\Delta(\cdot)|_{p(\cdot)}$ is equivalent to the norm $\|.\|_X$ (see [19, Theorem 4.4]).

Let us choose on $X$ the norm defined by

$$\|u\| := |\Delta u|_{p(\cdot)}.$$

Note that, $(X, \|\cdot\|)$ is also a separable and reflexive Banach space. Similar to Proposition 2.2, we have the following.

**Proposition 2.3.** *For all $u \in X$, denote $\Lambda_{p(\cdot)}(u) := \int_\Omega |\Delta u(x)|^{p(x)} dx$, then*

*1 For $u \in X$, we have*

*1.i* $\|u\| < 1$ *(*$= 1, > 1$*)* $\Leftrightarrow \Lambda_{p(\cdot)}(u) < 1$ *(*$= 1 > 1$*);*

*1.ii* $\|u\| \geq 1 \Rightarrow \|u\|^{p^-} \leq \Lambda_{p(\cdot)}(u) \leq \|u\|^{p^+}$;

*1.iii* $\|u\| \leq 1 \Rightarrow \|u\|^{p^+} \leq \Lambda_{p(\cdot)}(u) \leq \|u\|^{p^-}$.

*2 If $u, u_n \in X, n = 1, 2, \ldots$, then the following statements are equivalent:*

*2.i* $\lim_{n\to\infty} \|u_n - u\| = 0$;

*2.ii* $\lim_{n\to\infty} \Lambda_{p(\cdot)}(u_n - u) = 0$;

*2.iii* $u_n \to u$ *in measure in* $\Omega$ *and* $\lim_{n\to\infty} \Lambda_{p(\cdot)}(u_n) = \Lambda_{p(\cdot)}(u)$.

Denote for any $x \in \overline{\Omega}$,

$$p_2^*(x) := \begin{cases} \frac{Np(x)}{N-2p(x)} & \text{if } 2p(x) < N, \\ +\infty & \text{if } 2p(x) \geq N. \end{cases}$$

The following result [2, Theorem 3.2], which will be used later, is an embedding result between the spaces $X$ and $L^{q(\cdot)}(\Omega)$.

**Theorem 2.4.** *Let $p, q \in C_1^+(\Omega)$. Assume that $p(x) < \frac{N}{2}$ and $q(x) < p_2^*(x)$. Then there is a continuous and compact embedding $X$ into $L^{q(\cdot)}(\Omega)$.*

As in the case $p(x) \equiv p$ (constant), we consider the $p(\cdot)$-biharmonic operator

$$\Delta^2_{p(\cdot)} : X \to X^*,$$

defined by

$$\langle \Delta^2_{p(\cdot)} u, v \rangle = \int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta v dx, \quad \text{for all } u, v \in X.$$

**Proposition 2.5.** *(See. [8]) The following properties hold:*

*i* $\Delta^2_{p(\cdot)} : X \to X^*$ *is a homeomorphism.*

*ii* $\Delta^2_{p(\cdot)} : X \to X^*$ *is a strictly monotone operator, that is,*

$$\langle \Delta^2_{p(\cdot)} u - \Delta^2_{p(\cdot)} v, u - v \rangle > 0, \quad \textit{for all } u \neq v \in X.$$

*iii* $\Delta^2_{p(\cdot)} : X \to X^*$ *is a mapping of type* $(S_+)$, *that is,*

$$\text{if } u_n \rightharpoonup u \text{ in } X \text{ and } \limsup_{n\to\infty} \langle \Delta^2_{p(\cdot)} u_n, u_n - u\rangle \le 0,$$

*then* $u_n \to u$ *in* $X$.

Throughout this paper in a given Banach space $X$ we denote strong convergence by $\to$ and weak convergence by $\rightharpoonup$, we denote by $X^*$ its dual and by $\langle .,. \rangle$ the duality pairing between $X^*$ and $X$.

