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Impulsive fractional differential equations with state-dependent delay involving the Caputo-Hadamard derivative

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Abstract. In this paper, we investigate the existence of solutions for a class of initial value problems for impulsive Caputo-Hadamard fractional differential equations with state-dependent delay.

1. Introduction

This paper is concerned with the existence of solutions for the initial value problems, for the impulsive differential equations of the form

$$^{CH}D^r y(t) = f(t, y_{\rho(t,y_t)}), \text{ for a.e. } t \in J = [a, T],\ a > 0,\ t \neq t_k,\ k = 1, ..., m, 0 < r \leq 1, \tag{1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)),\ k = 1, ..., m, \tag{2}$$

$$y(t) = \varphi(t),\ t \in (-\infty, a], \tag{3}$$

where $^{CH}D^r$ is the Caputo-Hadamard fractional derivative , $f : J \times B \to \mathbb{R}$, $\varphi \in B$ are given functions, $I_k : \mathbb{R} \to \mathbb{R}$, $k = 1, ..., m$, are continuous functions, $a = t_0 < t_1 < ... < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim\limits_{\varepsilon \to 0^+} y(t_k + \varepsilon)$ and $y(t_k^-) = \lim\limits_{\varepsilon \to 0^-} y(t_k + \varepsilon)$ represent the right and left hand limits of $y(t)$ at $t = t_k$, $k = 1, ..., m$, and B is an abstract phase space to be specified later.

For any function y and any $t \in [a, T]$, we denote by y_t the element of B defined by $y_t(\theta) = y(t + \theta)$ for $\theta \in (-\infty, a]$. We assume that the histories y_t belong to B.

Fractional differential equations and fractional integral equations are valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, there are numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetism, etc. In the monographs [2, 5, 16, 22, 26–28], we can find background mathematics and various applications of fractional calculus. Recently, much research has been devoted to different fractional problems involving the Caputo and Hadamard derivatives; see, for example, the papers [3, 4, 6, 11].

Differential delay equations, or functional differential equations with or without impulse, have been used in modeling scientific phenomena for many years. Often, it has been assumed that the delay is either

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a fixed constant or is given as an integral in which case it is called a distributed delay; see for instance the books [10, 15, 23–25, 30, 33], and the papers [12, 14].

The concept of the phase space B plays an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a seminormed space satisfying suitable axioms, which was introduced by Hale and Kato [14] (see also Kappel and Schappacher [21] and Schumacher [31]). For a detailed discussion on this topic we refer the reader to the book by Hino et al. [11].

However, complicated situations in which the delay depends on the unknown functions have been proposed in modeling in recent years (see for instance [29, 32] and the references therein). These equations are frequently called equations with state-dependent delay. Existence results, among other things, were derived recently for functional differential equations when the solution depends on the delay on a bounded interval for impulsive problems. We refer the reader to the papers by Abada et al.[1], Ait Dads and Ezzinbi [7], Anguraj et al. [8], and Hernandez et al. [18, 19]. In [13], the authors considered a class of semilinear functional fractional order differential equations with state-dependent delay. in [9], the authors considered an initial problem for impulsive Caputo functional fractional order differential equations with state-dependent delay. As far as we know, no papers exist in the literature related to Caputo-Hadamard fractional order functional differential equations with state-dependet delay and impulses. The aim of this paper is to initiate that study.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper.

Let $C([a,T],\mathbb{R})$ be the Banach space of all continuous functions from $[a,T]$ into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : a \leq t \leq T\},$$

Let $AC([a,T],R)$ be the space of functions $y : [a,T] \to \mathbb{R}$, which are absolutely continuous, and $AC^n_\delta([a,T],\mathbb{R}) = \{h : [a,T] \to \mathbb{R} : \delta^{n-1}h(t) \in AC([a,T],\mathbb{R})\}$ where $\delta := t\dfrac{d}{dt}$.

