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Some properties of the condition pseudospectrum of multivalued linear operators

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Abstract. The main goal of this paper is to introduce the condition pseudospectrum of multivalued linear operators and prove several relations to the usual spectrum. We start by giving the definition then we focus on the characterization, the stability and some of their properties.

1. Introduction

The concept of condition pseudospectrum is an interesting subject by itself, which has become a useful tool in the numerical solutions of systems of linear equations and differential equations. In fact, it has convergence and approximation properties, carries more information than the spectrum and pseudospectrum. There are many generalizations of the concept of spectrum in literature such as Ransford spectrum [13], pseudospectrum([7, 8, 14]) and condition pseudospectrum ([9, 11]).

To have further details of the properties of the condition spectrum in finite dimensional space or in Banach algebras, we may refer to S. H. Kulkarni and D. Sukumar in [12] and G. K. Kumar and S. H. Lui in [10]. Recently, A. Ammar, K. Mahfoudhi and A. Jeribi in [5] extended some results of condition pseudospectrum to the case of bounded linear operators on Banach spaces and proved several relations to the usual spectrum. They defined the condition pseudospectrum of a linear operator T by:

$$\Lambda_\varepsilon(T) := \sigma(T) \bigcup \left\{ \lambda \in \mathbb{C} : \ \|\lambda - T\| \|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon} \right\},$$

where $0 < \varepsilon < 1$ and the convention that $\|\lambda - T\|\|(\lambda - T)^{-1}\| = \infty$, if $\lambda - T$ is not invertible.

In recent years, an important progress has been made in the study of linear relations. In this account, it seems interesting to extend the previous results of the condition pseudospectrum obtained in the case of bounded linear operators to the case of multivalued linear operators. The concept of linear relations appeared in Functional Analysis some decades ago not only as to the need to consider adjoints (conjugates) of non-densely defined linear differential operators but also the necessity to study the inverses of certain operators used, for example, in the study of some Cauchy problems associated with parabolic type equation in Banach spaces. The theory of linear relations is one of the most exciting and influential fields of research

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in modern mathematics. Applications of this theory can be found in economic theory, non-cooperative games, artificial intelligence, medicine and solutions for differential inclusions. We refer as examples to ([1, 2, 4]).

The main focus in this paper is to investigate a detailed treatment of the condition pseudospectrum of closed linear relations in Banach spaces and to study some of their properties.One of the central questions consists in the characterization of condition pseudospectrum. Our paper is organized as follows: In Section 2, we recall some basic notations and results from the theory of linear relations that we will need to prove the main results of other sections. Section 3 is devoted to investigating some properties and useful results for the condition pseudospectrum of multivalued linear operators. Finally, in Section 4, we will give a characterization of the condition pseudospectrum of multivalued linear operators.

2. Preliminary and auxiliary results

Before we state and prove our results, we first recall some definitions and give some preliminary results that will be useful in the sequel. We use the notations and terminology of the book [6].

Let X and Y be vector spaces over the same field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A linear relation T from X to Y is a mapping from a subspace

$$\mathcal{D}(T) = \{x \in X : Tx \neq \emptyset\} \subseteq X,$$

called the domain of T, into the collection of nonempty subsets of Y such that

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$$

for all nonzero scalars α_1, α_2 and $x_1,\ x_2 \in \mathcal{D}(T)$. If T maps the points of its domain to singletons, then T is said to be a single valued or simply an operator, that is equivalent to $T(0) = \{0\}$. We denote by $\mathcal{L}(X, Y)$ the set of bounded operators from X to Y. The collection of linear relations is denoted by $L\mathcal{R}(X, Y)$, and we write $L\mathcal{R}(X) = L\mathcal{R}(X, X)$. A linear relation $T \in L\mathcal{R}(X, Y)$ is uniquely determined by its graph $G(T)$ which is defined by

$$G(T) = \{(x, y) \in X \times Y : x \in \mathcal{D}(T) \text{ and } y \in Tx\}.$$

The inverse of $T \in L\mathcal{R}(X, Y)$ is the linear relation T^{-1} defined by

$$G(T^{-1}) = \{(y, x) \in Y \times X : (x, y) \in G(T)\}.$$

Let $T \in L\mathcal{R}(X, Y)$. The subspaces $R(T)$, $N(T)$ and $T(0)$ stand, respectively, for the range, the null space and the multivalued part of T which are defined by

$$\begin{aligned} R(T) &= \Big\{y : (x, y) \in G(T)\Big\}, \\ N(T) &= \Big\{x \in \mathcal{D}(T) : (x, 0) \in G(T)\Big\}, \text{ and} \\ T(0) &= \Big\{y : (0, y) \in G(T)\Big\}. \end{aligned}$$