## 3. Hardy-type inequality related to $p(\cdot)$-biharmonic operator

Let $1 < p < \frac{N}{2}$, we recall the classical Hardy's inequality, which says that

$$\int_\Omega \frac{|u|^p}{\delta(x)^{2p}}\, dx \le \frac{1}{H} \int_\Omega |\Delta u|^p\, dx, \quad \forall u \in X, \tag{4}$$

where $H := \left[\frac{N(p-1)(N-2p)}{p^2}\right]^p$; see, for instance, the paper [14].

**Theorem 3.1.** *Suppose that the functions* $p, q \in C(\overline{\Omega})$ *satisfy the following bounds*

$$1 < p^- \le p^+ < q^- \le q^+ < \min\left\{\frac{N}{2}, p_2^*(x)\right\}, \quad \text{for all } x \in \overline{\Omega}.$$

*Then there exists two positive constants* $H,\ C > 0$ *such that the* $q(.)$*-Hardy inequality*

$$\int_\Omega \frac{|u(x)|^{q(x)}}{\delta(x)^{2q(x)}} dx \le \frac{1}{H} \int_\Omega |\Delta u(x)|^{p(x)}\, dx + C^{q^+} \|u(x)\|^{q^+}, \tag{5}$$

*holds for all* $u \in X$.

***Proof.*** Let $u \in X$ such that

$$1 < p^- \le p^+ \le q^- \le q^+ < \min\left\{\frac{N}{2}, p_2^*(x)\right\}, \quad \text{for all } x \in \overline{\Omega}.$$

▹ **Case 1.** If $|\triangle u(x)| \le 1$ and $|u(x)| \ge \delta(x)^2$.

Since $|\triangle u(x)| \le 1$, we can write

$$\int_{\Omega_1} |\Delta u(x)|^{p(x)}\, dx \ge \int_{\Omega_1} |\Delta u(x)|^{q^+}\, dx, \tag{6}$$

where $\Omega_1 := \left\{x \in \Omega \middle| |\triangle u(x)| \le 1\right\}$. Using the Hardy inequality (4), we have the following

$$\int_{\Omega_1} |\Delta u(x)|^{q^+}\, dx \ge H_{q^+} \int_{\Omega_1} \frac{|u(x)|^{q^+}}{\delta(x)^{2q^+}}\, dx, \tag{7}$$

where $H_{q^+} := \left[\frac{N(q^+-1)(N-2q^+)}{(q^+)^2}\right]^{q^+}$. Relations (6) and (7) showed that

$$\int_{\Omega_1} |\Delta u(x)|^{p(x)}\, dx \ge H_{q^+} \int_{\Omega_1} \frac{|u(x)|^{q^+}}{\delta(x)^{2q^+}}\, dx.$$

For $|u(x)| \geq \delta(x)^2$, it easy to see that

$$\Big(\frac{|u(x)|}{\delta(x)^2}\Big)^{q^+} \geq \Big(\frac{|u(x)|}{\delta(x)^2}\Big)^{q(x)}.$$

Thus

$$H_{q^+}\int_{\Omega_1\cap\{|u(x)|\geq\delta(x)^2\}}\Big(\frac{|u(x)|}{\delta(x)^2}\Big)^{q^+}dx \geq H_{q^+}\int_{\Omega_1\cap\{|u(x)|\geq\delta(x)^2\}}\Big(\frac{|u(x)|}{\delta(x)^2}\Big)^{q(x)}dx. \tag{8}$$

Consequently, from 8 we obtain

$$\int_{\Omega_1\cap\{|u(x)|\geq\delta(x)^2\}}|\Delta u(x)|^{p(x)}\,dx \geq H_{q^+}\int_{\Omega_1\cap\{|u(x)|\geq\delta(x)^2\}}\frac{|u(x)|^{q(x)}}{\delta(x)^{2q(x)}}\,dx.$$