Definition 2.1. *([22]). The Hadamard fractional integral of order $r > 0$, for a function $h : [a,b] \to \mathbb{R}$, where $a, b \geq 0$, is defined by*

$$I^r_a h(t) := \frac{1}{\Gamma(r)} \int_a^t \left(\log \frac{t}{s}\right)^{r-1} \frac{h(s)}{s} ds,$$

provided the integral exists.

Definition 2.2. *([20]). Let $AC^n_\delta[a,b] = \{g : [a,b] \to \mathbb{C}, \delta^{n-1}g \in AC[a,b]\}$, $0 < a < b < \infty$, and let $r \in \mathbb{C}$ such that $Re(r) \geq 0$. For a function $g \in AC^n_\delta[a,b]$, the Caputo-Hadamard derivative of fractional order r is defined as follows:*

(i) *If $r \notin \mathbb{N}$ and $n = [Re(r)] + 1$, then*

$$({}^{CH}D^r_a g)(t) = \frac{1}{\Gamma(n-r)} \left(t\frac{d}{dt}\right)^n \int_a^t \left(\log \frac{t}{s}\right)^{n-r-1} \delta^n g(s) \frac{ds}{s},$$

where $[Re(r)]$ denotes the integer part of the real number $Re(r)$ and $\log(.) = \log_e(.)$.

(ii) *If $r = n \in \mathbb{N}$, then $({}^{CH}D^r_a g)(t) = \delta^n g(t)$.*

Lemma 2.3. *([20]) Let $y \in AC^n_\delta[a,b]$ or $C^n_\delta[a,b]$ and $r \in \mathbb{C}$. Then*

$$I^r_a({}^{CH}D^r_a y)(t) = y(t) - \sum_{k=0}^{n-1} \frac{\delta^k y(a)}{k!} \left(\log \frac{t}{a}\right)^k. \tag{4}$$

In this paper, we will employ an axiomatic definition of the phase B introduced by Hale and Kato in [14] and follow the terminology used in [17], but we will add some transformations. Thus, $(B, \|.\|_B)$ will be a seminormed linear space of functions mapping $(-\infty, a]$ *into* $\mathbb{R}$. The first two axioms on B are motivated by the fact that we want a solution of the problem (1) − (3) to be continuous on $(t_k, t_{k+1}]$ and the left hand limit exists for every t_k. We will assume that B satisfies the following axioms:

- (A_1) If $y : (-\infty, T] \to \mathbb{R}$, $T > 0$, $y_0 \in B$ and $y(t_k^-)$ *and* $y(t_k^+)$, $k = 1, ..., m$ exist with $y(t_k^-) = y(t_k)$, $k = 1, ..., m$, then for every $t \in [a, T] \setminus \{t_1, ..., t_m\}$ the following conditions hold :
  - (i) $y_t \in B$; and y_t is continuous on $[a, T] \setminus \{t_1, ..., t_m\}$,
  - (ii) There exists a positive constant H such that $|y(t)| \leq H\|y_t\|_B$,
  - (iii) There exist two functions $K(.), M(.) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of y, with K continuous and M locally bounded such that:
    $\|y_t\|_B \leq K(t) \sup\{|y(s)|;\ a \leq s \leq t\} + M(t)\|y_0\|_B$.
- (A_2)The space B is complete.
  Denote
  $$K_T = \sup\{K(t) :\ a \leq t \leq T\} \text{ and } M_T = \sup\{M(t) :\ a \leq t \leq T\}.$$

Consider the following space

$$AC'(J, \mathbb{R}) = \left\{ \begin{array}{l} y : J \to \mathbb{R},\ y \in AC_\delta((t_k, t_{k+1}], \mathbb{R}),\ k = 1, \ldots, m, \text{ and there exist} \\ y(t_k^+) \text{ and } y(t_k^-),\ k = 1, \ldots, m, \text{ with, } y(t_k^-) = y(t_k) \end{array} \right\}.$$