Notice that when $x \in \mathcal{D}(T)$, $y \in Tx$ if, and only if, $Tx = y + T(0)$. A linear relation T is said to be surjective, if $R(T) = Y$. Similarly, T is said to be injective, if the null spaces $N(T) = T^{-1}(0) = \{0\}$. When T is both injective and surjective, we say that T is bijective.

Remark 2.1. *(i) T injective if, and only if, $T^{-1}T = I_{\mathcal{D}(T)}$.*
(ii) T is a single valued if, and only if, $TT^{-1} = I_{R(T)}$.

For $T,\ S \in L\mathcal{R}(X, Y)$, the linear relation $T + S$ is defined by

$$G(T + S) = \{(x, u + v) \in X \times Y : (x, u) \in G(T) \text{ and } (x, v) \in G(S)\}.$$

For $T \in L\mathcal{R}(X,Y)$ and $S \in L\mathcal{R}(Y,Z)$, the composition or product $ST \in L\mathcal{R}(X,Z)$ is defined by

$$G(ST) = \{(x,z) \in X \times Z : (x,y) \in G(T) \text{ and } (y,z) \in G(S) \text{ for some } y \in Y\}.$$

The closure of a linear relation $T \in L\mathcal{R}(X,Y)$ is the linear relation $\overline{T}$ defined by

$$G(\overline{T}) = \overline{G(T)}.$$

A linear relation T is said to be closed if its graph is a closed subspace, continuous if $\|T\| < \infty$, bounded if it is continuous and $\mathcal{D}(T) = X$ and open if T^{-1} is continuous, equivalently $\gamma(T) > 0$, where $\gamma(T)$ is the minimum modulus of T defined by

$$\gamma(T) = \sup\{\lambda \geq 0 : \lambda\, d(x, N(T)) \leq \|Tx\|,\ x \in \mathcal{D}(T)\}.$$

We denote the set of all closed and bounded linear relations from X to Y by $C\mathcal{R}(X,Y)$ and $B\mathcal{R}(X,Y)$, respectively, and we write $C\mathcal{R}(X) = C\mathcal{R}(X,X)$ and $B\mathcal{R}(X) = B\mathcal{R}(X,X)$.
If M and N are subspaces of X and of the dual space X^*, respectively, then

$$M^{\perp} = \Big\{x' \in X^* :\ x'(x) = 0 \ \text{ for all } x \in M\Big\},$$

and

$$N^{\top} = \Big\{x \in X :\ x'(x) = 0 \ \text{ for all } x' \in N\Big\}.$$

The adjoint (or conjugate) T^* of a linear relation $T \in L\mathcal{R}(X,Y)$ is defined by

$$G(T^*) = G(-T^{-1})^{\perp} \subset Y^* \times X^*,$$

This means that $(y', x') \in G(T^*)$ if, and only if, $y'(y) = x'(x)$ for all $(x,y) \in G(T)$.

Proposition 2.2. [6, Propositions I.2.8] *Let X, Y be two linear spaces and let $T \in L\mathcal{R}(X,Y)$. Then for $x \in \mathcal{D}(T)$, we have the following equivalence:*
(i) $y \in Tx$ if, and only if, $Tx = y + T(0)$.
In particular,
(ii) $0 \in Tx$ if, and only if, $Tx = T(0)$.

Proposition 2.3. ([6, Proposition I.4.2] *and* [1, Lemma 2]) *Let $T \in L\mathcal{R}(Y,Z)$ and $S, R \in L\mathcal{R}(X,Y)$.*

(i) If $T(0) \subset N(S)$ (or $T(0) \subset N(R)$), then $(R+S)T = RT + ST$.

(ii) If $\mathcal{D}(T)$ contains the ranges of both R and S (in particular, $\mathcal{D}(T)$is the whole space), then $T(R+S) = TR + TS$.

(iii) If $S(0) \subset T(0)$ and $\mathcal{D}(T) \subset \mathcal{D}(S)$, then $T = T + S - S$.

