On the $g_z$-Kato decomposition and generalization of Koliha Drazin invertibility

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Abstract. In [24], Koliha proved that $T \in L(X)$ ($X$ is a complex Banach space) is generalized Drazin invertible operator iff there exists an operator $S$ commuting with $T$ such that $STS = S$ and $\sigma(T^2S - T) \subset \{0\}$ iff $0 \notin \text{acc} \sigma(T)$. Later, in [14, 34] the authors extended the class of generalized Drazin invertible operators and they also extended the class of pseudo-Fredholm operators introduced by Mbekhta [27] and other classes of semi-Fredholm operators. As a continuation of these works, we introduce and study the class of $g_z$-invertible (resp., $g_z$-Kato) operators which generalizes the class of generalized Drazin invertible operators (resp., the class of generalized Kato-meromorphic operators introduced by Živković-Zlatanović and Duggal in [35]). Among other results, we prove that $T$ is $g_z$-invertible iff $T$ is $g_z$-Kato with $\bar{\rho}(T) = \bar{q}(T) < \infty$ iff there exists a commuting operator $S$ with $T$ such that $STS = S$ and $\text{acc} \sigma(T^2S - T) \subset \{0\}$ iff $0 \notin \text{acc} \sigma(T)$. As application and using the concept of the Weak SVEP introduced at the end of this paper, we give new characterizations of Browder-type theorems.

1. Introduction

Let $T \in L(X)$, where $L(X)$ is the Banach algebra of bounded linear operators acting on an infinite dimensional complex Banach space $(X, ||||)$. Throughout this paper $T^{*}$, $\alpha(T)$ and $\beta(T)$ means respectively, the dual of $T$, the dimension of the kernel $N(T)$ and the codimension of the range $R(T)$. The ascent and the descent of $T$ are defined by $p(T) = \text{inf} \{ n \in \mathbb{N} : N(T^n) = N(T^{n+1}) \}$ (with $\text{inf} \emptyset = \infty$) and $q(T) = \text{inf} \{ n \in \mathbb{N} : R(T^n) = R(T^{n+1}) \}$. A subspace $M$ of $X$ is $T$-invariant if $T(M) \subset M$ and the restriction of $T$ on $M$ is denoted by $T_M$. $(M, N) \in \text{Red}(T)$ if $M, N$ are closed $T$-invariant subspaces and $X = M \oplus N$. $M \cap N = \{0\}$. Let $n \in \mathbb{N}$, denote by $T_{[n]} = T|_{R(T^n)}$ and by $m_T = \text{inf} \{ n \in \mathbb{N} : \beta(T_{[n]}) \leq \infty \}$ the essential degree of $T$. According to [10, 28], $T$ is called upper semi-B-Fredholm (resp., lower semi-B-Fredholm) if the essential ascent $p_T(T) = \text{inf} \{ n \in \mathbb{N} : \alpha(T_{[n]}) \leq \infty \} < \infty$ and $R(T^{p_T(T)+1})$ is closed (resp., the essential descent $q_T(T) = \text{inf} \{ n \in \mathbb{N} : \beta(T_{[n]}) \leq \infty \} < \infty$ and $R(T^{q_T(T)})$ is closed). If $T$ is an upper or a lower (resp., upper and lower) semi-B-Fredholm, then $T$ is called semi-B-Fredholm (resp., B-Fredholm) and its index is defined by $\text{ind}(T) = \beta(T_{[m_T]}) - \alpha(T_{[m_T]})$. $T$ is said to be an upper semi-B-Weyl (resp., lower semi-B-Weyl, B-Weyl, left Drazin invertible, right Drazin invertible, Drazin invertible) if $T$ is an upper semi-B-Fredholm with $\text{ind}(T) \leq 0$ (resp., $T$ is a lower semi-B-Fredholm with $\text{ind}(T) \geq 0$, $T$ is a B-Fredholm with
ind(T) = 0, T is an upper semi-B-Fredholm and \( p(T_{[m]}) < \infty \), T is a lower semi-B-Fredholm and \( q(T_{[m]}) < \infty \), \( p(T_{[m]}) = q(T_{[m]}) < \infty \). If T is upper semi-B-Fredholm (resp., lower semi-B-Fredholm, semi-B-Fredholm, B-Fredholm, upper semi-B-Weyl, lower semi-B-Weyl, B-Weyl, left Drazin invertible, right Drazin invertible, Drazin invertible) with essential degree \( m_T = 0 \), then T is said to be an upper semi-Fredholm (resp., lower semi-Fredholm, semi-Fredholm, Fredholm, upper semi-Weyl, lower semi-Weyl, Weyl, upper semi-Browder, lower semi-Browder, Browder) operator. T is said to be bounded below if T is upper semi-Fredholm with \( a(T) = 0 \).

The degree of stable iteration of T is defined by \( \text{dis}(T) = \inf \Delta(T) \), where

\[
\Delta(T) = \{ m \in \mathbb{N} : a(T_{[m]}) = a(T_{[r]}), \forall r \in \mathbb{N} \ r > m \}.
\]

T is said to be semi-regular if \( \mathcal{R}(T) \) is closed and \( \text{dis}(T) = 0 \), and is said to be quasi-Fredholm if there exists \( n \in \mathbb{N} \) such that \( \mathcal{R}(T^n) \) is closed and \( T_{[n]} \) is semi-regular, see [25, 27]. Note that every semi-B-Fredholm operator is quasi-Fredholm [10, Proposition 2.5].

According to [1], T is said to have the SVEP at \( \lambda \in \mathbb{C} \) if for every open neighborhood \( U_\lambda \) of \( \lambda \), \( \mathcal{R}(T - \mu I) \neq \emptyset \) is the only analytic solution of the equation \( (T - \mu I)f(\mu) = 0 \) \( \forall \mu \in U_\lambda \). T is said to have the SVEP on \( A \subseteq \mathbb{C} \) if T has the SVEP at every \( \lambda \in A \), and is said to have the SVEP if it has the SVEP on \( \mathbb{C} \). It is easily seen that \( T \oplus S \) has the SVEP at \( \lambda \) if and only if T and S have the SVEP at \( \lambda \), see [1, Theorem 2.15]. Moreover,

\[
p(T - \lambda I) < \infty \implies T \text{ has the SVEP at } \lambda \quad (A)
\]

\[
q(T - \lambda I) < \infty \implies T^* \text{ has the SVEP at } \lambda \quad (B)
\]

and these implications become equivalences if \( T - \lambda I \) has topological uniform descent [1, Theorem 2.97, Theorem 2.98]. For definitions and properties of operators which have topological uniform descent, see [18].

**Definition 1.1.** [1] (i) The local spectrum of T at \( x \in X \) is the set defined by

\[
\sigma_T(x) := \{ \lambda \in \mathbb{C} : \text{ for all open neighborhood } U_\lambda \text{ of } \lambda \text{ and analytic function } f : U_\lambda \to X \text{ there exists } \mu \in U_\lambda \text{ such that } (T - \mu I)f(\mu) \neq x \}.
\]

(ii) If \( F \) is a complex closed subset, then the local spectral subspace of T associated to F is defined by

\[
X_T(F) = \{ x \in X : \sigma_T(x) \subseteq F \}.
\]

A Banach space operator S is said to be nilpotent of degree \( d \) if \( S^d = 0 \) and \( S^{d-1} \neq 0 \) [with the degree of the null operator takes 0 if it acts on the space \( \{0\} \) and takes 1 otherwise]. S is a quasi-nilpotent (resp., Riesz, meromorphic) operator if \( S - \lambda I \) is invertible (resp., Browder, Drazin invertible) for all non-zero complex \( \lambda \). Note that S is nilpotent \( \implies \) S is quasi-nilpotent \( \implies \) S is Riesz \( \implies \) S is meromorphic. Denote by \( \mathcal{K}(T) \) the analytic core of T (see [27]):

\[
\mathcal{K}(T) = \{ x \in X : \exists \epsilon > 0, \exists (u_n)_n \subset X \text{ such that } x = u_0, Tu_{n+1} = u_n \text{ and } ||u_n|| \leq \epsilon ||x|| \forall n \in \mathbb{N} \},
\]

and by \( \mathcal{H}_0(T) \) the quasi-nilpotent part of T :

\[
\mathcal{H}_0(T) = \{ x \in X : \lim_{n \to \infty} ||T^nx||^{1/2} = 0 \}.
\]

In [23, Theorem 4, 1958], Kato proved that if T is a semi-Fredholm operator, then T is of Kato-type of degree \( d \), that is there exists \( (M, N) \in \text{Red}(T) \) such that:

(i) \( T_M \) is semi-regular.

(ii) \( T_N \) is nilpotent of degree \( d \).

Later, these operators are characterized by Labrousse [25, 1980] in the case of Hilbert space. The important results obtained by Kato and Labrousse opened the field to many researchers to work in this direction [7, 11, 14, 16, 27, 33–35]. In particular, Berkani [7] showed that T is B-Fredholm (resp., B-Weyl) if and only if there exists \( (M, N) \in \text{Red}(T) \) such that \( T_M \) is Fredholm (resp., Weyl) and \( T_N \) is nilpotent. On the other hand,
it is well known [16] that T is Drazin invertible if and only if there exists \((M, N) \in \text{Red}(T)\) such that \(T_M\) is invertible and \(T_N\) is nilpotent.

If the condition (ii) “\(T_N\) is nilpotent” mentioned in the Kato’s decomposition is replaced by “\(T_N\) is quasi-nilpotent” (resp., “\(T_N\) is Riesz”, “\(T_N\) is meromorphic”), we find the pseudo-Fredholm [27] (resp., generalized Kato-Riesz [34], generalized Kato-meromorphic [35]) decomposition. By the same argument the pseudo B-Fredholm [32, 33] (resp., generalized Drazin-Riesz Fredholm [11, 34], generalized Drazin-meromorphic Fredholm [35]) decomposition are obtained by substituting in the B-Fredholm decomposition the condition “\(T_N\) is nilpotent” by “\(T_N\) is quasi-nilpotent” (resp., “\(T_N\) is Riesz”, “\(T_N\) is meromorphic”). Similarly, the Drazin decomposition has been generalized [24, 34, 35].