This space is a Banach space with the norm

$$\|y\|_{AC'} = \sup\{\|y(t)\|_{\mathbb{R}} : a \leq t \leq T\}.$$

Set

$$B_T = \{y : (-\infty, T] \to \mathbb{R} \backslash y \in AC'(J, \mathbb{R}) \cap B\},$$

and

$$J' := J \setminus \{t_1, ..., t_m\}.$$

Definition 2.4. *A function $y \in B_T$ whose $\exists r^{th}$ derivative on J' is said to be a solution of (1)-(3) if y satisfies (1)-(3).*

As a consequence of Lemma 2.3, the next result will be useful in what follows.

Lemma 2.5. *Let $0 < r \leq 1$ and let $\rho \in AC(J, \mathbb{R})$. A function y is a solution of the fractional integral equation*

$$y(t) = \begin{cases} y_a + \dfrac{1}{\Gamma(r)} \displaystyle\int_a^t \left(\log \frac{t}{s}\right)^{r-1} \rho(s) \frac{ds}{s}, & \text{if } t \in [a, t_1] \\ y_a + \dfrac{1}{\Gamma(r)} \displaystyle\sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{r-1} \rho(s) \frac{ds}{s} & \\ +\dfrac{1}{\Gamma(r)} \displaystyle\int_{t_k}^t \left(\log \frac{t}{s}\right)^{r-1} \rho(s) \frac{ds}{s} & \\ + \displaystyle\sum_{k=1}^{m} I_k(y(t_k^-)), \ \text{if } t \in (t_k, t_{k+1}],\ k = 1, ..., m, & \end{cases} \tag{5}$$

if and only if y is a solution of the fractional IVP

$${}^{CH}D_a^r y(t) = \rho(t), \ \text{for each}, \ \ t \in J', \tag{6}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, ..., m, \tag{7}$$

$$y(a) = y_a. \tag{8}$$

We will need to introduce the following hypotheses

- (H_φ) The function $t \to \varphi_t$ is continuous from $\hbar(\rho^-) = \{\rho(s,\varphi) : (s,\varphi) \in J \times B, \rho(s,\varphi) \leq 0\}$ into β and there exists a continuous and bounded function $L^\varphi : \hbar(\rho^-) \to (0,\infty)$ such that $\|\varphi_t\|_\beta \leq L^\varphi(t)\|\varphi\|_\beta$ *for every* $t \in \hbar(\rho^-)$.
- (H_1) The function $f : J \times \beta \to \mathbb{R}$ is continuous.
- (H_2) There exists $p \in C(J, \mathbb{R}^+)$ and $\psi : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that

$$|f(t,u)| \leq p(t)\psi(|u|), \ \ for \ each \ t \in J, \ \ u \in B, \ \ with \ \|I^r P\|_\infty < \infty.$$

- (H_3) There exist numbers $M_1, M_2, ..., M_m > 0$ such that

$$\frac{M_k}{\dfrac{\psi(M_k)(\log T)^r \|p\|_\infty}{\Gamma(r+1)}} > 1, \ k = 1, ..., m. \tag{9}$$

The next result is a consequence of the phase space axioms.

Lemma 2.6. . *If* $y : (-\infty, T] \to \mathbb{R}$ *is a function such that* $y_0 = \varphi$ *and* $y|_J \in AC'(J, \mathbb{R})$, *then*

$$\|y_s\|_\beta \leq (M_T + L^\varphi)\|\varphi\|_\beta + K_T \sup\{\|y_\theta\|; \theta \in [0, \max\{0, s\}]\}, s \in \hbar(\rho^-) \cup J,$$

where

$$L^\varphi = \sup_{t \in \hbar(\rho^-)} L^\varphi(t).$$

Theorem 2.7. *Assume that the hypotheses* (H_φ) *and* $(H_1) - (H_3)$ *hold. Then the problem (1)-(3) has at least one solution on* $(-\infty, T]$.

Proof. The proof will be given in several steps.