▹ **Case 2.** If $|\triangle u(x)| \leq 1$ and $|u(x)| < \delta(x)^2$. Then,

$$\int_{\Omega_2}|\Delta u(x)|^{p(x)}\,dx \geq \int_{\Omega_2}|\Delta u(x)|^{q^-}\,dx, \tag{9}$$

where $\Omega_2 := \Big\{x\in\Omega\,\Big|\,|\triangle u(x)|\leq 1\Big\}$. Using again (4), we have the following

$$\int_{\Omega_2}|\Delta u(x)|^{q^-}\,dx \geq H_{q^-}\int_{\Omega_2}\frac{|u(x)|^{q^-}}{\delta(x)^{2q^-}}\,dx, \tag{10}$$

where $H_{q^-} := \Big[\frac{N(q^--1)(N-2q^+)}{(q^-)^2}\Big]^{q^-}$. Using (9) and (3), we deduce that

$$\int_{\Omega_2}|\Delta u(x)|^{p(x)}\,dx \geq H_{q^-}\int_{\Omega_2}\frac{|u(x)|^{q^-}}{\delta(x)^{2q^-}}\,dx.$$

For $|u(x)| < \delta(x)^2$, we get

$$\Big(\frac{|u(x)|}{\delta(x)^2}\Big)^{q^-} > \Big(\frac{|u(x)|}{\delta(x)^2}\Big)^{q(x)}.$$

Thus

$$H_{q^-}\int_{\Omega_2\cap\{|u(x)|<\delta(x)^2\}}\Big(\frac{|u(x)|}{\delta(x)^2}\Big)^{q^-}dx > H_{q^-}\int_{\Omega_2\cap\{|u(x)|<\delta(x)^2\}}\Big(\frac{|u(x)|}{\delta(x)^2}\Big)^{q(x)}dx.$$

Hence we deduce that

$$\int_{\Omega_2\cap\{|u(x)|<\delta(x)^2\}}|\Delta u(x)|^{p(x)}\,dx > H_{q^-}\int_{\Omega_2\cap\{|u(x)|<\delta(x)^2\}}\frac{|u(x)|^{q(x)}}{\delta(x)^{2q(x)}}\,dx.$$

▹ **Case 3.** If $|\triangle u(x)| > 1$ and $|u(x)| \leq \delta(x)^2$. Then,

$$\int_{\Omega_3}|\Delta u(x)|^{p(x)}\,dx \geq \int_{\Omega_3}|\Delta u(x)|^{p^-}\,dx, \tag{11}$$

where $\Omega_3 := \Big\{x\in\Omega\,\Big|\,|\triangle u(x)| > 1\Big\}$. By (4), we deduce that

$$\int_{\Omega_3}|\Delta u(x)|^{p^-}\,dx \geq H_{p^-}\int_{\Omega_3}\frac{|u(x)|^{p^-}}{\delta(x)^{2p^-}}\,dx, \tag{12}$$

where $H_{p^-} := \left[\frac{N(p^- -1)(N-2p^-)}{(p^-)^2}\right]^{p^-}$. Combining (11) and (12), we obtain

$$\int_{\Omega_3} |\Delta u(x)|^{p(x)}\, dx \geq H_{p^-} \int_{\Omega_3} \frac{|u(x)|^{p^-}}{\delta(x)^{2p^-}}\, dx.$$

For $|u(x)| \leq \delta(x)^2$, we obtain

$$\Big(\frac{|u(x)|}{\delta(x)^2}\Big)^{p^-} \geq \Big(\frac{|u(x)|}{\delta(x)^2}\Big)^{q(x)}.$$

Therefore

$$H_{p^-} \int_{\Omega_3 \cap \{|u(x)| \leq \delta(x)^2\}} \Big(\frac{|u(x)|}{\delta(x)^2}\Big)^{p^-} dx \geq H_{p^-} \int_{\Omega_3 \cap \{|u(x)| \leq \delta(x)^2\}} \Big(\frac{|u(x)|}{\delta(x)^2}\Big)^{q(x)} dx.$$