We summarize in the following definition several known decompositions.

**Definition 1.2.** [5, 7, 10–12, 14, 27, 33–35] T is said to be

(i) of Kato-type of order \(d\) [resp., quasi upper semi-B-Fredholm, quasi lower semi-B-Fredholm, quasi B-Fredholm, quasi upper semi-B-Weyl, quasi lower semi-B-Weyl, quasi semi-B-Weyl] if there exists \((M, N) \in \text{Red}(T)\) such that \(T_M\) is semi-regular [resp., upper semi-Fredholm, lower semi-Fredholm, Fredholm, upper semi-Weyl, lower semi-Weyl, Weyl] and \(T_N\) is nilpotent of degree \(d\). We write \((M, N) \in \text{KD}(T)\) if it is a Kato-type decomposition.

(ii) Pseudo-Fredholm [resp., upper pseudo semi-B-Fredholm, lower pseudo semi-B-Fredholm, pseudo B-Fredholm, upper pseudo semi-B-Weyl, lower pseudo semi-B-Weyl, pseudo B-Weyl, left generalized Drazin invertible, right generalized Drazin invertible, generalized Drazin invertible] if there exists \((M, N) \in \text{Red}(T)\) such that \(T_M\) is semi-regular [resp., upper semi-Fredholm, lower semi-Fredholm, Fredholm, upper semi-Weyl, lower semi-Weyl, Weyl, bounded below, surjective, invertible] and \(T_N\) is quasi-nilpotent. We write \((M, N) \in \text{GKD}(T)\) if it is a pseudo-Fredholm type decomposition.


As a continuation of the studies mentioned above, we define new classes of operators: one of them named \(g_z\)-Kato which generalizes the class of generalized Kato-meromorphic operators. We prove that the \(g_z\)-Kato spectrum \(\sigma_{g_z}(T)\) is compact and \(\text{acc} \sigma(T) \subset \sigma_{g_z}(T)\). Moreover, we show that if \(T\) is \(g_z\)-Kato, then \(a(T_M), \beta(T_M), p(T_M)\) and \(q(T_M)\) are independent of the choice of the decomposition \((M, N) \in g_z\text{KD}(T)\).

An other class named \(g_z\)-invertible which generalizes the class of generalized Drazin invertible operators introduced by Koliha. As a characterization of \(g_z\)-invertible operator, we prove that \(T\) is \(g_z\)-invertible iff \(0 \notin \text{acc} \text{ (acc } \sigma(T))\) iff there exists a Drazin invertible operator \(S\) such that \(TS = ST, STS = S\) and \(T^2S - T\) is zeroloid. These characterizations are analogous to those proved by Koliha [24] which established that \(T\) is generalized Drazin invertible operator iff \(0 \notin \sigma(T)\) iff there exists an operator \(S\) such that \(TS = ST, STS = S\) and \(T^2S - T\) is quasi-nilpotent. As application, using the new spectra studied in the present work and the concept of the Weak SVEP introduced at the end of this paper, we give new characterizations of Browder-type theorems.
The next list summarizes some notations and symbols that we will need later.

\[ r(T) \colon \text{the spectral radius of } T \]
\[ \text{iso } A \colon \text{isolated points of a complex subset } A \]
\[ \text{acc } A \colon \text{accumulation points of a complex subset } A \]
\[ \overline{A} \colon \text{the closure of a complex subset } A \]
\[ A^C \colon \text{the complementary of a complex subset } A \]
\[ B(\lambda, \epsilon) \colon \text{the open ball of radius } \epsilon \text{ centered at } \lambda \]
\[ D(\lambda, \epsilon) \colon \text{the closed ball of radius } \epsilon \text{ centered at } \lambda \]

(i) A zeroloid operator has at most a countable spectrum.

(ii) Since acc \( \mathcal{A} \) \(+\) shows that the converse is not true, where I is the identity operator and \( Q \) is the quasi-nilpotent operator defined on \( H \).

(iii) \( T \) is zeroloid if and only if \( T \) is \( \mathcal{A} \)-satisfying Browder's theorem \((T \in (B) \text{ if } \sigma_w(T) = \sigma_b(T))\).

(iv) Let \( Q \) \( \in \mathcal{A} \) be any subspace which complemented by \( M \), \( N \) \( \in \mathcal{K}(T) \), then \( T \oplus S \) is zeroloid if and only if \( T \) and \( S \) are zeroloid.

Here and elsewhere denote by \( \text{comm}(T) = \{ S \in L(X) : TS = ST \} \). So if \( Q \in \text{comm}(T) \) is a quasi-nilpotent or a power finite rank operator, then \( T \) is zeroloid if and only if \( T + Q \) is zeroloid.

According to [4], the p-ascent \( \tilde{p}(T) \) and the p-descent \( \tilde{q}(T) \) of a pseudo-Fredholm operator \( T \in L(X) \) are defined respectively, by \( \tilde{p}(T) = p(T_M) \) and \( \tilde{q}(T) = q(T_M) \), where \( M \) is any subspace which complemented by a subspace \( N \) such that \( (M, N) \in \mathcal{GKD}(T) \).

Proposition 2.3. If \( T \in L(X) \) is a pseudo-Fredholm operator, then the following statements are equivalent:

(a) \( \tilde{p}(T) < \infty \);
(b) \( T \) has the SVEP at 0;
(c) \( \mathcal{H}_0(T) \cap \mathcal{K}(T) = \{0\} \);
(d) \( \mathcal{H}_0(T) \) is closed.

dually, the following are equivalent:

(e) \( \tilde{q}(T) < \infty \);
\((f)\) \(T^*\) has the SVEP at 0;  
\((g)\) \(H_0(T) + \mathcal{K}(T) = X.\)

**Proof.** (a) \(\iff\) (b) Let \((M, N) \in GKD(T),\) then \(T_M\) is semi-regular and \(T_N\) is quasi-nilpotent. As \(p(T_M) = \rho(T)\) then by the implication (A) above, we deduce that \(\rho(T) < \infty\) if and only if \(T_M\) has the SVEP at 0. Hence \(\rho(T) < \infty\) if and only if \(T\) has the SVEP at 0. The equivalence (e) \(\iff\) (f) goes similarly. The equivalences (b) \(\iff\) (c), (c) \(\iff\) (d) and (f) \(\iff\) (g) are proved in [1, Theorem 2.79, Theorem 2.80]. \(\square\)

**Lemma 2.4.** For \(T \in L(X),\) the following statements are equivalent:

(i) \(T\) is zeroloid;

(ii) \(\sigma_\alpha(T) \subset \{0\},\) where \(\sigma_\alpha \in \{\sigma_{pf}, \sigma_{\text{upb}}, \sigma_{\text{qp}}, \sigma_{\text{gb}}, \sigma_{\text{r}}, \sigma_{\text{pb}}, \sigma_{\text{gb}}\}.\)

**Proof.** (i) \(\implies\) (ii) Obvious, since \(\sigma_{\text{ad}}(T) = \text{acc } \sigma(T).\)

(ii) \(\implies\) (i) If \(\sigma_\alpha(T) \subset \{0\},\) then \(C(\{0\}) \subset \Omega,\) where \(\Omega\) is the component of \(\sigma_{\text{pf}}(T)\). Suppose that there exists \(\lambda \in \text{acc } \sigma(T) \setminus \{0\},\) then \(\lambda \notin \sigma_\alpha(T)\) and hence \(\rho(T - \lambda I) = \infty\) or \(\rho(T - \lambda I) = \infty,\) but this is impossible. Indeed, assume that \(\rho(T - \lambda I) = \infty,\) as \(T - \lambda I\) is pseudo-Fredholm, from Proposition 2.3 we have \(H_0(T - \lambda I) \cap \mathcal{K}(T - \lambda I) = \{0\}.\)

And from [12, Corollary 4.3], we obtain \(H_0(T - \lambda I) \cap \mathcal{K}(T - \lambda I) = H_0(T - \mu I) \cap \mathcal{K}(T - \mu I)\) for every \(\mu \in \Omega.\) This implies that \(\rho(T - \mu I) = \infty\) for all \(\mu \in \Omega \setminus \{0\}\) [otherwise \(H_0(T - \mu I)\) becomes closed for some \(\mu \in \Omega \setminus \{0\}\) and then \(\mathcal{H}(T - \lambda I) \cap \mathcal{K}(T - \lambda I) = \{0\},\) which is impossible] and this is contradiction. Thus \(\rho(T - \lambda I) = \infty,\) but this leads (by the same argument) to a contradiction. Hence \(T\) is zeroloid. \(\square\)

**Proposition 2.5.** \(T \in L(X)\) is zeroloid if and only if \(T_M\) and \(T_{M^*}\) are zeroloid, where \(M\) is any closed \(T\)-invariant subspace.

**Proof.** If \(T\) is zeroloid, then its resolvent \((\sigma(T))^C\) is connected. From [15, Proposition 2.10], we obtain that \(\sigma(T) = \sigma(T_M) \cup \sigma(T_{M^*}).\) Thus \(T_M\) and \(T_{M^*}\) are zeroloid. Conversely, if \(T_M\) and \(T_{M^*}\) are zeroloid, then \(T\) is zeroloid, since the inclusion \(\sigma(T) \subset \sigma(T_M) \cup \sigma(T_{M^*})\) is always true. \(\square\)

**Definition 2.6.** Let \(T \in L(X).\) A pair of subspaces \((M, N) \in \text{Red}(T)\) is a generalized Kato zeroloid decomposition associated to \(T\) if \((M, N) \in gKD(T)\) for brevity if \(T_M\) is semi-regular and \(T_N\) is zeroloid. If such a pair exists, we say that \(T\) is a \(g_z\)-Kato operator.

**Example 2.7.** (i) Every zeroloid operator and every semi-regular operator are \(g_z\)-Kato.  
(ii) Every generalized Kato-meromorphic operator is \(g_z\)-Kato. But the converse is not true, see Example 4.13 below.