Step 1: Consider the following problem

$$^{CH}D^r y(t) = f(t, y_{\rho(t,y_t)}), \ \text{ for a.e. } \ t \in J = [a, t_1], \tag{10}$$

$$y(t) = \varphi(t), t \in (-\infty, a]. \tag{11}$$

Define the operator $N : B_{t_1} \to B_{t_1}$ by

$$Ny(t) = \begin{cases} \varphi(t), & \text{if } t \in (-\infty, a], \\ \varphi(a) + \frac{1}{\Gamma(r)} \displaystyle\int_a^t \left(\log \frac{t}{s}\right)^{r-1} f(s, y_{\rho(s,y_s)}) \frac{ds}{s}, & \text{if } t \in [a, t_1]. \end{cases} \tag{12}$$

Let $x(.) : (-\infty, t_1] \to \mathbb{R}$ be the function defined by

$$x(t) = \begin{cases} \varphi(t), & \text{if } t \in (-\infty, a], \\ \varphi(a), & \text{if } t \in [a, t_1]. \end{cases}$$

Then $x_0 = \varphi$. For each $z \in B_{t_1}$, with $z_0 = 0$, we denote by v the function defined by

$$v(t) = \begin{cases} 0, & \text{if } t \in (-\infty, a], \\ z(t), & \text{if } t \in [a, t_1]. \end{cases}$$

If $y(.)$ satisfies the integral equation

$$y(t) = \varphi(a) + \frac{1}{\Gamma(r)} \int_a^t \left(\log \frac{t}{s}\right)^{r-1} f(s, y_{\rho(s,y_s)}) \frac{ds}{s},$$

we can decompose $y(.)$ into $y(t) = v(t) + x(t)$, $t \in [a, t_1]$, which implies $y_t = v_t + x_t$, for every $t \in [a, t_1]$, and the function $z(.)$ satisfies

$$z(t) = \frac{1}{\Gamma(r)} \int_a^t \left(\log \frac{t}{s}\right)^{r-1} f(s, v_{\rho(s,v_s+x_s)} + x_{\rho(s,v_s+x_s)}) \frac{ds}{s}.$$

Set

$$C_0 = \{z \in B_{t_1} \mid z_0 = 0\}.$$

Let $\|.\|_0$ be the norm in C_0 defined by

$$\|z\|_0 = \|z_0\|_B + sup\{|z(s)| : s \in [a, t_1]\} = \sup\{|z(s)| : s \in [a, t_1]\}.$$

We define an operator $P : C_0 \to C_0$ by

$$P(z)(t) = \frac{1}{\Gamma(r)} \int_a^t \left(\log \frac{t}{s}\right)^{r-1} f(s, v_{\rho(s,v_s+x_s)} + x_{\rho(s,v_s+x_s)}) \frac{ds}{s}.$$

Obviously that the operator N has a fixed point is equivalent to P has a fixed point. So we need to prove that P has a fixed point. We shall use the Leray-Schauder alternative.

Claim 1: *P is continuous.*
Let $\{z_n\}$ be a sequence such that $z_n \to z$ in C_0. Then

$$\begin{array}{rcl} |P(z_n)(t) - P(z)(t)| & \leq & \frac{1}{\Gamma(r)} \int_a^t \left(\log \frac{t}{s}\right)^{r-1} |f(s, v_{n\rho(s,v_{ns}+x_s)} + x_{\rho(s,v_{ns}+x_s)}) \\ & - & f(s, v_{\rho(s,v_s+x_s)} + x_{\rho(s,v_s+x_s)})| \frac{ds}{s}. \end{array}$$

Since f is a continuous functions, we have

$$\|P(z_n) - P(z)\|_0 \to 0 \;\; as \;\; n \to \infty.$$

Claim 2: *P maps bounded sets into bounded sets in C_0.*

Indeed, it is enough to show that for any $\eta > 0$, there exists a positive constant l such that for each $z \in B_\eta = \{z \in C_0 : \|z\|_0 \leq \eta\}$ by (H_2) we have for each $t \in [a, t_1]$.