Hence, we conclude that

$$\int_{\Omega_3 \cap \{|u(x)| \leq \delta(x)^2\}} |\Delta u(x)|^{p(x)}\, dx \geq H_{p^-} \int_{\Omega_3 \cap \{|u(x)| \leq \delta(x)^2\}} \frac{|u(x)|^{q(x)}}{\delta(x)^{2q(x)}}\, dx.$$

▹ <u>**Case 4.**</u> If $|\triangle u(x)| > 1$ and $|u(x)| > \delta(x)^2$.

Since $q^+ < p_2^*(x)$, we deduce that $X$ is continuously embedded in $L^{q^+}(\Omega)$, there exist a positive constant $C$ such that

$$\int_{\Omega} |u|^{q^+} dx \leq C^{q^+} \|u(x)\|^{q^+}, \quad \forall u \in X. \tag{13}$$

According to the fact that $|u(x)| > \delta(x)^2$, and for $\delta(x)^2$ large enough such that $\delta(x)^2 > 1$ we deduce that for every $u \in X$,

$$\left(\frac{|u(x)|}{\delta(x)^2}\right)^{q(x)} \leq |u(x)|^{q(x)} \leq |u(x)|^{q^+}, \quad \forall x \in \overline{\Omega_4 \cap \{|u(x)| > \delta(x)^2\}},$$

where $\Omega_4 := \left\{x \in \Omega \middle| |\triangle u(x)| > 1\right\}$. Integrating the above inequality, for all $u \in X$, we have

$$\int_{\Omega_4 \cap \{|u(x)| > \delta(x)^2\}} \left(\frac{|u(x)|}{\delta(x)^2}\right)^{q(x)} dx \leq \int_{\Omega_4 \cap \{|u(x)| > \delta(x)^2\}} |u(x)|^{q^+} dx. \tag{14}$$

Combining (13) and (14), we get

$$\int_{\Omega_4 \cap \{|u(x)| > \delta(x)^2\}} \frac{|u(x)|^{q(x)}}{\delta(x)^{2q(x)}} dx \leq C^{q^+} \|u\|^{q^+},$$

for all $u \in X$.

If we divide a domain $\Omega$ into four sets such that

$$A := \left\{x \in \Omega : |\triangle u(x)| \leq 1\right\} \cap \left\{|u(x)| \geq \delta(x)^2\right\},$$

$$B := \left\{x \in \Omega : |\triangle u(x)| \leq 1\right\} \cap \left\{|u(x)| < \delta(x)^2\right\},$$

$$C := \left\{x \in \Omega : |\triangle u(x)| > 1\right\} \cap \left\{|u(x)| \leq \delta(x)^2\right\},$$

$$D := \Big\{x \in \Omega : |\triangle u(x)| > 1\Big\} \cap \Big\{|u(x)| > \delta(x)^2\Big\}.$$

Then we can write

$$\int_\Omega \frac{|u(x)|^{q(x)}}{\delta(x)^{2q(x)}}dx \leq \int_A \frac{|u(x)|^{q(x)}}{\delta(x)^{2q(x)}}dx + \int_B \frac{|u(x)|^{q(x)}}{\delta(x)^{2q(x)}}dx + \int_C \frac{|u(x)|^{q(x)}}{\delta(x)^{2q(x)}}dx + \int_D \frac{|u(x)|^{q(x)}}{\delta(x)^{2q(x)}}dx.$$

We deduce that there exists two positive constants $H, C > 0$ such that the following $q(.)$-Hardy inequality

$$\int_\Omega \frac{|u|^{q(x)}}{\delta(x)^{2q(x)}}dx \leq \frac{1}{H}\int_\Omega |\Delta u|^{p(x)}\,dx + C^{q^+}\|u\|^{q^+},$$

holds for all $u \in X$, where

$$H := \max\Big(H_{p^-}, H_{q^-}, H_{q^+}\Big),$$

and $C$ is the best constant of the embedding $X$ into $L^{q^+}(\Omega)$.