Our next result gives a punctured neighborhood theorem for \(g_z\)-Kato operators. Recall that the reduced minimal modulus \(\gamma(T)\) of an operator \(T\) is defined by \(\gamma(T) = \inf_{x \notin N(T)} \|Tx\| / \text{dist}(x, N(T))\), where \(d(x, N(T))\) is the distance between \(x\) and \(N(T)\).

**Theorem 2.8.** Let \(T \in L(X)\) be a \(g_z\)-Kato operator. For every \((M, N) \in g_zKD(T),\) there exists \(\epsilon > 0\) such that for all \(\lambda \in B(0, \epsilon) \setminus \{0\}\) we have

(i) \(T - \lambda I\) is pseudo-Fredholm.

(ii) \(\alpha(T_M) = \dim N(T - \lambda I) \cap \mathcal{K}(T - \lambda I) \leq \alpha(T - \lambda I).\)

(iii) \(\beta(T_M) = \text{codim } \overline{\mathcal{R}(T - \lambda I) + \mathcal{H}_\Omega(T - \lambda I)} \leq \beta(T - \lambda I).\)

**Proof.** Let \(\epsilon = \gamma(T_M) > 0\) and let \(\lambda \in B(0, \epsilon) \setminus \{0\}.\) From [18, Theorem 4.7], \(T_M - \lambda I\) is semi-regular, \(\alpha(T_M) = \alpha(T_M - \lambda I)\) and \(\beta(T_M) = \beta(T_M - \lambda I).\) As \(T_N\) is zeroloid then from [4], \(T_N - \lambda I\) is pseudo-Fredholm with \(N(T - \lambda I) \cap \mathcal{K}(T - \lambda I) = \{0\}\) and \(N = \mathcal{R}(T - \lambda I) + \mathcal{H}_\Omega(T - \lambda I).\) Hence \(T - \lambda I\) is pseudo-Fredholm, \(\alpha(T_M) = \dim N(T - \lambda I) \cap \mathcal{K}(T - \lambda I)\) and \(\beta(T_M) = \text{codim } \overline{\mathcal{R}(T - \lambda I) + \mathcal{H}_\Omega(T - \lambda I)}.\) \(\square\)

Since every pseudo-Fredholm operator is \(g_z\)-Kato, from Theorem 2.8 we immediately obtain the following corollary. Hereafter, we denote by \(\sigma_{g_K}(T) = \{\lambda \in \mathbb{C} : T - \lambda I\) is not \(g_z\)-Kato operator\} the \(g_z\)-Kato spectrum.

**Corollary 2.9.** The \(g_z\)-Kato spectrum \(\sigma_{g_K}(T)\) of an operator \(T \in L(X)\) is compact.
Proposition 2.10. If \( T \in L(X) \) is a \( g_z \)-Kato operator, then \( \alpha(T_M), \beta(T_M), p(T_M) \) and \( q(T_M) \) are independent of the choice of the generalized Kato zeroloid decomposition \((M, N) \in g_zKD(T)\).

Proof. Let \((M_1, N_1), (M_2, N_2) \in g_zKD(T)\) and let \( n \geq 1\). It is easily seen that \( T^n \) is also a \( g_z \)-Kato operator and \((M_1, N_1), (M_2, N_2) \in g_zKD(T^n)\). We put \( \epsilon_n = \min_{\lambda \in B(0, \epsilon_n)} |\gamma(T^n_{M_2})|\). If \( \lambda \in B(0, \epsilon_n) \), then by Theorem 2.8 we obtain \( \alpha(T^n_{M_2}) = \alpha(T^n_{M_1}) = \dim N(T^n_{M_1} - \lambda I) \cup K(T^n_{M_1} - \lambda I) \) and \( \beta(T^n_{M_2}) = \beta(T^n_{M_1}) = \text{codim} [R(T^n_{M_1} - \lambda I) + H_0(T^n_{M_1} - \lambda I)]\).

Hence \( p(T_M) = p(T_{M_{1}}) \) and \( q(T_M) = q(T_{M_{1}}) \). \( \square \)

Let \( T \in L(X) \) be a \( g_z \)-Kato operator. Following Proposition 2.10, we denote by \( \alpha(T) = \alpha(T_M), \beta(T) = \beta(T_M), p(T) = p(T_M) \) and \( q(T) = q(T_M) \), where \((M, N) \in g_zKD(T)\) be arbitrary. If in addition, \( T_M \) is semi-Fredholm, then for every \((M', N') \in g_zKD(T)\) the operator \( T_{M'} \) is also semi-Fredholm and \( \text{ind}(T_M) = \text{ind}(T_{M'}) \) (this result will be extended in Lemma 3.4).

The next lemma extends [30, Theorem A.16]. In the sequel, for \( T \in L(X) \) and \((M, N) \in \text{Red}(T)\), we define the operator \( T_{(M,N)} \in L(X) \) by \( T_{(M,N)} = TP_M + P_N \), where \( P_M \) is the projection operator on \( X \) onto \( M \).

Lemma 2.11. Let \( T \in L(X) \) and let \((M, N) \in \text{Red}(T)\). The following assertions are equivalent:

(i) \( \mathcal{R}(T_{MN}) \) is closed;
(ii) \( \mathcal{R}(T_{M}) \) is closed;
(iii) \( \mathcal{R}(T_{N_{M}}) \oplus M^\perp \) is closed in the weak*-topology \( \sigma(X^*, X) \) on \( X^* \).

Proof. As \((M, N) \in \text{Red}(T)\) then \( (P_N)^* = P_{M^\perp} \) and \( (TP_M)^* = T^*P_{M^\perp} \). So \( (T_{(M,N)})^* = (TP_M + P_N)^* = T^*P_{M^\perp} \oplus M^\perp \). Thus \( \mathcal{R}(T_{(M,N)}) = \mathcal{R}(T_{M}) \oplus N \) and \( \mathcal{R}(T_{(M,N)})^* = \mathcal{R}(T_{N_{M}})^* \oplus M^\perp \). Moreover, \( \mathcal{R}(T_{(M,N)}) \) is closed if and only if \( \mathcal{R}(T_{(M,N)})^* \) is closed. By applying [30, Theorem A.16] to the operator \( T_{(M,N)} \), the proof is complete. \( \square \)

From this Lemma and some known classical properties of pseudo-Fredholm and quasi-Fredholm operators, we immediately obtain:

Corollary 2.12. Let \( T \in L(X) \). The following statements hold:

(i) \( \mathcal{R}(T^n) + \mathcal{H}_0(T^n) \) is closed in \( \sigma(X^*, X) \).
(ii) \( \mathcal{R}(T^n) + \mathcal{H}_0(T^n) \) is closed in \( \sigma(X^*, X) \).

The following Lemma extends some well known results in spectral theory, as relation between nullity, deficiency and some other spectral quantities of a given operator \( T \) and its dual \( T^* \).

Lemma 2.13. Let \( T \in L(X) \) and let \((M, N) \in \text{Red}(T)\). The following statements hold:

(i) \( T_M \) is semi-regular if and only if \( T_{N_{M}} \) is semi-regular.
(ii) \( \mathcal{R}(T_{M}) \) is closed, then \( \alpha(T_M) = \beta(T_{N_{M}}), \beta(T_M) = \alpha(T_{N_{M}}), p(T_M) = q(T_{N_{M}}) \) and \( q(T_M) = p(T_{N_{M}}) \).
(iii) \( \sigma_0(T_M) = \sigma_0(T_{N_{M}}), \sigma_1(T_M) = \sigma_1(T_{N_{M}}), \sigma_2(T_M) = \sigma_2(T_{N_{M}}) \) and \( r(T_M) = r(T_{N_{M}}) \), where \( \sigma_n \in \{\sigma_{0n}, \sigma_{1n}, \sigma_{2n}, \sigma_0, \sigma_1, \sigma_2\} \). Moreover, if \( T_M \) is semi-Fredholm, then \( \text{ind}(T_{N_{M}}) = -\text{ind}(T_{N_{M}}) \).

Proof. (i) We have \( \mathcal{N}(T_{MN}) = \mathcal{N}(T_M) \) and \( \mathcal{N}(T_{MN}^n) = T_{MN}^n \) for every \( n \in \mathbb{N} \). It is easy to see that \( T_M \) is semi-regular if and only if \( T_{MN} \) is semi-regular. As \( T_{MN}^n = T_{MN}^n \) then \( T_M \) is semi-regular if and only if \( T_{N_{M}} \) is semi-regular.
(ii) We have \( \mathcal{N}(T_{MN}^n) = \mathcal{N}(T_{M}^n) \) and \( \mathcal{R}(T_{(M,N)}^n) = \mathcal{R}(T_{M}^n) \). The other equalities go similarly.
(iii) As \( T_M \oplus 0_{N} \) and \( T_{N_{M}} \oplus 0_{M} \), then \( \sigma_0(T_M) \cup \sigma_0(0_{N}) = \sigma_0(T_M) \cup 0_{N_{M}} = \sigma_0(T_{N_{M}}) \cup 0_{M_{M}} = \sigma_0(T_{N_{M}}) \cup \sigma_0(0_{M_{M}}) \). We know that \( \sigma_n(S) = \emptyset \) for every nilpotent operator \( S \) with \( \sigma_n \in \{\sigma_{0n}, \sigma_{1n}, \sigma_{2n}, \sigma_0, \sigma_1, \sigma_2\} \). Furthermore, the first and the second points imply that \( \emptyset \in \sigma_n(T_M) \) if and only if \( \emptyset \in \sigma_n(T_{N_{M}}) \), where \( \sigma_n \in \{\sigma_{0n}, \sigma_{1n}, \sigma_{2n}, \sigma_0, \sigma_1, \sigma_2\} \). So \( \sigma_n(T_M) = \sigma_n(T_{N_{M}}) \) and \( r(T_M) = r(T_{N_{M}}) \). The proof of the other equalities is obvious, see Lemma 2.11. Moreover, if \( T_M \) is semi-Fredholm, then \( T_{N_{M}} \) is also semi-Fredholm and \( \text{ind}(T_{N_{M}}) = -\text{ind}(T_{N_{M}}) \). \( \square \)

Corollary 2.14. Let \( T \in L(X) \) and let \((M, N) \in \text{Red}(T)\). Then \((M, N) \in g_zKD(T) \) if and only if \((N^\perp, M^\perp) \in g_zKD(T^*) \).