$$\begin{array}{rcl} |P(z)(t)| & \leq & \frac{1}{\Gamma(r)} \int_a^t \left(\log \frac{t}{s}\right)^{r-1} |f(s, v_{\rho(s,v_s+x_s)} + x_{\rho(s,v_s+x_s)})| \frac{ds}{s} \\ & \leq & \frac{1}{\Gamma(r)} \int_a^t \left(\log \frac{t}{s}\right)^{r-1} [p(s) + q(s) \|v_{\rho(s,v_s+x_s)} + x_{\rho(s,v_s+x_s)})\|_B] \frac{ds}{s} \\ & \leq & \frac{1}{\Gamma(r)} \int_a^t \left(\log \frac{t}{s}\right)^{r-1} p(s) \frac{ds}{s} \\ & & + \frac{1}{\Gamma(r)} \int_a^t \left(\log \frac{t}{s}\right)^{r-1} q(s) \|v_{\rho(s,v_s+x_s)} + x_{\rho(s,v_s+x_s)})\|_B \frac{ds}{s} \\ & \leq & \frac{(\log T)^r}{\Gamma(r+1)} \|p\|_\infty + \frac{1}{\Gamma(r)} \int_a^t \left(\log \frac{t}{s}\right)^{r-1} q(s) \left(K_T \eta + K_T |\varphi(a)| + M_T \|\varphi\|_B\right) \frac{ds}{s} \\ & \leq & \frac{(\log T)^r}{\Gamma(r+1)} \|p\|_\infty + \frac{(\log T)^r}{\Gamma(r+1)} \|q\|_\infty (K_T \eta + K_T |\varphi(a)| + M_T \|\varphi\|_B) := l. \end{array}$$

Claim 3: *P maps bounded sets into equicontinuous sets of C_0.*
Let $d_1, d_2 \in [a, t_1]$, $d_1 < d_2$, let B_η a bounded set of C_0 as in *Claim* 2, and let $z \in B_\eta$. Then,

$$\begin{aligned}
|P(z)(d_2)-P(z)(d_1)| &\leq \tfrac{1}{\Gamma(r)}\int_{d_1}^{d_2}\left(\log\frac{d_2}{s}\right)^{r-1}|f(s,v_{\rho(s,v_s+x_s)}+x_{\rho(s,v_s+x_s)})|\frac{ds}{s}\\
&\quad+\tfrac{1}{\Gamma(r)}\int_0^{d_1}\left[(\log\frac{d_2}{s})^{r-1}-(\log\frac{d_1}{s})^{r-1}\right]\\
&\quad\times|f(s,v_{\rho(s,v_s+x_s)}+x_{\rho(s,v_s+x_s)})|\tfrac{ds}{s}\\
&\leq \tfrac{\|p\|_\infty}{\Gamma(r)}|\int_{d_1}^{d_2}\left(\log\frac{d_2}{s}\right)^{r-1}\frac{ds}{s}+\int_0^{d_1}\left(\log\frac{d_2}{s}\right)^{r-1}\frac{ds}{s}|\\
&\quad+\tfrac{\|q\|_\infty(K_T\eta+K_T|\varphi(a)|+M_T\|\varphi\|_B)}{\Gamma(r)}|\int_{d_1}^{d_2}\left(\log\frac{d_2}{s}\right)^{r-1}\frac{ds}{s}\\
&\quad+\int_0^{d_1}\left(\log\frac{d_2}{s}\right)^{r-1}\frac{ds}{s}|\\
&\leq \tfrac{2\|p\|_\infty}{\Gamma(r+1)}(\log(d_2)-\log(d_1))^r\\
&\quad+\tfrac{2\|q\|_\infty(\log(d_2)-\log(d_1))^r(K_T\eta+K_T|\varphi(a)|+M_T\|\varphi\|_B)}{\Gamma(r+1)}.
\end{aligned}$$

As $d_1 \longrightarrow d_2$, the right-hand side of the above inequality tends to zero. As a consequence of Claims 1 to 3, together with the Arzela-Ascoli theorem, we can conclude that P in continuous and completely continuous.