## 4. Weak solvability of problem (1)

Our main tool is the following celebrated mountain pass theorem (see for example [13, 15, 17]) of Ambrosetti and Rabinowitz.

**Theorem 4.1.** *Let $X$ be a real infinite dimensional Banach space and $\varphi \in C^1(X, \mathbb{R})$ such that $\varphi(0_X) = 0$ and satisfying the (PS)-condition. Suppose that*

- ***i*** *There are constants $\beta, \rho > 0$ such that $\varphi(u) \geq \beta$ for all $u \in \partial B_\rho \cap X$,*
- ***ii*** *There exists $e \in X$ with $\|e\| > \rho$ such that $\varphi(e) < 0$.*

*Then $\varphi$ possesses a critical value $c \geq \beta$, which can be characterized as*

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\varphi(\gamma(t)) > 0,$$

*where*

$$\Gamma := \Big\{\gamma \in C([0,1], X)\Big|\gamma(0) = 0 \text{ and } \varphi(\gamma(1)) < 0\Big\}.$$

.

We seek the solution for problem 1 belonging to the space $X$ in the sense below.

**Definition 4.2.** *A function $u \in X \setminus \{0\}$ is termed a weak solution of problem 1, if and only if*

$$\int_\Omega |\Delta u(x)|^{p(x)-2}\Delta u(x)\Delta v(x)\,dx - \lambda\int_\Omega \frac{|u(x)|^{q(x)-2}u(x)}{\delta(x)^{2q(x)}}v(x)\,dx = 0, \tag{15}$$

*for any $v \in X$.*

The energy functional $\mathcal{A} : X \to \mathbb{R}$ associated to problem (1) is defined as:

$$\mathcal{A}(u) = \int_\Omega \frac{|\Delta u(x)|^{p(x)}}{p(x)}dx - \lambda\int_\Omega \frac{1}{q(x)}\frac{|u(x)|^{q(x)}}{\delta(x)^{2q(x)}}\,dx.$$

Then, $\mathcal{A} \in C^1(X, \mathbb{R})$ and

$$\begin{aligned}\left\langle d\mathcal{A}(u), v\right\rangle = &\int_\Omega |\Delta u(x)|^{p(x)-2}\Delta u(x)\Delta v(x)\,dx\\ &- \lambda \int_\Omega \frac{|u(x)|^{q(x)-2}u(x)}{\delta(x)^{2q(x)}} v(x)\,dx,\end{aligned}$$

for all $u, v \in X$. Thus the weak solution of problem (1) are exactly the critical points of $\mathcal{A}$.

Now, we establish the main abstract result of this paper.

**Theorem 4.3.** *Let $p, q \in C_1^+(\overline{\Omega})$. Assume that*

$$p^+ < q^- \leq q^+ < \min\left\{\frac{N}{2}, p_2^*(x)\right\} \quad \textit{for all } x \in \overline{\Omega}.$$

*Then for any $\lambda \in (0, H)$, problem* (1) *possesses a nontrivial weak solution.*

Recall now the definition of the Palais-Smale compactness condition.

**Definition 4.4.** *A Gâteaux differentiable function I satisfies the Palais-Smale condition (in short (PS)-condition) if any sequence $\{u_n\}$ such that*

*i* $\{I(u_n)\}$ *is bounded,*

*ii* $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$*, where $X^*$ denote the dual space of X,*

*has a convergent subsequence.*

Next, we investigate the compactness conditions for the functional $\mathcal{A}$.