Lemma 2.4. ([6, Propositions II.1.4] *and* [6, Theorem II.2.5]) *Let X and Y be normed spaces and $T \in L\mathcal{R}(X,Y)$. Then, for $x \in \mathcal{D}(T)$,*
(a) $\|Tx\| = d(y, T(0))$ for any $y \in Tx$.
(b) $\|Tx\| = d(Tx, T(0)) = d(Tx, 0)$.
(c) $\|T\| = sup_{x \in B_X}\|Tx\|$ with $B_X := \{x \in X : \|x\| \leq 1\}$.
(d) $\gamma(T) = \|T^{-1}\|^{-1}$.

Proposition 2.5. [6, Propositions II.1.5 and II.3.13] *Let $S, T \in L\mathcal{R}(X,Y)$ and $R \in L\mathcal{R}(Y,Z)$.*

(i) For $x \in \mathcal{D}(S+T)$, we have

$$\|S + T\| \leq \|S\| + \|T\|,$$

if additionally $S(0) \subset \overline{T(0)}$ then,

$$\|T\| - \|S\| \leq \|T - S\|.$$

(ii) If $S(0) \subset \mathcal{D}(R)$, then, we have

$$\|RS\| \leq \|R\|\|S\|.$$

Let X be a normed space over the complex field $\mathbb{C}$. We shall write $\lambda - T := \lambda I_X - T$.

Definition 2.6. *Let $T \in L\mathcal{R}(X)$ and $\lambda \in \mathbb{C}$. Then,*

$$R(\lambda, T) := (\lambda - T)^{-1}$$

is called the resolvent of T (corresponding to λ) and

$$T_\lambda := (\lambda - \widetilde{T})^{-1}$$

is called the complete resolvent of T. The resolvent set of T is the set

$$\rho(T) := \{\lambda \in \mathbb{C} : \ T_\lambda \text{ is everywhere defined and single valued}\}.$$

The spectrum of T is the set $\sigma(T) := \mathbb{C}\backslash\rho(T)$.

Remark 2.7. [6] *Let $T \in L\mathcal{R}(X)$. Then, we have*
(i) $\rho(T) := \{\lambda \in \mathbb{C} : \ \lambda - T$ *is injective, open and with dense range on* $X\}$.
(ii) $\sigma(T) = \sigma(\widetilde{T}) = \sigma(\overline{T})$.

Proposition 2.8. ([6, Proposition III.1.13] *and* [6, Proposition VI.1.11]) *Let $T \in L\mathcal{R}(X)$. Then,*
(i) $\|T'\| \leq \|T\|$ *and if T is continuous, then* $\|T'\| = \|T\| < \infty$.
(ii) $\sigma(T) = \sigma(T')$.

Lemma 2.9. [6, Corollary III.7.7] *Let T be open and injective with dense range. Then, for any relation S such that:*
(i) $S \subset \overline{T(0)}$,
(ii) $\mathcal{D}(S) \supset \mathcal{D}(T)$,
(ii) $\|S\| < \gamma(T)$,
we have $T + S$ is open injective with dense range.

Definition 2.10. [3] *Let $\varepsilon > 0$. We define the pseudospectra of a linear relation $T \in L\mathcal{R}(X)$ by*

$$\sigma_\varepsilon(T) = \sigma(T) \cup \left\{\lambda \in \mathbb{C} : \ \|(\lambda - \widetilde{T})^{-1}\| > \frac{1}{\varepsilon}\right\}.$$

The pseudoresolvent of T is denoted by $\rho_\varepsilon(T)$ and is defined as

$$\rho_\varepsilon(T) = \rho(T) \cap \left\{\lambda \in \mathbb{C} : \ \|(\lambda - \widetilde{T})^{-1}\| \leq \frac{1}{\varepsilon}\right\}.$$

Remark 2.11. *Let $T \in L\mathcal{R}(X)$. Observe that if $T \in C\mathcal{R}(X)$, where X complete, then $T = \widetilde{T}$ and*

$$\sigma_\varepsilon(T) = \sigma(T) \cup \left\{\lambda \in \mathbb{C} : \ \|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon}\right\}.$$

3. Some properties of condition pseudospectrum $\Sigma_\varepsilon(T)$.

In this section, we define the condition pseudospectrum of a linear relation in $L\mathcal{R}(X)$, where X is a normed space over the complex field $\mathbb{C}$, and be consider some basic properties in order to put this definition in its due place. We begin with the following definition.