In particular, if \( T \) is \( g_z \)-Kato, then \( T^* \) is \( g_z \)-Kato.
Proposition 2.15. If \( T \in L(X) \) is \( g_{z}\)-Kato, then
(a) There exist \( S, R \in L(X) \) such that:
   (i) \( T = S + R \), \( RT = TR = 0 \), \( S \) is quasi-Fredholm of degree \( d \leq 1 \) and \( R \) is zero-lid.
   (ii) \( N(S) + N(R) = X \) and \( R(S) \oplus \overline{R(R)} \) is closed.
(b) There exist \( S, R \in L(X) \) such that \( SR = RS = (S + R) - I = T \), \( S \) is semi-regular and \( R \) is zero-lid.

Proof. (a) Let \((M, N) \in g_{z} KD(T)\). The operators \( S = TP_{M} \) and \( R = TP_{N} \) respond to the statement (a). Indeed, as \( T_{N} \) is zero-lid and \( \text{acc} \sigma(R) = \text{acc} \sigma(T_{N}) \) then \( R \) is zero-lid. Suppose that \( M \notin \{0, X\} \) (the other case is trivial) and let \( n \in \mathbb{N} \geq 1 \), then \( N(S^{n}) = N \oplus N(T_{N}^{n}) \) and \( R(S) = \overline{R(T_{M})} \) is closed. As \( T_{M} \) is semi-regular, it follows that \( N(S^{n}) + R(S) = N \oplus N(T_{N}^{n}) + \overline{R(T_{M})} = N \oplus N(T_{M}) = N(S) + R(S) \). Consequently, \( S \) is quasi-Fredholm of degree \( d \leq 1 \). Moreover, \( N(S) + N(R) = X \) and \( R(S) \oplus \overline{R(R)} = \overline{R(T_{M})} \) is closed.
(b) Let \((M, N) \in g_{z} KD(T)\). If we take \( S = T_{(M,N)} \) and \( R = T_{(N,M)} \), then \( SR = RS = (S + R) - I = T \), \( S = T_{M} \oplus I_{N} \) is semi-regular and \( R = I_{M} \oplus T_{N} \) is zero-lid. \( \square \)

In the case of Hilbert space operator \( T \), the next proposition shows that the statement (a) of Proposition 2.15 is equivalent to say that \( T \) is \( g_{z}\)-Kato.

Proposition 2.16. If \( H \) is a Hilbert space, then \( T \in L(H) \) is \( g_{z}\)-Kato if and only if there exist \( S, R \in L(H) \) such that \( T = S + R \) and
(i) \( RT = TR = 0 \), \( S \) is quasi-Fredholm of degree \( \text{dis}(S) \leq 1 \), \( R \) is a zero-lid operator;
(ii) \( N(S) + N(R) = H \) and \( R(S) \oplus \overline{R(R)} \) is closed.

Proof. Assume that \( S \) is quasi-Fredholm of degree \( 1 \) (the case of semi-regular is obvious), then from the proof of [27, Theorem 2.2], there exists \((M, N) \in GKD(S)\) such that \( T_{M} = S_{M} \) and \( T_{N} = R_{N} \). As \( R \) is zero-lid then Proposition 2.5 entails that \( T_{N} \) is zero-lid. Thus \( T \) is \( g_{z}\)-Kato. For the converse, see Proposition 2.15. \( \square \)

3. \( g_{z}\)-Fredholm operators

Definition 3.1. \( T \in L(X) \) is said to be an upper semi-\( g_{z}\)-Fredholm (resp., lower semi-\( g_{z}\)-Fredholm, \( g_{z}\)-Fredholm) operator if there exists \((M, N) \in \text{Red}(T)\) such that \( T_{M} \) is an upper semi-Fredholm (resp., lower semi-Fredholm, Fredholm) operator and \( T_{N} \) is zero-lid. \( T \) is said a semi-\( g_{z}\)-Fredholm if it is an upper or a lower semi-\( g_{z}\)-Fredholm.

Every zero-lid operator is \( g_{z}\)-Fredholm. Every generalized Draizin-meromorphic semi-Fredholm is a semi-\( g_{z}\)-Fredholm, and we show by Example 4.13 that the converse is generally not true.

The next proposition gives some relations between semi-\( g_{z}\)-Fredholm and \( g_{z}\)-Kato operators.

Proposition 3.2. Let \( T \in L(X) \). The following statements are equivalent:
(i) \( T \) is semi-\( g_{z}\)-Fredholm [resp., upper semi-\( g_{z}\)-Fredholm, lower semi-\( g_{z}\)-Fredholm, \( g_{z}\)-Fredholm];
(ii) \( T \) is \( g_{z}\)-Kato and \( \min\{\alpha(T), \beta(T)\} < \infty \) [resp., \( T \) is \( g_{z}\)-Kato and \( \beta(T) < \infty \), \( T \) is \( g_{z}\)-Kato and \( \max\{\alpha(T), \beta(T)\} < \infty \)];
(iii) \( T \) is \( g_{z}\)-Kato and \( 0 \notin \sigma_{upf}(T) \) [resp., \( T \) is \( g_{z}\)-Kato and \( 0 \notin \sigma_{upf}(T) \), \( T \) is \( g_{z}\)-Kato and \( 0 \notin \sigma_{upf}(T) \)] where \( \sigma_{upf}(T) := \sigma_{upf}(T) \cup \sigma_{upf}(T) \).

Proof. (i) \( \iff \) (ii) Assume that \( T \) is semi-\( g_{z}\)-Fredholm, then there exists \((A, B) \in \text{Red}(T)\) such that \( T_{A} \) is semi-Fredholm and \( T_{B} \) is zero-lid. From [5, Corollary 3.7], there exists \((M, N) \in g_{z} KD(T)\) such that \( T_{M} \) is semi-Fredholm. Thus \( T \) is \( g_{z}\)-Kato operator and \( \min\{\alpha(T), \beta(T)\} = \min\{\alpha(T_{M}), \beta(T_{M})\} < \infty \). The converse is obvious. The other equivalence cases go similarly.
(ii) \( \iff \) (iii) Is a consequence of Theorem 2.8. \( \square \)

Corollary 3.3. \( T \in L(X) \) is \( g_{z}\)-Fredholm if and only if \( T \) is an upper and a lower semi-\( g_{z}\)-Fredholm.

The following lemma will allow us to define the index for semi-\( g_{z}\)-Fredholm operators.

Lemma 3.4. Let \( T \in L(X) \). If there exist two pair of closed \( T \)-invariant subspaces \((M, N) \) and \((M', N') \) such that \( M \oplus N = M' \oplus N' \) is closed, \( T_{M} \) and \( T_{M'} \) are semi-Fredholm, \( T_{N} \) and \( T_{N'} \) are zero-lid, then \( \text{ind}(T_{M}) = \text{ind}(T_{M'}) \).
Proof. As $T_M$ and $T_M'$ are semi-Fredholm operators then from the punctured neighborhood theorem for semi-Fredholm operators, there exists $\epsilon > 0$ such that $B(0, \epsilon) \subset \sigma_f(T_M)^c \cap \sigma_{sf}(T_M')^c$, $\ind(T_M - \lambda I) = \ind(T_M)$ and $\ind(T_M' - \lambda I) = \ind(T_M')$ for every $\lambda \in B(0, \epsilon)$. From [4, Remark 2.4] and the fact that $T_N$ and $T_N'$ are zeroold, we conclude that $B_0 := B(0, \epsilon) \setminus \{0\} \subset \sigma_f(T_M)^c \cap \sigma_{sf}(T_M')^c \cap \sigma_{sf}(T_N)^c \cap \sigma_{sf}(T_N')^c \subset \sigma_{sf}(T_{M \oplus N})^c$. Let $\lambda \in B_0$, then $(T - \lambda I)_{M \oplus N}$ is pseudo-B-Fredholm and $\ind((T - \lambda I)_{M \oplus N}) = \ind(T_M - \lambda I) + \ind(T_N - \lambda I) = \ind(T_M' - \lambda I) + \ind(T_N' - \lambda I)$. Thus $\ind(T_M) = \ind(T_M')$. \qed

Definition 3.5. Let $T \in L(X)$ be a semi-$g_z$-Fredholm. We define its index $\ind(T)$ as the index of $T_M$, where $M$ is a closed $T$-invariant subspace which has a complementary closed $T$-invariant subspace $N$ such that $T_M$ is semi-Fredholm and $T_N$ is zeroold. From Lemma 3.4, the index of $T$ is independent of the choice of the pair $(M, N)$ appearing in Definition 3.1 of $T$ as a semi-$g_z$-Fredholm. In addition, we have from Proposition 3.2, $\ind(T) = \check{\alpha}(T) = \bar{\beta}(T)$.

We say that $T \in L(X)$ is an upper semi-$g_z$-Weyl (resp., lower semi-$g_z$-Weyl, $g_z$-Weyl) operator if $T$ is an upper semi-$g_z$-Fredholm (resp., lower semi-$g_z$-Fredholm, $g_z$-Fredholm) with $\ind(T) \leq 0$ (resp., $\ind(T) \geq 0$, $\ind(T) = 0$).

Remark 3.6. (i) Every zeroold operator $T$ is $g_z$-Fredholm with $\check{\alpha}(T) = \check{\beta}(T) = \ind(T) = 0$. A pseudo-B-Fredholm operator $T$ is semi-$g_z$-Fredholm and its usual index coincides with its index as a semi-$g_z$-Fredholm.

(ii) $T$ is $g_z$-Fredholm if and only if $T$ is semi-$g_z$-Fredholm with an integer index. And $T$ is $g_z$-Weyl if and only if $T$ is upper and lower semi-$g_z$-Weyl.

