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Existence of the solution for hybrid differential equation with Caputo-Fabrizio fractional derivative

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Abstract. In this manuscript, we are interested in the existence result of the solution of hybrid nonlinear differential equations. involving fractional Caputo Fabrizio derivatives of arbitrary order $\alpha \in]0,1[$. By applying Dhage's fixed point theorem and some fractional analysis techniques, we prove our main result. As an application, A non-trivial example is given to demonstrate the effectiveness of our theoretical result.

1. Introduction

Fractional differential equations are a generalization of ordinary differential equations. They are used to describe many phenomena in several fields, engineering, physics, economics and science. There are several concepts of fractional derivatives, some classical, such as Riemann-Liouville , Caputo and Caputo-Fabrizio Derivative.

Fractional hybrid differential equations are the quadratic perturbations of nonlinear differential equations, for the information we refer [5, 7, 8, 10–14].

Dhage and Lakshmikantham [11] discussed the hybrid differential equation of the following form

$$\begin{cases} \frac{d}{dt}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)) \quad a.e. \quad t \in J = [0,T] \\ x(t_0) = x_0 \end{cases} \tag{1}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ and $g \in C(J \times \mathbb{R}, \mathbb{R})$. They established the existence, uniqueness results, and some fundamental differential inequalities for hybrid differential equations initiating the study of the theory of such systems and proved to utilize the theory of inequalities, its existence of extremal solutions, and comparison results.

Hilal and Kajouni [15] have studied boundary fractional hybrid differential equations involving Caputo differential operators of order $0 < \alpha < 1$,

$$\begin{cases} D^{\alpha}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)) \quad a.e. \quad t \in J = [0,T] \quad , \\ \\ a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c \quad . \end{cases} \tag{2}$$

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where $f \in C(J \times \mathbb{R}, \mathbb{R}\backslash\{0\})$, $g \in C(J \times \mathbb{R}, \mathbb{R})$ and a, b, c are real constants with $a + b \neq 0$. They proved the existence result for boundary fractional hybrid differential equations under mixed Lipschitz and Carathéodory conditions. Some fundamental fractional differential inequalities are also established which are utilized to prove the existence of extremal solutions. Necessary tools are considered and the comparison principle is proved which will be useful for further study of the qualitative behavior of solutions.
Here, we are going to study the Cauchy problem with fractional derivatives. Precisely, we consider the hybrid fractional problem of the from :

$$\begin{cases} D^{\alpha}_{0,a,b}\Big(\frac{x(t)}{f(t,x(t))}\Big) = g(t,x(t)) \quad a.e. \quad t \in J = [0,T] \quad 0 < \alpha < 1, \quad a > 0, b \geq 0 \\ x_0 = \frac{x(0)}{f(0,x(0))} \end{cases} \tag{3}$$

Where $D^{\alpha}_{0,a,b}$ denotes generalized Caputo-Fabrizo fractional derivative, $0 < \alpha < 1$. $X = C(J, \mathbb{R})$ is Banach espace of continuous fonctions with $\|y\| = sup\{|y(t)| \ , \ t \in J\}$.

2. Preliminaries

Definition 2.1. *[1] Given $a > 0$, $b \geq 0$, $0 < \alpha < 1$, $n \in \mathbb{N} \bigcup \{0\}$ and $f \in C^{n+1}[0, +\infty)$. The fractional derivative of order $\alpha + n$ of f with respect to Kernel function $K_{a,b}$ $\Big(K_{a,b}(t) = \Big(\frac{a^2+b^2}{a}\Big)e^{-at}cos(bt) \ , \ t \geq 0\Big)$ is defined by*

$$D^{\alpha+n}_{0,a,b}\Big(f\Big)(t) = \Big(\frac{1}{1-\alpha}\Big)\Big(\frac{a^2+b^2}{a}\Big)\int_0^t e^{\frac{-a\alpha(t-s)}{1-\alpha}} cos\Big(\frac{b\alpha(t-s)}{1-\alpha}\Big)f^{n+1}(s)ds \quad t > 0.$$