Claim 4: *A priori bounds.*

Let z be a possible solution of the equation $z = \lambda P(z)$ for some $\lambda \in (0, 1)$. Then for each $t \in [a, t_1]$, we have

$$P(z)(t) = \frac{1}{\Gamma(r)}\int_a^t\left(\log\frac{t}{s}\right)^{r-1} f(s,v_{\rho(s,v_s+x_s)}+x_{\rho(s,v_s+x_s)})\frac{ds}{s}.$$

This implies by (H_3)

$$\begin{aligned}
|z(t)| &\leq \tfrac{1}{\Gamma(r)}\int_a^t\left(\log\frac{t}{s}\right)^{r-1}|f(s,v_{\rho(s,v_s+x_s)}+x_{\rho(s,v_s+x_s)})|\frac{ds}{s}\\
&\leq \tfrac{1}{\Gamma(r)}\int_a^t\left(\log\frac{t}{s}\right)^{r-1}p(s)\psi(\|v_{\rho(s,v_s+x_s)}+x_{\rho(s,v_s+x_s)})\|)\frac{ds}{s}\\
&\leq \tfrac{(\log T)^{r-1}\|p\|_\infty}{\Gamma(r+1)}\psi(\|v_{\rho(s,v_s+x_s)}+x_{\rho(s,v_s+x_s)})\|)\tfrac{ds}{s}\\
&\leq \tfrac{(\log T)^{r-1}\|p\|_\infty}{\Gamma(r+1)}\psi(w).
\end{aligned}$$

But

$$\begin{aligned}
\|v_{\rho(s,v_s+x_s)}+x_{\rho(s,v_s+x_s)}\|_B &\leq \|v_{\rho(s,v_s+x_s)}\|_B+\|x_{\rho(s,v_s+x_s)}\|_B\\
&\leq K_T\sup\{|z(s)|: a\leq s\leq t\}+M_T\|z_0\|_B\\
&\quad+K_T\sup\{|x(s)|: a\leq s\leq t\}+M_T\|x_0\|_B\\
&\leq K_T\sup\{|z(s)|: a\leq s\leq t\}\\
&\quad+M_T\|\varphi\|_B+K_T\|\varphi(a)\| := w(t).
\end{aligned}$$

Thus

$$\frac{\|z\|_\infty}{\dfrac{\psi(w)(\log T)^r\|p\|_\infty}{\Gamma(r+1)}}\leq 1.$$

Then by condition (9), there exists $M_1 > 0$ such that $\|z\|_\infty \neq M_1$.. Let $U_1 = \{z \in C_0 : \|z\|_\infty < M_1\}$.. The operator $P : \bar{U}_1 \to C_0$ is completely continuous. From the choice of U_1, there is no $z \in \partial U_1$ such that $z = \lambda P(z)$, $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Leray-Shauder, we deduce that P has a fixed point $z \in \bar{U}_1$ which is a solution of the problem (1)-(3).

Step 2: Consider the following problem

$$^{CH}D^r y(t) = f(t, y_{\rho(t,y_t)}), \text{ for a.e., } t \in J = [t_1, t_2], \tag{13}$$

$$y(t_1^+) - y(t_1^-) = I_1(y_a(t_1^-)), \tag{14}$$

$$y(t) = y_a(t), t \in (-\infty, t_1]. \tag{15}$$

Let $C_1 = \{y \in B_{t_2} : y(t_1^+) \text{ exists}\}$,
and define the operator $N_1 : C_1 \to C_1$ by

$$N_1 y(t) = \begin{cases} y_a(t), & \text{if } t \in (-\infty, t_1], \\ y_a(t_1^-) + I_a(y_a(t_1^-)) + \frac{1}{\Gamma(r)} \int_{t_1}^{t} \left(\log \frac{t}{s}\right)^{r-1} f(s, y_{\rho(s,y_s)}) \frac{ds}{s}, & \text{if } t \in [t_1, t_2]. \end{cases} \tag{16}$$