Proposition 3.7. If $T \in L(X)$ and $S \in L(Y)$ are semi-$g_z$-Fredholm, then

(i) $T^n$ is semi-$g_z$-Fredholm and $\ind(T^n) = n\ind(T)$ for every integer $n \geq 1$.

(ii) $T \in S$ is semi-$g_z$-Fredholm and $\ind(T + S) = \ind(T) + \ind(S)$.

Proof. (i) As $T$ is semi-$g_z$-Fredholm, then there exists $(M, N) \in \text{Red}(T)$ such that $T_M$ is semi-Fredholm and $T_N$ is zeroold. So $(M, N) \in \text{Red}(T^n)$, $T_M^n$ is semi-Fredholm and $T_N^n$ is zeroold. Thus $\ind(T^n) = \ind(T_M^n) = n\ind(T_M) = n\ind(T)$.

(ii) Since $T \in L(X)$ and $S \in L(Y)$ are semi-$g_z$-Fredholm, then there exist $(M_1, N_1) \in \text{Red}(T)$ and $(M_2, N_2) \in \text{Red}(S)$ such that $T_M$ and $T_M'$ are semi-Fredholm, $T_{N_1}$ and $T_{N_2}$ are zeroold. Hence $T_M \oplus T_M' \oplus T_{N_1} \oplus T_{N_2}$ is zeroold. Moreover, $(M_1 \oplus M_2, N_1 \oplus N_2) \in \text{Red}(T \oplus S)$. Hence $\ind(T \oplus S) = \ind((T \oplus S)M_1 \oplus \text{Red}(S)) = \ind(T_M) + \ind(S)$.

Denote by $\sigma_{sf}(T)$, $\sigma_{gf}(T)$, $\sigma_{gwf}(T)$, $\sigma_{gwf}(T)$, $\sigma_{gwf}(T)$ and $\sigma_{gwf}(T)$ respectively, the upper semi-$g_z$-Fredholm spectrum, the lower semi-$g_z$-Fredholm spectrum, the semi-$g_z$-Fredholm, the $g_z$-Fredholm spectrum, the upper semi-$g_z$-Weyl spectrum, the lower semi-$g_z$-Weyl spectrum, the semi-$g_z$-Weyl spectrum and the $g_z$-Weyl spectrum of $T$.

Corollary 3.8. For every $T \in L(X)$, we have $\sigma_{gf}(T) = \sigma_{gwf}(T) \cup \sigma_{gwf}(T)$ and $\sigma_{gwf}(T) = \sigma_{gwf}(T) \cup \sigma_{gwf}(T)$.

Proposition 3.9. Let $T \in L(X)$ be a semi-$g_z$-Fredholm operator which is semi-$g_z$-Fredholm. Then $T$ is quasi semi-$g_z$-Fredholm and its index as a semi-$g_z$-Fredholm coincides with its index as a semi-$g_z$-Fredholm.

Proof. Let $(M, N) \in \text{Red}(T)$ such that $T_M$ is semi-Fredholm and $T_N$ is zeroold. Since $T$ is semi-$g_z$-Fredholm then $T_N$ is Drazin invertible. So there exists $(A, B) \in \text{Red}(T_N)$ such that $T_A$ is invertible and $T_B$ is nilpotent. It is easy to get that $M \oplus A$ is closed, so that $T_{M \oplus A}$ is semi-Fredholm. Consequently, $T = T_{M \oplus A} \oplus T_B$ is quasi semi-$g_z$-Fredholm. Furthermore, the punctured neighborhood theorem for semi-Fredholm operators implies that $\ind(T_M) = \ind(T_{M \oplus A})$.

From [29, Theorem 7] and the previous proposition, we obtain the following corollary.

Corollary 3.10. Every $g_z$-Fredholm operator $T \in L(X)$ is $g_z$-Fredholm and its usual index coincides with its index as a $g_z$-Fredholm operator.

Proposition 3.11. If $T \in L(X)$ is a semi-$g_z$-Fredholm operator, then $T^*$ is semi-$g_z$-Fredholm, $\check{\alpha}(T) = \check{\beta}(T^*)$, $\check{\beta}(T) = \check{\alpha}(T^*)$ and $\ind(T) = -\ind(T^*)$.
Definition 3.12. We say that $T \in L(X)$ is an upper semi-$g_{z}$-Browder (resp., lower semi-$g_{z}$-Browder, $g_{z}$-Browder) if $T$ is a direct sum of an upper semi-Browder (resp., lower semi-Browder, Browder) operator and a zeroloid operator.

Proposition 3.13. Let $T \in L(X)$. The following statements are equivalent:

(i) $T$ is an upper semi-$g_{z}$-Browder [resp., lower semi-$g_{z}$-Browder, $g_{z}$-Browder];

(ii) $T$ is an upper $g_{z}$-Weyl and $T$ has the SVEP at 0 [resp., $T$ is a lower semi-$g_{z}$-Weyl and $T^*$ has the SVEP at 0, $T$ is $g_{z}$-Weyl and $T$ or $T^*$ has the SVEP at 0];

(iii) $T$ is an upper semi-$g_{z}$-Fredholm and $T$ has the SVEP at 0 [resp., $T$ is a lower semi-$g_{z}$-Fredholm and $T^*$ has the SVEP at 0, $T$ is $g_{z}$-Fredholm and $T \oplus T^*$ has the SVEP at 0].

Proof. (i) $\iff$ (ii) Suppose that $T$ is $g_{z}$-Browder, then there exists $(M, N) \in g_{z}KD(T)$ such that $T_M$ is Browder. So $T_M, (T_M)^*$, $T_N$ and $(T_N)^*$ have the SVEP at 0. Thus $T$ and $T^*$ have the SVEP at 0. Conversely, if $T$ is $g_{z}$-Weyl and $T$ or $T^*$ has the SVEP at 0, then there exists $(M, N) \in g_{z}KD(T)$ such that $T_M$ is Weyl and $T_M$ or $(T_M)^*$ has the SVEP at 0. So $\max\{\sigma(T), \sigma(T^*)\} < \infty$ and $\min\{\sigma(T), \sigma(T^*)\} > \infty$. This implies from [1, Lemma 1.22] that $\max\{\sigma(T), \sigma(T^*)\} < \infty$ and then $T_M$ is Browder. Therefore $T$ is $g_{z}$-Browder. The other equivalence cases go similarly.

(i) $\iff$ (iii) Suppose that $T$ is $g_{z}$-Fredholm and $T \oplus T^*$ has the SVEP at 0. Let $(M, N) \in g_{z}KD(T)$ such that $T_M$ is Fredholm and $T_N$ is zeroloid. Hence $T_M \oplus (T_M)^*$ has the SVEP at 0. From the implications (A) and (B) mentioned in the introduction, we deduce that $T_M$ is Browder and then $T$ is $g_{z}$-Browder. The converse is clear and the other equivalence cases go similarly.

The proofs of the following results are obvious and are left to the reader.

Proposition 3.14. If $T \in L(X)$ is semi-$g_{z}$-Fredholm, then there exists $\epsilon > 0$ such that $B_0 := B(0, \epsilon) \setminus \{0\} \subset (\sigma_{spbw}(T))^C$ and $\text{ind}(T) = \text{ind}(T - \lambda I)$ for every $\lambda \in B_0$.

Corollary 3.15. For every $T \in L(X)$, the following assertions hold:

(i) $\sigma_{spbw}(T), \sigma_{w}(T), \sigma_{sp}(T), \sigma_{spz}(T), \sigma_{spbw}(T), \sigma_{z}(T)$, and $\sigma_{spbw}(T)$ are compact.

(ii) If $\Omega$ is a component of $(\sigma_{sp}(T))^C$ or $(\sigma_{w}(T))^C$, then the index $\text{ind}(T - \lambda I)$ is constant as $\lambda$ ranges over $\Omega$.

Corollary 3.16. Let $T \in L(X)$. The following statements are equivalent:

(i) $T$ is semi-$g_{z}$-Weyl [resp., upper semi-$g_{z}$-Weyl, lower semi-$g_{z}$-Weyl, $g_{z}$-Weyl];

(ii) $T$ is $g_{z}$-Kato and $0 \notin \sigma_{spbw}(T)$ [resp., $T$ is $g_{z}$-Kato and $0 \notin \sigma_{spbw}(T)$, $T$ is $g_{z}$-Kato and $0 \notin \sigma_{w}(T)$, $T$ is $g_{z}$-Kato and $0 \notin \sigma_{sp}(T)$, where $\sigma_{spbw}(T) := \sigma_{upbw}(T) \cup \sigma_{lpbw}(T)$, $\sigma_{upbw}(T) := \sigma_{wpbw}(T)$).

4. $g_{z}$-invertible operators

Recall [1] that $T \in L(X)$ is said to be Drazin invertible if there exists an operator $S \in L(X)$ which commutes with $T$ with $STS = S$ and $T^nS = T^n$ for some integer $n \in \mathbb{N}$. The index of a Drazin invertible operator $T$ is defined by $i(T) = \min\{n \in \mathbb{N} : \exists S \in L(X) \text{ such that } ST = TS, STS = S \text{ and } T^nS = T^n\}$.

Proposition 4.1. Let $T \in L(X)$. $p(T) < \infty \Rightarrow \text{tr}(T) = \text{dis}(T)$ (resp., $q(T) < \infty \Rightarrow \text{tr}(T) = \text{dis}(T)$). Moreover, if $T$ is Drazin invertible, then $i(T) = \text{dis}(T)$.

Proof. Suppose that $p(T) < \infty$, then $N(T_{[d]}) = \{0\}$ for every $n \geq p(T)$. This implies that $N(T_{[d]}) = \{0\}$, where $d := \text{dis}(T)$. Thus $p(T) \leq d$, and as we always have $d \leq \min\{p(T), q(T)\}$ then $p(T) = d$. If $q(T) < \infty$, then $X = \mathcal{R}(T) + N(T^n)$ for every $n \geq q(T)$. Since $\mathcal{R}(T) + N(T^n) = \mathcal{R}(T) + N(T^n)$ for every integer $m \geq d$, then $X = \mathcal{R}(T) + N(T^n)$. Hence $T_{[d]}$ is surjective and consequently $q(T) = d$. If in addition $T$ is Drazin invertible, then the proof of the equality desired is an immediate consequence of [1, Theorem 1.134].