Definition 2.2. *[1] Let $g \in C[0, T]$. The fractional integral of order α of g is defined by*

$$I^{\alpha}_{0,a,b}(g)(t) = \frac{a(1-\alpha)}{a^2+b^2}g(t) + \alpha\Big(\int_0^t g(s)ds - \frac{b^2}{a^2+b^2}\int_0^t e^{\frac{-a\alpha(t-s)}{1-\alpha}} g(s)ds\Big).$$

We prove the existence of a solution for the problem (3) by a fixed point theorem in Banach algebra due to Dhage [8].

Lemma 2.3. *[8] Let S be a non-empty, closed convex and bounded subset of the Banach algebra X and let $A : X \longrightarrow X$ and $B : X \longrightarrow X$ be two operators such that*

(a) A is Lipschitzian with a Lipschitz constant α.
(b) B is completely continuous.
(c) $x = AxBy \Longrightarrow x \in S$ for all $y \in S$.
(d) $\alpha M < 1$, where $M = \|B(S)\| = sup\{\|B(x)\| : x \in S\}$.

Then the operator equation $AxBx = x$ has a solution in S .

3. Existence result

Before presenting our main results, we introduce the following assumptions :
H_0) The function $x \longrightarrow \frac{x}{f(t,x)}$ is increasing in $\mathbb{R}$ almost every where for $t \in J$.
H_1) $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}\backslash\{0\}$ is continuous and also there exists a constant $L > 0$, such that

$$|f(t,x_1) - f(t,x_2)| \leq L_1|x_1 - x_2| \quad \forall x_1, x_2 \in \mathbb{R}.$$

H_2) $g : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there exists a function $h \in L^1(J, \mathbb{R})$ such that

$$|g(t,x)| \leq h(t) \quad \forall x \in \mathbb{R}.$$

H_3)There exists a constant $L_2 > 0$, such that

$$|g(t_1,x) - g(t_2,x)| \leq L_2|t_1 - t_2| \quad \forall t_1, t_2 \in J.$$

Lemma 3.1. *The function $x \in \mathbb{C}([0,T],\mathbb{R})$ is a solution of the problem*

$$\begin{cases} D^{\alpha}_{0,a,b}\Big(\frac{x(t)}{f(t,x(t))}\Big) = g(t,x(t)) \quad a.e. \quad t \in J = [0,T] \quad 0<\alpha<1, \quad a>0, b\geq 0 \\ x_0 = \frac{x(0)}{f(0,x(0))} \end{cases}$$

If x solves the integral equation

$$x(t) = f(t,x(t))\Big(x_0 + \Big(I^{\alpha}_{0,a;b}g(.,x(.))\Big)(t)\Big) \quad t \in [0,T].$$