**Lemma 4.5.** *The functional $\mathcal{A}$ satisfies the condition* (*PS*)*.*

***Proof*.** Let $(u_n)_{n\geq 1} \subset X$ be a sequence such that

$$c := \sup_n \mathcal{A}(u_n) < \infty \textit{ and } d\mathcal{A}(u_n) \to 0 \text{ in } X^*. \tag{16}$$

Assume by contradiction the contrary. Then

$$\|u_n\| \to +\infty \text{ as } n \to \infty \quad \text{and} \quad \|u_n\| > 1 \quad \text{for any } n.$$

Thus,

$$\begin{aligned}&c + \|u_n\|\\ &\geq \mathcal{A}(u_n) - \frac{1}{q^-}\langle d\mathcal{A}(u_n), u_n\rangle\\ &= \int_\Omega \frac{1}{p(x)}\big|\Delta u_n|^{p(x)}dx - \frac{1}{q^-}\int_\Omega |\Delta u_n|^{p(x)}dx + (\frac{\lambda}{q^-} - \frac{\lambda}{q^-})\int_\Omega \frac{|u_n|^{q(x)}}{\delta(x)^{2q(x)}}\,dx\\ &\geq (\frac{1}{p^+} - \frac{1}{q^-})\int_\Omega |\Delta u_n|^{p(x)}dx\\ &\geq (\frac{1}{p^+} - \frac{1}{q^-})\|u_n\|^{p^-}.\end{aligned}$$

This contradicts the fact that $p^- > 1$. So, the sequence $(u_n)$ is bounded in $X$. Hence, we may extract a subsequence $(u_n) \subset X$ and $u \subset X$ such that

$$u_n \rightharpoonup u \ \ in \ \ X.$$

Since $q^+ < p_2^*(x)$, it follows from theorem 2.4 that $u_n \to u$ in $L^{q(\cdot)}(\Omega)$. taking into account 16, we have $\langle d\mathcal{A}(u_n), u_n - u\rangle \to 0$. Thus

$$\Big\langle d\mathcal{A}(u_n), u_n - u\Big\rangle = \int_\Omega |\Delta u_n(x)|^{p(x)-2}\Delta u_n(x)(\Delta u_n(x) - \Delta u(x))\,dx - \lambda \int_\Omega \frac{|u_n(x)|^{q(x)-2}u_n(x)}{\delta(x)^{2q(x)}}(u_n(x) - u(x))\,dx \to 0.$$

By Hölder's inequality 3, it follows that

$$\Big|\int_\Omega \frac{|u_n(x)|^{q(x)-2}u_n(x)}{\delta(x)^{2q(x)}}(u_n(x) - u(x))\,dx\Big| \leq 2\Big|\frac{|u_n(x)|^{q(x)-1}}{\delta(x)^{2q(x)}}\Big|_{\frac{q(\cdot)}{q(\cdot)-1}}\Big|u_n(x) - u(x)\Big|_{q(\cdot)}.$$

Because $(u_n)$ converges strongly to $u$ in $L^{q(\cdot)}(\Omega)$, that is, $|u_n - u|_{q(\cdot)} \to 0$ as $n \to \infty$, we get

$$\int_\Omega \frac{|u_n(x)|^{q(x)-2}u_n(x)}{\delta(x)^{2q(x)}}(u_n(x) - u(x))\,dx \to 0.$$

Hence,

$$\int_\Omega |\Delta u_n(x)|^{p(x)-2}\Delta u_n(x)(\Delta u_n(x) - \Delta u(x))\,dx \to 0.$$

Eventually, by 2.5, we have $u_n \to u$ in $X$. The proof is complete.

Now, we prove that the functional $\mathcal{A}$ has the geometric features required by the mountain pass theorem.