Definition 4.2. We say that $T$ is quasi left Drazin invertible (resp., quasi right Drazin invertible) if there exists $(M, N) \in KD(T)$ such that $T_M$ is bounded below (resp., surjective).
Proposition 4.3. Let $T \in L(X)$. The following hold:
(i) $T$ is Drazin invertible if and only if $T$ is quasi left and quasi right Drazin invertible.
(ii) If $T$ is quasi left Drazin invertible, then $T$ is left Drazin invertible.
(iii) If $T$ is quasi right Drazin invertible, then $T$ is right Drazin invertible.
Furthermore, the converses of (ii) and (iii) are true in the case of Hilbert space.

Proof. (i) Assume that $T$ is Drazin invertible, then $n := p(T) = q(T) < \infty$. It is well known that $(\mathcal{R}(T^n), N(T^n)) \in \text{Red}(T)$, $T_{\mathcal{R}(T^n)}$ is invertible and $T_{N(T^n)}$ is nilpotent. So $T$ is quasi left and quasi right Drazin invertible. Conversely, if $T$ is quasi left and quasi right Drazin invertible, then $\alpha(T) = \beta(T) = 0$. Therefore $\alpha(T_M) = \alpha(T) = \beta(T_M) = 0$ for every $(M, N) \in \text{KD}(T)$. Thus $T$ is Drazin invertible.
(ii) Let $(M, N) \in \text{Red}(T)$ such that $T_M$ is bounded below and $T_N$ is nilpotent of degree $d$. As a bounded below operator is semi-regular, we deduce from [5, Theorem 2.21] that $d = \text{dis}(T)$. Clearly, $\mathcal{R}(T^n)$ is closed and $T_{[n]} = (T_M)_{[n]}$ is bounded below for every integer $n \geq d$. Hence $T$ is left Drazin invertible. Conversely, assume that $T$ is left Drazin invertible Hilbert space operator. Then $T$ is upper semi-B-Fredholm, which entails from [10, Theorem 2.6] and [5, Corollary 3.7] that there exists $(M, N) \in \text{KD}(T)$ such that $T_M$ is upper semi-Browder. Using [4, Lemma 2.17], we conclude that $T_M$ is bounded below and then $T$ is quasi left Drazin invertible.
(iii) Goes similarly with (ii). \qed

Proposition 4.4. $T \in L(X)$ is an upper semi-Browder [resp., lower semi-Browder] if and only if $T$ is a quasi left Drazin invertible [resp., quasi right Drazin invertible] and $\dim N < \infty$ for every (or for some) $(M, N) \in \text{KD}(T)$.

Proof. If $T$ is an upper semi-Browder, then $T$ is upper semi-Fredholm. From [5, Corollary 3.7], there exists $(M, N) \in \text{KD}(T)$ with $T_M$ is upper semi-Browder. It follows from [4, Lemma 2.17] that $T_M$ is bounded below. Let $(A, B) \in \text{KD}(T)$ be arbitrary. Since a nilpotent operator $S \in L(Y)$ is semi-Fredholm iff $\dim Y < \infty$, then $\dim B < \infty$. The converse is obvious and the other case goes similarly. \qed

Definition 4.5. $T \in L(X)$ is said to be left $g_\varepsilon$-invertible (resp., right $g_\varepsilon$-invertible) if there exists $(M, N) \in g_\varepsilon \text{KD}(T)$ such that $T_M$ is bounded below (resp., surjective). $T$ is called $g_\varepsilon$-invertible if it is left and right $g_\varepsilon$-invertible.

Remark 4.6. (i) It is clear that $T$ is $g_\varepsilon$-invertible if and only if there exists $(M, N) \in g_\varepsilon \text{KD}(T)$ such that $T_M$ is invertible.
(ii) Every generalized Drazin-meromorphic invertible operator is $g_\varepsilon$-invertible.

We prove in the following result that the class of $g_\varepsilon$-invertible operators preserves some properties of Drazin invertibility [16, 24].

Theorem 4.7. Let $T \in L(X)$. The following statements are equivalent:
(i) $T$ is $g_\varepsilon$-invertible;
(ii) $T$ is $g_\varepsilon$-Browder;
(iii) There exists $(M, N) \in g_\varepsilon \text{KD}(T)$ such that $T_M$ is Drazin invertible;
(iv) There exists a Drazin invertible operator $S \in L(X)$ such that $TS = ST$, $STS = S$ and $T^2S - T$ is zeroloid. A such $S$ is called a $g_\varepsilon$-inverse of $T$;
(v) There exists a bounded projection $P$ on $X$ which commutes with $T$, $T + P$ is generalized Drazin invertible and $(T + P)P = 0$ is zeroloid;
(vi) There exists a bounded projection $P$ on $X$ commuting with $T$ such that there exist $U, V \in L(X)$ which satisfy $P = TU = VT$ and $T(I - P)$ is zeroloid;
(vii) $T$ is $g_\varepsilon$-Kato and $\varphi(T) = \varphi(T) < \infty$.

Proof. The equivalences (i) $\iff$ (ii) and (i) $\iff$ (iii) are immediate consequences of Propositions 4.3 and 4.4.
(i) $\implies$ (iv) Assume that $T$ is $g_\varepsilon$-invertible and let $(M, N) \in g_\varepsilon \text{KD}(T)$ such that $T_M$ is invertible. The operator $S = (T_M)^{-1} \oplus 0_N$ is Drazin invertible. Moreover, $TS = ST = I_M \oplus 0_N$, $STS = S$ and $T^2S - T = 0_M \oplus (-T_N)$. As $T_N$ is zeroloid then $T^2S - T$ is also zeroloid. Conversely, suppose that there exists a Drazin invertible operator $S$ such that $TS = ST$, $STS = S$ and $T^2S - T$ is zeroloid. Then $TS$ is a projection. If we take $M = \mathcal{R}(TS)$ and $N = \mathcal{N}(TS)$, then $(M, N) \in \text{Red}(T) \cap \text{Red}(S)$. We have $T_M$ is one-to-one. Indeed, $x \in \mathcal{N}(T_M)$ implies
that \( x = TSy \) and \( Tx = 0 \), so \( x = (TS)^2y = STx = 0 \). Since \( \mathcal{R}(T_M) = M \) then \( T_M \) is invertible. Let us to show that \( S = (T_M)^{-1} \oplus 0_N \). We have \( S_N = 0_N \), since \( S = STS \). Let \( x = TSy \in M \), as \( Sy = STSy \in M \) then \( Sx = Sy = (T_M)^{-1}T_MSy = (T_M)^{-1}x \). Hence \( S = (T_M)^{-1} \oplus 0_N \) and \( T^2S - T = 0_M \oplus (-T_N) \). Thus \( T_N \) is zeroloid and then \( T \) is \( g_z \)-invertible.

(i) \( \iff \) (v) Suppose that there exists a bounded projection \( P \) on \( X \) which commutes with \( T \), \( T + P \) is generalized Drazin invertible and \( TP \) is zeroloid. Then \( (A, B) := (N(P), \mathcal{R}(P)) \in \text{Red}(T) \), \( T_A = (T + P)_A \) is generalized Drazin invertible and \( T_B = (TP)_B \) is zeroloid. Thus there exists \( (C, D) \in \text{Red}(T_A) \) such that \( T_C \) is invertible and \( T_D \) is quasi-nilpotent. Hence \( (C, D \oplus B) \in g_zKD(T) \) and then \( T \) is \( g_z \)-invertible. Conversely, let \( (M, N) \in g_zKD(T) \) such that \( T_M \) is invertible. Clearly, \( P := 0_M \oplus I_N \) is a projection and \( TP = PT \). Furthermore, \( TP = 0_M \oplus T_N \) is zeroloid and \( T + P = T_M \oplus (T + I)_N \) is generalized Drazin invertible, since \(-1 \not\in \text{acc} \sigma(T_N) = \sigma_{sf}(T_N) \).

(vi) \( \implies \) (i) Suppose that there exists a bounded projection \( P \) on \( X \) commuting with \( T \) such that there exist \( U, V \in L(X) \) which satisfy \( P = TU = VT \) and \( T(I - P) \) is zeroloid. In addition, we assume that \( U, V \in \text{comm}(T) \) (for the general case, one can see the proof of the implication (v) \( \implies \) (vi) of [35, Theorem 2.4]). Then \( I_M \oplus 0_N = T_MUM \oplus T_MUN = VM TM \oplus V_NT_N \), where \( (M, N) := (\mathcal{R}(P), N(P)) \in \text{Red}(T) \), and thus \( T_MUM = VM TM = I_M \) and \( T_MUN = V_NT_N = 0_N \). Hence \( T_M \) is invertible. Moreover, \( T_N \) is zeroloid, since \( T(I - P) = 0_M \oplus T_N \) is zeroloid. Consequently, \( T \) is \( g_z \)-invertible.

(vi) \( \iff \) (v) and (i) \( \iff \) (vi) are clear. \( \Box \)

The next two theorems are analogous to the previous one.

**Theorem 4.8.** Let \( T \in L(X) \). The following statements are equivalent:

(i) \( T \) is left \( g_z \)-invertible;

(ii) \( T \) is upper semi-\( g_z \)-Browder;

(iii) There exists \( (M, N) \in g_zKD(T) \) such that \( T_M \) is quasi left Drazin invertible;

(iv) \( T \) is \( g_z \)-Kato and \( p(T) = 0 \);

(v) \( T \) is \( g_z \)-Kato and \( 0 \not\in \text{acc} \sigma_{sf}(T) \).

**Theorem 4.9.** Let \( T \in L(X) \). The following statements are equivalent:

(i) \( T \) is right \( g_z \)-invertible;

(ii) \( T \) is lower semi-\( g_z \)-Browder;

(iii) There exists \( (M, N) \in g_zKD(T) \) such that \( T_M \) is quasi right Drazin invertible;

(iv) \( T \) is \( g_z \)-Kato and \( \bar{q}(T) = 0 \);

(v) \( T \) is \( g_z \)-Kato and \( 0 \not\in \text{acc} \sigma_{sf}(T) \).