Proof. We have,

$$\begin{aligned} \frac{d}{dt}\Big(D^{\alpha}_{0,a,b}\Big(\frac{x(t)}{f(t,x(t))}\Big)\Big) &= \Big(\frac{1}{1-\alpha}\Big)\Big(\frac{a^2+b^2}{a}\Big)\Big\{\Big(\frac{x(t)}{f(t,x(t))}\Big)^{'} \\ &+ \int_0^t \frac{d}{dt}\Big(e^{\frac{-a\alpha(t-s)}{1-\alpha}}cos\Big(\frac{b\alpha(t-s)}{1-\alpha}\Big)\Big(\frac{x(s)}{f(s,x(s))}\Big)^{'}\Big)ds\Big\} \\ &= \Big(\frac{1}{1-\alpha}\Big)\Big(\frac{a^2+b^2}{a}\Big)\Big(\frac{x(t)}{f(t,x(t))}\Big)^{'} \\ &- \Big(\frac{1}{1-\alpha}\Big)\Big(\frac{a^2+b^2}{a}\Big)\Big(\frac{\alpha a}{1-\alpha}\Big)\int_0^t e^{\frac{-a\alpha(t-s)}{1-\alpha}}cos\Big(\frac{b\alpha(t-s)}{1-\alpha}\Big)\Big(\frac{x(s)}{f(s,x(s))}\Big)^{'}ds \\ &- \Big(\frac{1}{1-\alpha}\Big)\Big(\frac{a^2+b^2}{a}\Big)\Big(\frac{\alpha b}{1-\alpha}\Big)\int_0^t e^{\frac{-a\alpha(t-s)}{1-\alpha}}sin\Big(\frac{b\alpha(t-s)}{1-\alpha}\Big)\Big(\frac{x(s)}{f(s,x(s))}\Big)^{'}ds \\ &= \Big(\frac{1}{1-\alpha}\Big)\Big(\frac{a^2+b^2}{a}\Big)\Big(\frac{x(t)}{f(t,x(t))}\Big)^{'} \qquad (*) \\ &- \Big(\frac{\alpha a}{1-\alpha}\Big)g(t,x(t)) - \Big(\frac{\alpha b}{1-\alpha}\Big)\Big(\frac{1}{1-\alpha}\Big)\Big(\frac{a^2+b^2}{a}\Big)\gamma(t) \end{aligned}$$

Where,

$$\gamma(t) = \int_0^t e^{\frac{-a\alpha(t-s)}{1-\alpha}}sin\Big(\frac{b\alpha(t-s)}{1-\alpha}\Big)\Big(\frac{x(s)}{f(s,x(s))}\Big)^{'}ds.$$

In the other hand

$$\frac{d}{dt}\gamma(t) = \int_0^t \frac{d}{dt}\Big(e^{\frac{-a\alpha(t-s)}{1-\alpha}}sin\Big(\frac{b\alpha(t-s)}{1-\alpha}\Big)\Big(\frac{x(s)}{f(s,x(s))}\Big)^{'}\Big)ds.$$

Then,

$$\begin{aligned} \frac{d\gamma(t)}{dt} &= \frac{-a\alpha}{1-\alpha}\int_0^t e^{\frac{-a\alpha(t-s)}{1-\alpha}}sin\Big(\frac{b\alpha(t-s)}{1-\alpha}\Big)\Big(\frac{x(s)}{f(s,x(s))}\Big)^{'}ds + \frac{b\alpha}{1-\alpha}\int_0^t e^{\frac{-a\alpha(t-s)}{1-\alpha}}cos\Big(\frac{b\alpha(t-s)}{1-\alpha}\Big)\Big(\frac{x(s)}{f(s,x(s))}\Big)^{'}ds \\ &= \frac{-a\alpha}{1-\alpha}\gamma(t) + \frac{ab\alpha}{a^2+b^2}g(t,x(t)). \end{aligned}$$

Integrating the above equality and using that $\gamma(0) = 0$, we obtains

$$\gamma(t) = \frac{ab\alpha}{a^2+b^2}\int_0^t e^{\frac{-a\alpha(t-s)}{1-\alpha}}g(s,x(s))ds.$$

Hence by (∗), we deduce that

$$\begin{aligned}\frac{d}{dt}\Big(D^{\alpha}_{0,a,b}\Big(\frac{x(t)}{f(t,x(t))}\Big)\Big) &= \Big(\frac{1}{1-\alpha}\Big)\Big(\frac{a^2+b^2}{a}\Big)\Big\{\Big(\frac{x(t)}{f(t,x(t))}\Big)' - \Big(\frac{\alpha a}{1-\alpha}\Big)g(t,x(t))\\ &- \Big(\frac{\alpha b}{1-\alpha}\Big)^2\int_0^t e^{\frac{-a\alpha(t-s)}{1-\alpha}}g(s,x(s))ds.\end{aligned}$$

By using

$$\frac{d}{dt}\Big(D^{\alpha}_{0,a,b}\Big(\frac{x(t)}{f(t,x(t))}\Big)\Big) = \frac{d}{dt}g(t,x(t)) \ , \qquad 0 < t < T.$$