Definition 3.1. *Let $T \in L\mathcal{R}(X)$ and $0 < \varepsilon < 1$. The condition pseudospectrum of T is denoted by $\Sigma_\varepsilon(T)$ and is defined as*

$$\Sigma_\varepsilon(T) = \sigma(T) \cup \left\{\lambda \in \mathbb{C} : \ \|\lambda - \widetilde{T}\|\|(\lambda - \widetilde{T})^{-1}\| > \frac{1}{\varepsilon}\right\}.$$

The condition pseudoresolvent of T is denoted by $\rho_\varepsilon(T)$ and is defined as

$$\rho_\varepsilon(T) = \rho(T) \cap \left\{\lambda \in \mathbb{C} : \ \|\lambda - \widetilde{T}\|\|(\lambda - \widetilde{T})^{-1}\| \leq \frac{1}{\varepsilon}\right\}.$$

Remark 3.2. *Let $T \in L\mathcal{R}(X)$ and $0 < \varepsilon < 1$. Observe that if $T \in C\mathcal{R}(X)$, where X complete then $T = \widetilde{T}$ and*

$$\Sigma_\varepsilon(T) = \sigma(T) \cup \left\{\lambda \in \mathbb{C} : \ \|\lambda - T\|\|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon}\right\},$$

with the convention that $\|\lambda - T\|\|(\lambda - T)^{-1}\| = \infty$, if $\lambda - T$ is unbounded or nonexistent, i.e., if λ is in the spectrum $\sigma(T)$.

In the next proposition, we will establish the relationship between condition pseudospectrum and pseudospectrum of a bounded linear relation $T \in \mathcal{L}R(X)$.

Proposition 3.3. *Let $T \in B\mathcal{R}(X)$, such that $\varepsilon < \|\lambda - \widetilde{T}\|$ and $0 < \varepsilon < 1$. Then,*
(i) $\lambda \in \Sigma_\varepsilon(T)$ if, and only if, $\lambda \in \sigma_{\varepsilon\|\lambda - \widetilde{T}\|}(T)$.
(ii) $\lambda \in \sigma_\varepsilon(T)$ if, and only if, $\lambda \in \Sigma_{\frac{\varepsilon}{\|\lambda - \widetilde{T}\|}}(T)$.
(iii) $\sigma(T) = \bigcap\limits_{\varepsilon > 0} \Sigma_\varepsilon(T)$.

Proof. (*i*) If $\lambda \in \Sigma_\varepsilon(T)$, then,

$$\lambda \in \sigma(T) \ \text{ and } \ \|\lambda - \widetilde{T}\|\|(\lambda - \widetilde{T})^{-1}\| \geq \frac{1}{\varepsilon},$$

hence,

$$\lambda \in \sigma(T) \ \text{ and } \ \|(\lambda - \widetilde{T})^{-1}\| \geq \frac{1}{\varepsilon\|\lambda - \widetilde{T}\|},$$

which implies that $\lambda \in \sigma_{\varepsilon\|\lambda - \widetilde{T}\|}(T)$. The converse is similar.
(*ii*) Let $\lambda \in \sigma_\varepsilon(T)$, then,

$$\lambda \in \sigma(T) \ \text{ and } \ \|(\lambda - \widetilde{T})^{-1}\| \geq \frac{1}{\varepsilon},$$

thus

$$\lambda \in \sigma(T) \ \text{ and } \ \|\lambda - \widetilde{T}\|\|(\lambda - \widetilde{T})^{-1}\| \geq \frac{\|\lambda - \widetilde{T}\|}{\varepsilon}.$$

This proves that $\lambda \in \Sigma_{\frac{\varepsilon}{\|\lambda - \widetilde{T}\|}}(T)$. The converse is similar.
(*iii*) It is clear that $\sigma(T) \subset \Sigma_\varepsilon(T)$, and so $\sigma(T) \subset \bigcap\limits_{\varepsilon > 0} \Sigma_\varepsilon(T)$. Conversely, if $\lambda \in \bigcap\limits_{\varepsilon > 0} \Sigma_\varepsilon(T)$, then for all, $\varepsilon > 0$, we have $\lambda \in \Sigma_\varepsilon(T)$. We will discuss these two cases:
First case: If $\lambda \in \sigma(T)$, we get the desired result.
Second case: If $\lambda \in \left\{\lambda \in \mathbb{C} : \ \|\lambda - \widetilde{T}\|\|(\lambda - \widetilde{T})^{-1}\| > \frac{1}{\varepsilon}\right\}$, taking limits as $\varepsilon \to 0^+$, we get $\|\lambda - \widetilde{T}\|\|(\lambda - \widetilde{T})^{-1}\| = \infty$. Thus $\lambda \in \sigma(T)$. □