**Corollary 4.10.** If \( T \in L(X) \) is \( g_z \)-invertible and \( S \) is a \( g_z \)-inverse of \( T \), then \( TST \) is the Drazin inverse of \( S \) and \( p(S) = q(S) = \text{dis}(S) \leq 1 \).

**Proof.** Obvious. \( \Box \)

Hereafter, \( \sigma_{g_z}(T) \), \( \sigma_{gZ}(T) \) and \( \sigma_{g_{sf}}(T) \) are respectively, the left \( g_z \)-invertible spectrum, the right \( g_z \)-invertible spectrum and the \( g_z \)-invertible spectrum of \( T \).

**Theorem 4.11.** For every \( T \in L(X) \) we have \( \sigma_{g_{sf}}(T) = \text{acc} \sigma(T) \).

**Proof.** Let \( \mu \not\in \text{acc} \sigma(T) \). Without loss of generality we assume that \( \mu = 0 \) [note that \( \text{acc} \text{acc} \sigma(T - \alpha I) = \text{acc} \sigma(T) - \alpha \), for every complex \( \alpha \)]. If \( 0 \not\in \text{acc} \sigma(T) \), then \( T \) is generalized Drazin invertible and in particular \( g_z \)-invertible. If \( 0 \in \text{acc} \sigma(T) \) then \( 0 \in \text{acc} \sigma(T) \). We distinguish two cases:

**Case 1:** \( \text{acc} \sigma(T) \neq \emptyset \). It follows that \( \varepsilon := \inf_{\lambda \in \text{acc} \sigma(T) \setminus 0} |\lambda| > 0 \). Moreover, the sets \( F_2 := D(0, \frac{\varepsilon}{2}) \cap \text{iso } \sigma(T) \)

and \( F_1 := (\text{acc} \sigma(T) \setminus \{0\}) \cup (\text{iso } \sigma(T) \setminus F_2) \) are closed and disjoint. Indeed, \( F_1 \cap F_2 = F_2 \cap (\text{acc } \sigma(T) \setminus \{0\}) \subset [\text{acc } \sigma(T) \setminus \{0\}] \cap D(0, \frac{\varepsilon}{2}) = \emptyset \). As \( 0 \not\in \text{acc } \sigma(T) \) then \( (\text{acc } \sigma(T) \setminus \{0\}) \) is closed. Let us to show that \( C := (\text{iso } \sigma(T) \setminus F_2) \) is closed. If \( \lambda \in \text{acc } C \) (the case of \( \text{acc } C = \emptyset \) is obvious), then \( \lambda \in \text{iso } \sigma(T) \). Let \( (\lambda_n)_n \subset C \) be a non stationary sequence that converges to \( \lambda \), it follows that \( \lambda \neq 0 \). We have \( \lambda \notin F_2 \). Otherwise, \( \lambda \in D(0, \frac{\varepsilon}{2}) \) and then \( \lambda \not\in \text{acc } \sigma(T) \). So \( \lambda \in \text{iso } \sigma(T) \) and this is a contradiction. Therefore \( C \) is closed and then \( F_1 \) is
closed. As $\sigma(T) = F_1 \cup F_2$ then there exists $(M, N) \in \text{Red}(T)$ such that $\sigma(T_M) = F_1$ and $\sigma(T_N) = F_2$. So $T_M$ is invertible and $0 \in \text{acc}(T_N)$. Let $\nu \in F_2$, then $\nu \notin \text{acc}(T_N) \setminus \{0\}$, since $F_1 \cap F_2 = F_2 \cap (\text{acc}(T) \setminus \{0\}) = \emptyset$. Hence $\text{acc}(T_M) = \{0\}$ and $T$ is $g_2$-invertible.

Case 2: $\text{acc} (\text{iso}(\sigma(T))) = \emptyset$. Then $F_2 := D(0, 1) \cap \text{iso}(\sigma(T))$ and $F_1 := (\text{acc}(\sigma(T)) \setminus \{0\}) \cup (\text{iso}(\sigma(T)) \setminus F_2)$ are closed disjoint subsets and give the desired result. For this, if $\lambda \in \mathbb{C}$, where $\mathbb{C} := \text{iso}(\sigma(T)) \setminus F_2$, then there exists a sequence $(\lambda_n) \subset C$ that converges to $\lambda$. As $\text{acc}(\text{iso}(\sigma(T))) = \{0\}$ and $\lambda(\neq 0) \notin \text{iso}(\sigma(T))$ then $\lambda \in \text{iso}(\sigma(T))$. Therefore $(\lambda_n)_n$ is stationary and so $\lambda \in C$. Thus $F_1$ is closed and hence there exists $(M, N) \in \text{Red}(T)$ such that $\sigma(T_M) = F_1$ and $\sigma(T_N) = F_2$. Conclusion, $T$ is $g_2$-invertible.

Conversely, if $T$ is $g_2$-invertible, then $T = T_1 \oplus T_2$, where $T_1$ is invertible and $T_2$ is zeroloid. And then there exists $\varepsilon > 0$ such that $\delta(0, \varepsilon) \setminus \{0\} \subset (\text{acc}(T_1))^c \cap (\text{acc}(T_2))^c \subset (\text{acc}(\sigma(T)))^c$. Thus $0 \notin \text{acc}(\sigma(T))$. ☐

From the previous theorem and some well known results in perturbation theory, we obtain the following corollary.

**Corollary 4.12.** Let $T \in L(X)$. The following statements hold:

(i) $\sigma_{g_2}(T), \sigma_{g_2}(T)$ and $\sigma_{g_2}(T)$ are compact.

(ii) $\sigma_{g_2}(T) = \sigma_{g_2}(T)$.

(iii) If $S \in L(Y)$, then $T \oplus S$ is $g_2$-invertible if and only if $T$ and $S$ are $g_2$-invertible.

(iv) $T$ is $g_2$-invertible if and only if $T^n$ is $g_2$-invertible for some (equivalently for every) integer $n \geq 1$.

(v) If $Q \in \text{comm}(T)$ is quasi-nilpotent, then $\sigma_{g_2}(T) = \sigma_{g_2}(T + Q)$.

(vi) If $F \in \mathcal{F}(X) \cap \text{comm}(T)$, then $\sigma_{g_2}(T) = \sigma_{g_2}(T + F)$, where $\mathcal{F}(X)$ is the set of all power finite rank operators.

**Example 4.13.** Let $T \in L(X)$ be the operator such that $\sigma(T) = \sigma(T) = \{1^{-1} \}$. Then $T$ is $g_2$-invertible and not generalized Drazin-meromorphic invertible, since $0 \in \text{acc}(\sigma(T))$ (see [35, Theorem 5]). Note also that $T$ is not generalized Kato-meromorphic. Otherwise, we get $\bar{\alpha}(T) = \bar{\beta}(T) = 0$, since $T$ is $g_2$-invertible. Hence $T$ is generalized Drazin-meromorphic invertible and this is a contradiction.

**Proposition 4.14.** Let $T \in L(X)$, the statements are equivalent:

(i) $0 \in \text{iso} (\text{acc}(\sigma(T)))$ (i.e. $T$ is $g_2$-invertible and not generalized Drazin invertible);

(ii) $T = T_1 \oplus T_2$, where $T_1 \oplus T_2$ and $\text{acc}(T_2) = \{0\}$;

(iii) $T$ is $g_2$-Kato and there exists a non stationary sequence of isolated points of $\sigma(T)$ that converges to $0$.

**Proof.** (i) $\implies$ (ii) Follows directly from the proof of Theorem 4.11. Note here that $\text{acc}(T_M) = \{0\}$ for every $(M, N) \in g_2 \text{KD}(T)$.

(ii) $\implies$ (iii) As $T = T_1 \oplus T_2$, $T_1$ is invertible and $\text{acc}(T_2) = \{0\}$, then $0 \in \text{iso}(\text{acc}(\sigma(T)))$ and there exists a non stationary sequence $(\lambda_n)_n \subset \text{iso}(\sigma(T_2))$ that converges to $0$. Thus $T$ is $g_2$-invertible and there exists $N \in \mathbb{N}$ such that $\lambda_n \in \sigma(T) \cap \text{acc}(\sigma(T))$ for all $n \geq N$.

(iii) $\implies$ (i) Assume that $T = T_1 \oplus T_2$, $T_1$ is semi-regular, $T_2$ is zeroloid and there exists a non stationary sequence $(\lambda_n)_n$ of isolated point of $\sigma(T)$ that converges to $0$. Hence $0 \in \text{acc}(\sigma(T))$ and $T \oplus T^*$ has the SVEP at 0. This entails that $T$ is $g_2$-invertible and then $0 \in \text{iso}(\text{acc}(\sigma(T)))$. ☐

Recall that $\sigma \subset \sigma(T)$ is called a spectral set (called also isolated part) of $T$ if $\sigma$ and $\sigma(T) \setminus \sigma$ are closed, see [17]. Let $T$ be a $g_2$-invertible operator which is not generalized Drazin invertible. From Proposition 4.14, we conclude that there exists a non-zero strictly decreasing sequence $(\lambda_n)_n \subset \sigma(T)$ that converges to 0 such that $\sigma := \{\lambda_n : n \in \mathbb{N}\}$ is a spectral set of $T$. If $P_\sigma$ is the spectral projection associated to $\sigma$, then $(M_\sigma, N_\sigma) := (N(P_\sigma), \mathcal{R}(P_\sigma)) \in g_2 \text{KD}(T), \sigma(T_N) = \sigma$ and $\sigma(T_M) = \sigma(T) \setminus \sigma$. Thus $T + rP_\sigma = T_M \oplus (T + rI)_{N_\sigma}$ is invertible for every $|r| > |\lambda_0|$ and then the operator $T_\sigma^n := (T + rP_\sigma)^{-1}(T - P_\sigma) = (T_M)^{-1} \oplus N_\sigma$ is a $g_2$-inverse of $T$ and depends only on $\sigma$. Note that $P_\sigma = I - TT_\sigma^\prime \in \text{comm}(T) := \{S \in \text{comm}(L) : L \in \text{comm}(T), \delta \}$, so that $(M_\sigma, N_\sigma) \in \text{Red}(S)$ for every operator $S \in \text{comm}(T)$ and $T_\sigma \in \text{comm}(T)$. Note also that $T + P_\sigma$ is generalized Drazin invertible and $P_\sigma$ is zeroloid.