We obtain that

$$\begin{aligned}\frac{d}{dt}g(t,x(t)) &= \Big(\frac{1}{1-\alpha}\Big)\Big(\frac{a^2+b^2}{a}\Big)\Big\{\Big(\frac{x(t)}{f(t,x(t))}\Big)' - \Big(\frac{\alpha a}{1-\alpha}\Big)g(t,x(t))\\ &- \Big(\frac{\alpha b}{1-\alpha}\Big)^2\int_0^t e^{\frac{-a\alpha(t-s)}{1-\alpha}}g(s,x(s))ds\end{aligned}$$

Then,

$$\Big(\frac{x(t)}{f(t,x(t))}\Big)' = \frac{a(1-\alpha)}{a^2+b^2}g'(t,x(t)) + \frac{\alpha a^2}{a^2+b^2}g(t,x(t)) + \frac{ab^2\alpha^2}{(a^2+b^2)(1-\alpha)}\int_0^t e^{\frac{-a\alpha(t-s)}{1-\alpha}}g(s,x(s))ds.$$

Integrating the above equality,using that $\frac{x(0)}{f(t,x(0))} = x_0$ and $g(0,x(0)) = 0$ it holds

$$\frac{x(t)}{f(t,x(t))} - x_0 = \frac{\alpha a^2}{a^2+b^2}\int_0^t g(\tau,x(\tau))d\tau + \frac{a(1-\alpha)}{a^2+b^2}g(t,x(t)) + \frac{ab^2\alpha^2}{(a^2+b^2)(1-\alpha)}\int_0^t\int_0^{\tau} e^{\frac{-a\alpha(\tau-s)}{1-\alpha}}g(s,x(s))dsd\tau$$

On the other hand using Fubini's theorem, we gets

$$\begin{aligned}\int_0^t\int_0^{\tau} e^{\frac{-a\alpha(\tau-s)}{1-\alpha}}g(s,x(s))dsd\tau &= \int_0^t e^{\frac{a\alpha s}{1-\alpha}}g(s,x(s))\Big(\int_s^t e^{\frac{-a\alpha\tau}{1-\alpha}}d\tau\Big)ds\\ &= \Big(\frac{1-\alpha}{a\alpha}\Big)\int_0^t g(s,x(s))ds - \Big(\frac{1-\alpha}{a\alpha}\Big)\int_0^t e^{\frac{-a\alpha(t-s)}{1-\alpha}}g(s,x(s))ds\end{aligned}$$

Then,

$$\begin{aligned}\frac{x(t)}{f(t,x(t))} - x_0 &= \frac{\alpha a^2}{a^2+b^2}\int_0^t g(\tau,x(\tau))d\tau + \frac{a(1-\alpha)}{a^2+b^2}g(t,x(t))\\ &+ \frac{ab^2\alpha^2}{(a^2+b^2)(1-\alpha)}\Big(\Big(\frac{1-\alpha}{a\alpha}\Big)\int_0^t g(s,x(s))ds - \Big(\frac{1-\alpha}{a\alpha}\Big)\int_0^t e^{\frac{-a\alpha(t-s)}{1-\alpha}}g(s,x(s))ds\Big)\\ &= \frac{\alpha a^2}{a^2+b^2}\int_0^t g(\tau,x(\tau))d\tau + \frac{a(1-\alpha)}{a^2+b^2}g(t,x(t)) + \frac{b^2\alpha}{a^2+b^2}\int_0^t g(s,x(s))ds\\ &- \frac{b^2\alpha}{a^2+b^2}\int_0^t e^{\frac{-a\alpha(t-s)}{1-\alpha}}g(s,x(s))ds\\ &= \frac{a(1-\alpha)}{a^2+b^2}g(t,x(t)) + \alpha\Big(\int_0^t g(s,x(s))ds - \frac{b^2}{a^2+b^2}\int_0^t e^{\frac{-a\alpha(t-s)}{1-\alpha}}g(s,x(s))ds\Big)\\ &= \Big(I^{\alpha}_{0,a;b}g(.,x(.))\Big)(t)\end{aligned}$$