Proposition 3.4. *Let $T \in L\mathcal{R}(X)$ and $0 < \varepsilon < 1$.*
(i) If $0 < \varepsilon_1 < \varepsilon_2 < 1$, then $\sigma(T) \subset \Sigma_{\varepsilon_1}(T) \subset \Sigma_{\varepsilon_2}(T)$.
(ii) For any $\alpha, \beta \in \mathbb{C}$, with $\beta \neq 0$, we have $\Sigma_\varepsilon(\alpha I + \beta T) = \alpha + \Sigma_\varepsilon(T)\beta$.

Proof. (*i*) Let $\lambda \in \Sigma_{\varepsilon_1}(T)$. Then $\|\lambda - \widetilde{T}\|\|(\lambda - \widetilde{T})^{-1}\| > \dfrac{1}{\varepsilon_1} > \dfrac{1}{\varepsilon_2}$. Hence, $\lambda \in \Sigma_{\varepsilon_2}(T)$.
(*ii*) Let $\alpha, \beta \in \mathbb{C}$, such that $\beta \neq 0$. Then $\lambda \notin \Sigma_\varepsilon(\alpha I + \beta T)$ if, and only if,

$$\lambda \in \rho(\alpha I + \beta T) \text{ and } \|(\lambda - \alpha)I - \beta\widetilde{T}\|\|((\lambda - \alpha)I - \beta\widetilde{T})^{-1}\| \leq \frac{1}{\varepsilon},$$

if, and only if, $(\lambda - \alpha)I - \beta T$ is injective, open with dense range and

$$\|(\lambda - \alpha)I - \beta\widetilde{T}\|\|((\lambda - \alpha)I - \beta\widetilde{T})^{-1}\| \leq \frac{1}{\varepsilon},$$

if, and only if, $\beta^{-1}(\lambda - \alpha)I - T$ is injective, open with dense range and

$$\begin{aligned} &\|\beta^{-1}(\lambda - \alpha)I - \widetilde{T}\|\|(\beta^{-1}(\lambda - \alpha)I - \widetilde{T})^{-1}\| \\ = &\|\beta^{-1}((\lambda - \alpha)I - \beta\widetilde{T})\|\|(\beta^{-1}((\lambda - \alpha)I - \beta\widetilde{T})^{-1}\| \\ = &\|(\lambda - \alpha)I - \beta\widetilde{T}\|\|((\lambda - \alpha)I - \beta\widetilde{T})^{-1}\| \leq \frac{1}{\varepsilon}, \end{aligned}$$

if, and only if, $\beta^{-1}(\lambda - \alpha) \in \rho(T)$ and $\|\beta^{-1}(\lambda - \alpha)I - \widetilde{T}\|\|(\beta^{-1}(\lambda - \alpha)I - \widetilde{T})^{-1}\| \leq \dfrac{1}{\varepsilon}$,
if, and only if, $\beta^{-1}(\lambda - \alpha) \notin \Sigma_\varepsilon(T)$, if, and only if, $\lambda \notin \alpha + \Sigma_\varepsilon(T)\beta$. □

In this sequel of this section, X is a Banach space over the complex field $\mathbb{C}$.

Theorem 3.5. *Let $T \in C\mathcal{R}(X)$ be continuous and $0 < \varepsilon < 1$. Then,*

$$\Sigma_\varepsilon(T) = \Sigma_\varepsilon(T').$$

Proof. Let $\lambda \in \rho_\varepsilon(T')$. Then

$$\lambda \in \rho(T') \text{ and } \|\lambda - T'\|\|(\lambda - T')^{-1}\| \leq \frac{1}{\varepsilon}.$$

So, $\lambda \in \rho(T)$, and therefore $(\lambda - T)^{-1}$ is continuous. We also have $\lambda - T$ is continuous, hence from Proposition 2.8, it follows that

$$\|\lambda - T\|\|(\lambda - T)^{-1}\| = \|\lambda - T'\|\|(\lambda - T')^{-1}\| \leq \frac{1}{\varepsilon},$$

furthermore,

$$\lambda \in \rho_\varepsilon(T).$$

However, the opposite inclusion follows by symmetry. □

Proposition 3.6. *Let $T \in C\mathcal{R}(X)$ such that $0 \notin \Sigma_\varepsilon(T)$, $k = \|T\|\|T^{-1}\|$ and $0 < \varepsilon < 1$.*
(i) If $\lambda \in \Sigma_\varepsilon(T)$, then $\frac{1}{\lambda} \in \Sigma_{\varepsilon k}(T^{-1})$.