Let $x(.) : (-\infty, t_2] \to \mathbb{R}$ be the function defined by:

$$x(t) = \begin{cases} y_a(t), & \text{if } t \in (-\infty, t_1], \\ y_a(t_1^-) + I_a(y_a(t_1^-)), & \text{if } t \in [t_1, t_2]. \end{cases}$$

Then $x_{t_1} = y_a$. For each $z \in C_1$, with $z_{t_1} = 0$, we denote by v the function defined by:

$$v(t) = \begin{cases} 0, & \text{if } t \in (-\infty, t_1], \\ z(t), & \text{if } t \in [t_1, t_2]. \end{cases}$$

If $y(.)$ satisfies the integral equation

$$y(t) = y_a(t_1^-) + I_a(y_a(t_1^-)) + \frac{1}{\Gamma(r)} \int_{t_1}^{t} \left(\log \frac{t}{s}\right)^{r-1} f(s, y_{\rho(s,y_s)}) \frac{ds}{s},$$

we can decompose $y(.)$ into $y(t) = v(t) + x(t)$, $t \in [t_1, t_2]$, which implies $y_t = v_t + x_t$, for every $t \in [t_1, t_2]$, and the function $z(.)$ satisfies

$$z(t) = \frac{1}{\Gamma(r)} \int_{a}^{t} \left(\log \frac{t}{s}\right)^{r-1} f(s, v_{\rho(s,v_s+x_s)} + x_{\rho(s,v_s+x_s)}) \frac{ds}{s}.$$

Set

$$C_{t_1}^* = \{z \in C_1; z_{t_1} = 0\},$$

and consider the operator $P_1 : C_{t_1}^* \to C_{t_1}^*$ defined by

$$P_1(z)(t) = \frac{1}{\Gamma(r)} \int_{a}^{t} \left(\log \frac{t}{s}\right)^{r-1} f(s, v_{\rho(s,v_s+x_s)} + x_{\rho(s,v_s+x_s)}) \frac{ds}{s}.$$

As in Step 1, we can show that P_1 is continuous and completely continuous, and if z is a possible solution of the equation $z = \lambda P_1(z)$ for some $\lambda \in (0, 1)$, then there exists $M_2 > 0$ such that

$$\|z\|_{t_i} \neq M_2.$$

Set

$$U_2 = \{z \in C_{t_1}^* : \|z\|_0 < M_2\}.$$

From the choice of U_2, there is no $z \in \partial U_2$ such that $z = \lambda P_1(z)$, for $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Leray-Schauder type, we deduce that P_1 has a fixed point $z \in \bar{U}_2$. Hence N_1 has a fixed point that is solution to the problem (1)-(3). Denote this solution by y_1.

Step 3: We continue this process and taking into account that $y_m := y|_{[t_m,T]}$ is a solution to the problem

$$^{CH}D^r y(t) = f(t, y_{\rho(t,y_t)}), \text{ for a.e., } t \in J = [t_m, T], \tag{17}$$

$$y(t_m^+) - y_{m-1}(t_{m-1}^-) = I_m(y_{m-1}(t_m^-)), \tag{18}$$

$$y(t) = y_{m-1}(t), t \in (-\infty, t_{m-1}]. \tag{19}$$

The solution of problem (1) − (3) is then defined by

$$y(t) = \begin{cases} y_a(t), & \text{if } t \in (-\infty, t_1], \\ y_1(t), & \text{if } t \in [t_1, t_2], \\ \ldots\ldots\ldots\ldots & \\ y_m(t), & \text{if } t \in (t_m, T]. \end{cases}$$