Then,

$$x(t) = f(t,x(t))\Big(x_0 + \Big(I^{\alpha}_{0,a;b}g(.,x(.))\Big)(t)\Big).$$

□

Theorem 3.2. *Assume that hypotheses $(H_0)-(H_3)$ hold and if*

$$L\Big(|x_0| + \frac{a(1-\alpha)L_2T}{a^2+b^2} + \alpha\|h\|_{L^1} + \frac{b^2\alpha}{a^2+b^2}\|h\|_{L^1}\Big) < 1.$$

Then the hybrid fractional differential equation (3) has a solution.

Proof. We define a subset B_r of X by

$$B_r = \{x \in X \mid \|x\| \leq R\}$$

Where

$$R = \frac{F_0\Big(|x_0| + \frac{a(1-\alpha)L_2T}{a^2+b^2} + \alpha\|h\|_{L^1} + \frac{b^2\alpha}{a^2+b^2}\|h\|_{L^1}\Big)}{1 - L\Big(|x_0| + \frac{a(1-\alpha)L_2T}{a^2+b^2} + \alpha\|h\|_{L^1} + \frac{b^2\alpha}{a^2+b^2}\|h\|_{L^1}\Big)}$$

and $F_0 = \sup_{t\in[0,T]} |f(t,0)|$.

Claim 1, Let $x, y \in X$. Then by hypothesis (H_1),

$$|Ax(t) - Ay(t)| = |f(t,x(t)) - f(t,y(t))| \leq L|x(t) - y(t)| \leq L\|x-y\| \quad ,$$

for all $t \in J$. Taking supremum over t, we obtain

$$\|Ax - Ay\| \leq L\|x-y\| \qquad \forall x, y \in X$$

Claim 2, We show that B is continuous in B_r .

Let (x_n) be a sequence in B_r converging to a point $x \in B_r$. Then by Lebesgue dominated convergence theorem.

$$\begin{aligned} -\lim_{n\to\infty} g(t,x_n(t)) &= g(t,x(t)). \\ -\lim_{n\to\infty}\int_0^t g(s,x_n(s))ds &= \int_0^t g(s,x(s))ds. \\ -\lim_{n\to\infty}\int_0^t e^{\frac{-a\alpha(t-s)}{1-\alpha}} g(s,x_n(s))ds &= \int_0^t e^{\frac{-a\alpha(t-s)}{1-\alpha}} g(s,x(s))ds. \end{aligned}$$

Then,

$$\lim_{n\to\infty} Bx_n(t) = Bx(t).$$

for all $t \in J$. This shows that B is a continuous operator on B_r.

Claim 3, B is compact operator on B_r .

First, we show that $B(B_r)$ is a uniformly bounded set in X.

Let $x \in B_r$. Then for $0 \leq s \leq t$ we have

$$\begin{aligned} |B(x)| &\leq |x_0| + \frac{a(1-\alpha)}{a^2+b^2}|g(t,x(t))| + \alpha\Big(\int_0^t |g(s,x(s))|ds + \frac{b^2}{a^2+b^2}\int_0^t e^{\frac{-a\alpha(t-s)}{1-\alpha}}|g(s,x(s))|ds\Big) \\ &\leq |x_0| + \frac{a(1-\alpha)}{a^2+b^2}|g(t,x(t)) - g(0,x(0))| + \alpha\Big(\int_0^t |h(s)|ds + \frac{b^2}{a^2+b^2}\int_0^t e^{\frac{-a\alpha(t-s)}{1-\alpha}}|h(s)|ds\Big) \\ &\leq |x_0| + \frac{a(1-\alpha)L_2t}{a^2+b^2} + \alpha\Big(\int_0^t |h(s)|ds + \frac{b^2}{a^2+b^2}\int_0^t |h(s)|ds\Big) \\ &\leq \|x_0\| + \frac{a(1-\alpha)L_2T}{a^2+b^2} + \alpha\Big(\|h\|_{L^1} + \frac{b^2}{a^2+b^2}\|h\|_{L^1}\Big) \end{aligned}$$

This shows that B is uniformly bounded on B_r.