(ii) If $\lambda \in \Sigma_\varepsilon(T^{-1})$, then $\frac{1}{\lambda} \in \Sigma_{\varepsilon k}(T)$.

Proof. (*i*) Let $\lambda \in \Sigma_\varepsilon(T)$. Then,

$$\begin{aligned}\frac{1}{\varepsilon} < \|\lambda - T\|\|(\lambda - T)^{-1}\| &= \|-\lambda(\frac{1}{\lambda} - T^{-1})T\|\|-\lambda^{-1}T^{-1}(\frac{1}{\lambda} - T^{-1})^{-1}\|, \\ &\leq \|T\|\|T^{-1}\|\|(\frac{1}{\lambda} - T^{-1})\|\|(\frac{1}{\lambda} - T^{-1})^{-1}\|.\end{aligned}$$

Hence, $\frac{1}{\lambda} \in \Sigma_{\varepsilon k}(T^{-1})$.

(*ii*)Let $\lambda \in \Sigma_\varepsilon(T)$. Then,

$$\begin{aligned}\frac{1}{\varepsilon} < \|\lambda - T^{-1}\|\|(\lambda - T^{-1})^{-1}\| &= \|-\lambda T^{-1}(\frac{1}{\lambda} - T)\|\|-\lambda^{-1}(\frac{1}{\lambda} - T)^{-1}T\|, \\ &\leq \|T\|\|T^{-1}\|\|(\frac{1}{\lambda} - T)\|\|(\frac{1}{\lambda} - T)^{-1}\|.\end{aligned}$$

Hence, $\frac{1}{\lambda} \in \Sigma_{\varepsilon k}(T)$. □

4. Characterization of condition pseudospectrum.

In this section, we give a characterization of the condition pseudospectrum of a linear relation in $L\mathcal{R}(X)$, where X is a Banach space over the complex field $\mathbb{C}$. Our first result is the following.

Lemma 4.1. *Let $T \in B\mathcal{R}(X)$ and $0 < \varepsilon < 1$. Then, $\lambda \in \Sigma_\varepsilon(T) \setminus \sigma(T)$ if, and only if, there exists $x \in X$ such that*

$$\|(\lambda - T)x\| < \varepsilon\|\lambda - T\|\|x\|.$$

Proof. " $\Longrightarrow$ " Let $\lambda \in \Sigma_\varepsilon(T) \setminus \sigma(T)$. Then,

$$\|\lambda - T\|\|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon},$$

and thus we have

$$\|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon\|\lambda - T\|}.$$

Moreover,

$$\sup_{y \in X\setminus\{0\}} \frac{\|(\lambda - T)^{-1}y\|}{\|y\|} > \frac{1}{\varepsilon\|\lambda - T\|}.$$

Hence, there exists a nonzero $y \in X$, such that

$$\|(\lambda - T)^{-1}y\| > \frac{\|y\|}{\varepsilon\|\lambda - T\|}. \qquad (1)$$

Put $x = (\lambda - T)^{-1}y$, so

$$\begin{aligned}(\lambda - T)x &= (\lambda - T)(\lambda - T)^{-1}y \\ &= y + (\lambda - T)(0).\end{aligned}$$

Knowing that $(\lambda - T)(0) = T(0)$, allows us to deduce that

$$\begin{aligned}\|(\lambda - T)x\| &= d(y, (\lambda - T)(0)) \\ &= d(y, T(0)) \\ &\leq d(y, 0) \ \ (\textit{since} \ \ 0 \in T(0)) \\ &\leq \|y\|.\end{aligned}$$

Therefore, from Eq (1), we infer that

$$\|x\| > \frac{\|y\|}{\varepsilon\|\lambda - T\|} \geq \frac{\|(\lambda - T)x\|}{\varepsilon\|\lambda - T\|}.$$