□

3. An Example

We conclude this paper with an example to illustrate our main result. We apply Theorem 2.7 to the the following impulsive fractional differential equation,

$$^{H}D^r y(t) = \frac{e^{-t} y(t - \sigma(y(t))}{(1 + y^2(t - \sigma(y(t)))}, \text{ for a.e. } t \in J = [1, e], \quad 0 < r \leq 1, \tag{20}$$

$$I_k(y(t_k)) = \int_{-\infty}^{t_k} (t_k - s) y(s) ds, \quad k = 1, ..., m, \tag{21}$$

$$y(t) = \phi(t), \ t \in (-\infty, 1], \tag{22}$$

where $^{HC}D^r$ is the Caputo-Hadamard fractional derivative of order $0 < r \leq 1$, $\sigma \in C(\mathbb{R}, [0, \infty))$, $\phi \in B_\sigma$, and

$$B = \{y \in C((-\infty, 1], \mathbb{R}) : \lim_{\theta \to -\infty} e^{\sigma\theta} y(\theta) \text{ exists in } \mathbb{R}\}.$$

The norm of B_c is defined by

$$||y||_\sigma = \sup_{-\infty < \theta \leq 0} e^{\sigma\theta} |y(\theta)|.$$

We show that B is the phase space:

Step 1: $y_t \in B_\sigma$.

Let $y : (-\infty, e] \to \mathbb{R}$ such that $y_0 \in B_\sigma$. Then

$$\lim_{\theta \to -\infty} e^{\sigma\theta} y_t(\theta) = \lim_{\theta \to -\infty} e^{\sigma\theta} y(t + \theta) \lim_{\theta \to -\infty} e^{\sigma(\theta - t)} y(\theta) = e^{-\sigma} \lim_{\theta \to -\infty} e^{\sigma\theta} y_0(\theta) < \infty.$$

Step 2: *A priori bounds*
Now, we prove that

$$||y_t||_\sigma \leq K(t) \sup\{|y(s)| : 1 \leq s \leq e\} + M(t) ||y_0||_\sigma,$$

where $K = M = 1$ and $H = 1$. We have $|y_t(\theta)| = |y(t + \theta)|$.
If $t + \theta \leq 1$, we have

$$|y_t(\theta)| \leq \sup\{|y(s)| : -\infty < s \leq 1\}.$$

If $t + \theta \geq 1$, we have

$$|y_t(\theta)| \leq \sup\{|y(s)| : 1 < s \leq e\}.$$

For $t+\theta \in [1,e]$, we have

$$|y_t(\theta)| \leq \sup\{|y(s)| : -\infty < s \leq 1\} + \sup\{|y(s)| : 1 < s \leq e\}.$$

Thus

$$\|y_t\|_\sigma \leq \sup\{|y(s)| : 1 \leq s \leq e\} + \|y_0\|_\sigma.$$

It clear that $(B_\sigma, \|.\|_\sigma)$ is a Banach space. We conclude that B_σ is a phase space.

Set

$$\rho(t,\varphi) = t - \sigma(\varphi(t)), \ \ (t,\varphi) \in J \times B,$$

$$f(t,\varphi) = \frac{e^{-t}\varphi}{1+\varphi}, \text{ for a.e. } (t,\varphi) \in J \times B_\sigma,$$

$$I_k(y(t_k)) = \int_{-\infty}^{t_k} (t_k - s) y(s) ds.$$

It is clear that (H_φ) and $(H_1) - (H_2)$ are satisfied with

$$|f(t,u)| \leq e^{-t} \|\varphi\|_{B_\sigma}, \ \ for\ each\ (t,\varphi) \in J \times B_\varphi,$$

and there exists numbers $M_1, M_2, ..., M_m > 0$ such that

$$\frac{\Gamma(r+1)M_k}{M_k} > 1, \ \ k = 1, ..., m. \tag{23}$$

Then by Theorem 2.7 the problem (20)-(22) has a solution on $(-\infty, e]$.

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