Then let us show that $B(B_r)$ is an equicontinuous set on B_r.

We pose that $\phi(t) = \int_0^t h(s)ds$.
Let $t_1, t_2 \in J$ with $t_1 < t_2$ then for every $x \in B_r$, we have

$$\begin{aligned}
|Bx(t_1) - Bx(t_2)| &\leq \frac{a(1-\alpha)}{a^2+b^2}|g(t_1, x(t_1)) - g(t_2, x(t_2))| \\
&+ \alpha\Big|\Big(\int_0^{t_1} g(s, x(s))ds - \int_0^{t_2} g(s, x(s))ds\Big)\Big| \\
&+ \frac{b^2\alpha}{a^2+b^2}\Big|\Big(\int_0^{t_1} e^{\frac{-a\alpha(t_1-s)}{1-\alpha}} g(s, x(s))ds - \int_0^{t_2} e^{\frac{-a\alpha(t_2-s)}{1-\alpha}} g(s, x(s))ds\Big)\Big| \\
&\leq \frac{a(1-\alpha)L_2}{a^2+b^2}|t_1 - t_2| + \alpha\int_{t_1}^{t_2} |g(s, x(s))|ds + \frac{b^2\alpha}{a^2+b^2}\int_{t_1}^{t_2} |g(s, x(s))|ds \\
&\leq \frac{a(1-\alpha)L_2}{a^2+b^2}|t_1 - t_2| + \alpha\int_{t_1}^{t_2} |h(s)|ds + \frac{b^2\alpha}{a^2+b^2}\int_{t_1}^{t_2} |h(s)|ds \\
&\leq \frac{a(1-\alpha)L_2}{a^2+b^2}|t_1 - t_2| + \alpha|\phi(t_1) - \phi(t_2)| + \frac{b^2\alpha}{a^2+b^2}|\phi(t_1) - \phi(t_2)|
\end{aligned}$$

Since ϕ is continuous on compact $[0, T]$, it is uniformly continuous. Hence

$$\forall > 0 \ \exists\eta > 0 \ : \ |t_1 - t_2| < \eta \Rightarrow |Bx(t_1) - Bx(t_2)| < \varepsilon.$$

for all $t_1, t_2 \in J$ and for all $x \in X$.
This shows that $B(B_r)$ is an equicontinuous set in X.
Then, by the Arzela-Ascoli theorem, B is a continuous and compact operator on B_r.
claim 4 Let $x \in X$ and $y \in B_r$ be arbitrary such that $x = AxBy$.
Then,

$$\begin{aligned}
|x(t)| &= |Ax(t)By(t)| \\
&= \Big|f(t, x(t))\Big(x_0 + \big(I^{\alpha}_{0,a;b} g(., x(.))\big)(t)\Big)\Big| \\
&\leq |f(t, x(t))|\Big(|x_0| + \frac{a(1-\alpha)}{a^2+b^2}|g(t, x(t))| + \alpha\int_0^t |g(s, x(s))|ds + \frac{b^2\alpha}{a^2+b^2}\int_0^t e^{\frac{-a\alpha(t-s)}{1-\alpha}}|g(s, x(s))|ds\Big) \\
&\leq \Big(|f(t, x(t)) - f(t, 0)| + |f(t, 0)|\Big)\Big(|x_0| + \frac{a(1-\alpha)L_2T}{a^2+b^2} + \alpha\int_0^t |h(s)|ds + \frac{b^2\alpha}{a^2+b^2}\int_0^t |h(s)|ds\Big) \\
&\leq \Big(L|x(t)| + F_0\Big)\Big(|x_0| + \frac{a(1-\alpha)L_2T}{a^2+b^2} + \alpha\|h\|_{L^1} + \frac{b^2\alpha}{a^2+b^2}\|h\|_{L^1}\Big)
\end{aligned}$$