Finally, we have as a result,

$$\|(\lambda - T)x\| < \varepsilon\|\lambda - T\|\|x\|.$$

$'' \Longleftarrow ''$ We assume that there exists $x \in X$ such that

$$\|(\lambda - T)x\| < \varepsilon\|\lambda - T\|\|x\|.$$

Since $\lambda \in \rho(T)$, then $\lambda - T$ is injective and open. Moreover, we have

$$\gamma(\lambda - T)\|x\| \leq \|(\lambda - T)x\| < \varepsilon\|\lambda - T\|\|x\|,$$

therefore,

$$0 < \gamma(\lambda - T) < \varepsilon\|\lambda - T\|.$$

We already have

$$\gamma(\lambda - T) = \|(\lambda - T)^{-1}\|^{-1},$$

which shows that

$$\|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon\|\lambda - T\|},$$

and, consequently

$$\lambda \in \sigma_{\varepsilon\|\lambda - T\|}(T),$$

equivalently, in virtue of Proposition 3.3, we obtain $\lambda \in \Sigma_\varepsilon(T)$. □

Theorem 4.2. *Let $T \in \mathcal{CR}(X)$. Assume that V is a bounded and closed operator such that $0 \in \rho(V)$. Let $R = VTV^{-1}$ and $k = \|V\|\|V^{-1}\|$. Then for all $0 < \varepsilon < 1$, and $k^2\varepsilon > 0$, we have*

$$\Sigma_{\frac{\varepsilon}{k^2}}(T) \subseteq \Sigma_\varepsilon(R) \subseteq \Sigma_{k^2\varepsilon}(T).$$

Proof. First of all, we have from [4, Theorem 3.1] R is closed and $\sigma(T) = \sigma(R)$.
Now, we start with the first inclusion. So, we can write,

$$\lambda - T = V^{-1}(\lambda - R)V \text{ and } \lambda - R = V(\lambda - T)V^{-1},$$

then it is obvious that

$$(\lambda - T)^{-1} = V^{-1}(\lambda - R)^{-1}V \text{ and } (\lambda - R)^{-1} = V(\lambda - T)^{-1}V^{-1}.$$

Thus,

$$\begin{aligned}
\|\lambda - R\|\|(\lambda - R)^{-1}\| &= \|V(\lambda - T)V^{-1}\|\|V(\lambda - T)^{-1}V^{-1}\| \\
&\leq \|V(\lambda - T)\|\|V^{-1}\|\|V(\lambda - T)^{-1}\|\|V^{-1}\| \\
&\leq \|V\|\|(\lambda - T)\|\|V^{-1}\|\|V\|\|(\lambda - T)^{-1}\|\|V^{-1}\| \\
&\leq \left(\|V\|\|V^{-1}\|\right)^2\|(\lambda - T)\|\|(\lambda - T)^{-1}\| \\
&\leq k^2\|(\lambda - T)\|\|(\lambda - T)^{-1}\|.
\end{aligned}$$

In the similar way,

$$\begin{aligned}
\|\lambda - T\|\|(\lambda - T)^{-1}\| &= \|V^{-1}(\lambda - R)V\|\|V^{-1}(\lambda - R)^{-1}V\| \\
&\leq \left(\|V\|\|V^{-1}\|\right)^2\|(\lambda - R)\|\|(\lambda - R)^{-1}\| \\
&\leq k^2\|(\lambda - R)\|\|(\lambda - R)^{-1}\|.
\end{aligned}$$

For $\lambda \in \Sigma_{\varepsilon/k^2}(T)$, we have

$$\lambda \in \sigma(T) \ \ and \ \ \|\lambda - T\|\|(\lambda - T)^{-1}\| > \frac{k^2}{\varepsilon},$$

then,

$$\lambda \in \sigma(R) \ \text{ and } \ \|\lambda - R\|\|(\lambda - R)^{-1}\| \geq \frac{1}{k^2}\|\lambda - T\|\|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon},$$

hence,

$$\lambda \in \Sigma_{\varepsilon}(R).$$

Therefore,

$$\Sigma_{\varepsilon/k^2}(T) \subseteq \Sigma_{\varepsilon}(R).$$

For the second inclusion, let $\lambda \in \Sigma_{\varepsilon}(R)$. Then

$$\lambda \in \sigma(R) \ \text{ and } \ \|\lambda - R\|\|(\lambda - R)^{-1}\| > \frac{1}{\varepsilon}.$$