**Lemma 4.6.** *There exist $\rho, \beta > 0$ such that $\mathcal{A}(u) \geq \beta$, for $u \in X$ with $\|u\| = \rho$.*

***Proof.*** Let $\rho \in (0, 1)$ and $u \in X$ be such that $\|u\| = \rho$. By considering Theorem 3.1, we deduce that

$$\begin{aligned}
\mathcal{A}(u) &= \int_\Omega \frac{1}{p(x)}|\Delta u(x)|^{p(x)}dx - \lambda\int_\Omega \frac{1}{q(x)}\frac{|u(x)|^{q(x)}}{\delta(x)^{2q(x)}}dx,\\
&\geq \frac{1}{p^+}\int_\Omega |\Delta u(x)|^{p(x)}dx - \frac{\lambda}{q^-}\int_\Omega \frac{|u(x)|^{q(x)}}{\delta(x)^{2q(x)}},\\
&\geq \frac{1}{p^+}\int_\Omega |\Delta u(x)|^{p(x)}dx - \frac{\lambda}{Hq^-}\int_\Omega |\Delta u(x)|^{p(x)}dx - \frac{\lambda C^{q^+}}{q^-}\|u(x)\|^{q^+},\\
&\geq \left(\frac{1}{p^+} - \frac{\lambda}{Hq^-}\right)\int_\Omega |\Delta u(x)|^{p(x)}dx - \frac{\lambda C^{q^+}}{q^-}\|u(x)\|^{q^+},\\
&\geq \left(\frac{Hq^- - \lambda p^+}{p^+Hq^-}\right)\|u(x)\|^{p^+} - \frac{\lambda C^{q^+}}{q^-}\|u(x)\|^{q^+},\\
&\geq \left[\left(\frac{Hq^- - \lambda p^+}{p^+Hq^-}\right) - \frac{\lambda C^{q^+}}{q^-}\|u(x)\|^{q^+-p^+}\right]\|u(x)\|^{p^+}.
\end{aligned}$$

Since $p^+ < q^- \leq q^+$, then for any $\lambda \in (0, H)$

$$\lambda^* = \left(\frac{Hq^- - \lambda p^+}{p^+Hq^-}\right) - \frac{\lambda C^{q^+}}{q^-}\rho^{q^+-p^+},$$

is positive on neighborhood of the origin. Then, there exists $\beta = \lambda^*\rho^{p^+} > 0$ such that for any $u \in X$ with $\|u\| = \rho$ we have $\mathcal{A}(u) \geq \beta > 0$. So, the result of lemma 4.6 follows.

**Lemma 4.7.** *There exists $e \in X$ with $\|e\| \geq \rho$ such that $\mathcal{A}(e) < 0$, where $\rho$ is given in lemma 4.6.*

***Proof.*** Let $\Psi \in C_0^\infty(\Omega)$, $\Psi \geq 0$ and $\Psi \neq 0$. For $t > 1$, we have

$$\mathcal{A}(t\Psi) \leq \frac{t^{p^+}}{p^-}\int_\Omega \left|\Delta\Psi(x)\right|^{p(x)}dx - \frac{\lambda t^{q^-}}{q^+}\int_\Omega \frac{|\Psi(x)|^{q(x)}}{\delta(x)^{2q(x)}}dx.$$

Then, since $p^+ < q^-$, we obtain

$$\lim_{t\to\infty}\mathcal{A}(t\Psi) = -\infty.$$

Therefore, for $t > 1$ large enough, we can take $e = t\Psi$ such that $\|e\| \geq \rho$ and $\mathcal{A}(e) < 0$. This completes the proof.

***Proof of Theorem 4.3.*** From Lemmas 4.6 and 4.7, we deduce

$$\max(\mathcal{A}(0), \mathcal{A}(e)) = \mathcal{A}(0) < \inf_{\|u\|=\rho}\mathcal{A}(u) =: \beta.$$

By lemma 4.5 and the Mountain pass Theorem, we deduce the existence of critical points $u$ of $\mathcal{A}$ associated of the critical value given by

$$c := \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]}\mathcal{A}(\gamma(t)) \geq \beta, \tag{17}$$

where $\Gamma := \left\{\gamma \in C([0,1], X)\middle|\gamma(0) = 0 \text{ and } \gamma(1) = e\right\}$. This completes the proof.

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