and so,

$$|x(t)|\Big(1 - L\Big(|x_0| + \frac{a(1-\alpha)L_2T}{a^2+b^2} + \alpha\|h\|_{L^1} + \frac{b^2\alpha}{a^2+b^2}\|h\|_{L^1}\Big)\Big) \leq F_0\Big(|x_0| + \frac{a(1-\alpha)L_2T}{a^2+b^2} + \alpha\|h\|_{L^1} + \frac{b^2\alpha}{a^2+b^2}\|h\|_{L^1}\Big)$$

Hence,

$$|x(t)| \leq \frac{F_0\Big(|x_0| + \frac{a(1-\alpha)L_2T}{a^2+b^2} + \alpha\|h\|_{L^1} + \frac{b^2\alpha}{a^2+b^2}\|h\|_{L^1}\Big)}{1 - L\Big(|x_0| + \frac{a(1-\alpha)L_2T}{a^2+b^2} + \alpha\|h\|_{L^1} + \frac{b^2\alpha}{a^2+b^2}\|h\|_{L^1}\Big)}$$

Then,

$$\begin{aligned} \|x\| &\leq \frac{F_0\Big(|x_0| + \frac{a(1-\alpha)L_2T}{a^2+b^2} + \alpha\|h\|_{L^1} + \frac{b^2\alpha}{a^2+b^2}\|h\|_{L^1}\Big)}{1 - L\Big(|x_0| + \frac{a(1-\alpha)L_2T}{a^2+b^2} + \alpha\|h\|_{L^1} + \frac{b^2\alpha}{a^2+b^2}\|h\|_{L^1}\Big)} \\ &\leq R \end{aligned}$$

□

4. Exemple

In this section, we give an example to illustrate our main result.
Consider the following hybrid fractional equation

$$\begin{cases} D^{\alpha}_{0,a,b}\Big(\frac{x(t)}{f(t,x(t))}\Big)\Big) = g(t, x(t)) \quad a.e. \quad t \in J = [0, 1] \quad 0 < \alpha < 1, \quad a > 0, \quad b \geq 0 \\ x_0 = \frac{x(0)}{f(0,x(0))} \end{cases} \tag{4}$$

Here $T = 1$, $\alpha = \frac{1}{2}$ and $a = b = 1$.
Set $f(t, x(t)) = \frac{|x(t)|}{1+|x(t)|}$ and $g(t, x(t)) = \frac{t}{cos(x(t))}$.
It is clear that assumption (H_0) is satisfied. Let $u, v \in \mathbb{R}$ and $t \in [0, 1]$. Then we get

$$\begin{aligned} |f(t, u(t)) - f((t, v(t))| &= |\frac{|u(t)|}{1 + |u(t)|} - \frac{|v(t)|}{1 + |v(t)|}| \\ &\leq |\frac{|u(t)| - |v(t)|}{\Big(1 + |u(t)|\Big)\Big(1 + |v(t)|\Big)}| \\ &\leq ||u(t)| - |v(t)|| \\ &\leq |u(t) - v(t)| \end{aligned}$$

Then,

$$|f(t, u) - f((t, v)| \leq |u - v|$$

Thus, the assumption (H_1) holds true.
Moreover, for $u \in \mathbb{R}$ and $t \in [0, 1]$, we get

$$|g(t, u(t))| =|\frac{t}{cos(u(t))}| \leq t$$

And

$$\begin{aligned} |g(t_1, u(t)) - g(t_2, u(t))| &= |\frac{t_1}{cos(u(t))} - \frac{t_2}{cos(u(t))}| \\ &\leq |t_1 - t_2| \end{aligned}$$

Then,

$$|g(t_1, u) - g(t_2, u)| \leq|t_1 - t_2|$$

Thus all the condition of Theorem 3.2 are fulfilled, the hybride fractional problem (4) has a solution on [0, 1].

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