This induces that

$$\lambda \in \sigma(T) \ \text{ and } \ \|\lambda - T\|\|(\lambda - T)^{-1}\| \geq \frac{1}{k^2}\|\lambda - R\|\|(\lambda - R)^{-1}\| > \frac{1}{k^2\varepsilon},$$

hence,

$$\lambda \in \Sigma_{k^2\varepsilon}(T).$$

Consequently,

$$\Sigma_{\varepsilon}(R) \subseteq \Sigma_{k^2\varepsilon}(T).$$

□

In the sequel of this section, we suppose that X is a Banach space and A is a linear relation satisfying the following property $(\mathcal{P})$:

$$(\mathcal{P}) : \forall\, A \in L\mathcal{R}(X) \text{ with } 0 \in \rho(A),\ \exists\, B \in L\mathcal{R}(X) \text{ with } 0 \notin \rho(B) \ \text{ such that } \|A - B\| = \frac{1}{\|A^{-1}\|}.$$

The following example shows the above property.

Example 4.3. *Let X be a Banach space, and consider*

$$A = \begin{pmatrix} I & 0 \\ 0 & \frac{I}{2} \end{pmatrix} \ \ and \ \ B = \begin{pmatrix} \frac{I}{2} & 0 \\ 0 & 0 \end{pmatrix}.$$

So, we have

$$A^{-1} = \begin{pmatrix} I & 0 \\ 0 & 2I \end{pmatrix} \ \ and \ \ A - B = \begin{pmatrix} \frac{I}{2} & 0 \\ 0 & \frac{I}{2} \end{pmatrix}.$$

This implies that

$$\|A^{-1}\| = \max\left\{\|I\|, \|2I\|\right\} = 2 \ \ and \ \ \|A - B\| = \max\left\{\|\frac{I}{2}\|, \|\frac{I}{2}\|\right\} = \frac{1}{2}.$$

Hence we have,

$$\|A - B\| = \frac{1}{2} = \frac{1}{\|A^{-1}\|}.$$

Theorem 4.4. *Let X be a Banach space. Let $0 < \varepsilon < 1$ and $T \in C\mathcal{R}(X)$ satisfying property $(\mathcal{P})$. Suppose that there exists a non invertible linear relation R such that*
(i) $\mathcal{D}(R) \subset \mathcal{D}(T)$,
(ii) $T(0) \subset R(0)$,
then, $\lambda \in \Sigma_{\varepsilon}(T)$ if, and only, if $\|R\| < \varepsilon\|\lambda - T\|$ and $\lambda \in \sigma(T + R)$.

Proof. Assume that $\lambda \in \Sigma_\varepsilon(T)$. There are two cases to consider:
First case: If $\lambda \in \sigma(T)$, then it is sufficient to take $R = 0$.

Second case: If $\lambda \in \Sigma_\varepsilon(T) \setminus \sigma(T)$, then $\lambda - T$ is invertible. Hence, by property $(\mathcal{P})$, there exists a non invertible linear relation D such that

$$\|\lambda - T - D\| = \frac{1}{\|(\lambda - T)^{-1}\|}.$$

Putting $R = \lambda - T - D$, we have

$$\|R\| = \frac{1}{\|(\lambda - T)^{-1}\|} < \varepsilon\|\lambda - T\|.$$

Also $D = \lambda - T - R$ is non invertible, that is, $\lambda \in \sigma(T + R)$. For the reverse inclusion, we suppose $\lambda \in \sigma(T + R)$. We derive a contradiction from the assumption that $\lambda \notin \Sigma_\varepsilon(T)$, which is equivalent to

$$\lambda \in \rho(T) \text{ and } \|\lambda - T\|\|(\lambda - T)^{-1}\| \leq \frac{1}{\varepsilon}.$$

Since $\lambda \in \rho(T)$, then $\lambda - T$ is an invertible linear relation, and we have $\lambda - T - R$ is not invertible. So, from property $(\mathcal{P})$ we have

$$\|R\| = \|(\lambda - T - R) - (\lambda - T)\| = \frac{1}{\|(\lambda - T)^{-1}\|}.$$

Therefore,

$$\frac{1}{\|(\lambda - T)^{-1}\|} = \|R\| < \varepsilon\|\lambda - T\|.$$

Consequently,

$$\|\lambda - T\|\|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon},$$

which is a contraction. □

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