**Lemma 4.15.** Let $T \in L(X)$ be a $g_2$-invertible operator and $(M, N) \in g_2 \text{KD}(T)$ such that $T_M$ invertible and $\sigma(T_M) \cap \sigma(T_N) = \emptyset$. Then $\sigma(T_N) \setminus \{0\} \subset \text{iso}(\sigma(T))$ and for every $S \in \text{comm}(T)$ we have $(M, N) \in \text{Red}(S)$. 

Z. Azmay et al. / Filomat 37:7 (2023), 2087–2103

2098
Proof. If $T$ is generalized Drazin invertible, then $0 \notin \text{acc} \sigma(T)$ and so $\text{acc} \sigma(T_N) = \emptyset$, hence $\sigma(T_N)$ is a finite set of isolated points of $\sigma(T)$. Let $P_{\sigma}$ be the spectral projection associated to $\sigma = \sigma(T_N)$. From [17, Proposition 2.4] and the fact that $P_{\sigma} \in \text{comm}(T)$ we deduce that $(M, N) = (N(P_{\sigma}), \mathcal{R}(P_{\sigma})) \in \text{Red}(S)$ for every $S \in \text{comm}(T)$. If $T$ is not generalized Drazin invertible, then there exists a strictly decreasing sequence $(\lambda_n)_{n \in \mathbb{N}}$ of isolated point of $\sigma(T)$ that converges to 0 and such that $\sigma(T_N) = [\lambda_n : n \in \mathbb{N}]$. Thus $\sigma(T_N) \setminus \{0\} \subset \text{iso} \sigma(T)$. Let $P$ be the spectral projection associated to the spectral set $\sigma(T_N)$, then $(M, N) = (N(P), \mathcal{R}(P))$ and so $(M, N) \in \text{Red}(S)$ for every $S \in \text{comm}(T)$. \hfill \Box

Remark 4.16. It is not difficult to see that the following assertions are equivalent:
(i) $\exists (M, N) \in \text{Red}(S)$ such that $T_M$ is invertible for every $S \in \text{comm}(T)$;
(ii) $\exists L \in \text{comm}^2(T)$ such that $L = L^2T$.

Theorem 4.17. Let $T \in L(X)$. The following statements are equivalent:
(i) $T$ is $g_z$-invertible;
(ii) $0 \notin \text{acc}(\text{acc} \sigma(T))$;
(iii) There exists $(M, N) \in g_zKD(T)$ such that $T_M$ is invertible and $\sigma(T_M) \cap \sigma(T_N) = \emptyset$;
(iv) There exists a spectral set $\sigma$ of $T$ such that $0 \notin \sigma(T) \setminus \sigma$ and $\sigma \setminus \{0\} \subset \text{iso} \sigma(T)$;
(v) There exists a bounded projection $P \in \text{comm}^2(T)$ such that $T + P$ is generalized Drazin invertible and $TP$ is zerooid.

Proof. For the equivalence (i) $\iff$ (ii), see Theorem 4.11. For the equivalences (i) $\iff$ (iii) and (i) $\iff$ (v), see Theorem 4.7 and the paragraph preceding Lemma 4.15 (the case of $T$ is generalized Drazin invertible is clear). The proof of the equivalence (iii) $\iff$ (iv) is a consequence of Lemma 4.15 and the spectral decomposition theorem. \hfill \Box

Proposition 4.18. For every $g_z$-invertible operator $T \in L(X)$, the following statements hold:
(i) Let $(M, N), (M', N') \in g_zKD(T)$ such that $T_M, T_{M'}$ are invertible and $\sigma(T_M) \cap \sigma(T_N) = \emptyset$. If $(T_M)^{-1} \oplus 0_N = (T_{M'})^{-1} \oplus 0_{N'}$, then $(M, N) = (M', N')$.
(ii) Let $\sigma, \sigma'$ be two spectral sets of $T$ such that $0 \notin \sigma(T) \setminus \sigma$ and $\sigma \cup \sigma' \setminus \{0\} \subset \text{iso} \sigma(T)$, then $(T + rP_{\sigma'})^{-1}(I - P_{\sigma'}) \in \text{comm}(T)$, where $P_{\sigma}$ is the spectral projection of $T$ associated to $\sigma$, $|r| > \max |\lambda|$, then $\sigma = \sigma'$.

Proof. (i) From the proof of Lemma 4.15, we have $(M, N) = (N(P_{\sigma}), \mathcal{R}(P_{\sigma}))$ and $(M', N') = (N(P_{\sigma'}), \mathcal{R}(P_{\sigma'}))$, where $\sigma = \sigma(T_N)$ and $\sigma' = \sigma(T_N')$. As $(T_{M'})^{-1} \oplus 0_N = (T_M)^{-1} \oplus 0_{N'}$, then $\sigma(T_M) = \sigma(T_{M'})$ and thus $\sigma(T_N) = \sigma(T_{N'})$. This proves that $(M, N) = (M', N')$.
(ii) Follows from (i). \hfill \Box

The previous Proposition 4.18 gives a sense to the next remark.

Remark 4.19. If $T \in L(X)$ is $g_z$-invertible, then
(i) For every $(M, N) \in g_zKD(T)$ such that $T_M$ is invertible and $\sigma(T_M) \cap \sigma(T_N) = \emptyset$, the $g_z$-inverse operator $T_D^{(MN)} := (T_M)^{-1} \oplus 0_N \in \text{comm}(T)$, and we call $T_D^{(MN)}$ the $g_z$-inverse of $T$ associated to $(M, N)$.
(ii) If $\sigma$ is a spectral set of $T$ such that $0 \notin \sigma(T) \setminus \sigma$ and $\sigma \setminus \{0\} \subset \text{iso} \sigma(T)$, then the operator $T_{D_{\sigma}} := (T + rP_{\sigma})^{-1}(I - P_{\sigma}) \in \text{comm}(T)$ is a $g_z$-inverse of $T$, where $|r| > \max |\lambda|$, and we call $T_{D_{\sigma}}$ the $g_z$-inverse of $T$ associated to $\sigma$.

Note that if $T \in L(X)$ is generalized Drazin invertible which is not invertible, then by [24, Lemma 2.4] and Proposition 4.18 we conclude that the Drazin inverse of $T$ is exactly the $g_z$-inverse of $T$ associated to $\sigma = \{0\}$, in other words $T_D = T_{D_{\sigma}}^{(\{0\})}$.

Proposition 4.20. Let $T, S \in L(X)$ two commuting $g_z$-invertible. If $\sigma$ and $\sigma'$ are spectral sets of $T$ and $S$, respectively such that $0 \notin (\sigma(T) \setminus \sigma) \cup (\sigma(S) \setminus \sigma')$, $\sigma \setminus \{0\} \subset \text{iso} \sigma(T)$ and $\sigma' \setminus \{0\} \subset \text{iso} \sigma(S)$, then $T, S, T_D, S_D$ are mutually commutative.
Proof. As \( TS = ST \) then the previous remark entails that \( T^D_\sigma = (T + rP_\sigma)^{-1}(I - P_\sigma) \in \text{comm}(S^D_\sigma) \), and analogously for other operators. \( \square \)

The following proposition describes the relation between the \( g_\sigma \)-inverse of a \( g_\sigma \)-invertible operator \( T \) associated to \((M, N)\) and the \( g_\sigma \)-inverse of \( T \) associated to a spectral set \( \sigma \). Its proof is clear.

**Proposition 4.21.** If \( T \in L(X) \) is \( g_\sigma \)-invertible and \((M, N) \in g_\sigma KD(T) \) such that \( T_M \) is invertible and \( \sigma(T_M) \cap \sigma(T_N) = \emptyset \), then \( T^D_{\sigma(T_M)} = T^D_{\sigma} \), where \( \sigma = \sigma(T_N) \). In other words \( T^D_{\sigma(T_M)} = (T_M)^{-1} \oplus 0_N \).

Our next theorem gives a generalization of [24, Theorem 4.4] in the case of the complex Banach algebra \( L(X) \). Denote by \( \text{Hol}(T) \) the set of all analytic functions defined on an open neighborhood of \( \sigma(T) \).

**Theorem 4.22.** If \( 0 \in \sigma(T) \setminus \text{acc}(\sigma(T)) \), then for every spectral set \( \sigma \) such that \( 0 \in \sigma \) and \( \sigma \setminus \{0\} \subset \text{iso}(\sigma(T)) \) we have

\[
T^D_\sigma = f_\sigma(T),
\]

where \( f_\sigma \) is defined by \( f_\sigma = 0 \) in a neighborhood of \( \sigma \) and \( f_\sigma(\lambda) = \lambda^{-1} \) in a neighborhood of \( \sigma(T) \setminus \sigma \). Moreover \( \sigma(T^D_\sigma) = \{0\} \cup \{\lambda^{-1} : \lambda \in \sigma(T) \setminus \sigma\} \).

Proof. Let \( \Omega_1 \) and \( \Omega_2 \) be disjoint open sets such that \( \sigma \subset \Omega_1 \) and \( \sigma(T) \setminus \sigma \subset \Omega_2 \) (for the construction of \( \Omega_1 \) and \( \Omega_2 \), see the paragraph below) and let \( g \in \text{Hol}(T) \) be the function defined by

\[
g(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \Omega_1 \\ 0 & \text{if } \lambda \in \Omega_2 \end{cases}
\]

It is clear that \( P_\sigma = g(T) \) and as \( T^D_\sigma = (T + rP_\sigma)^{-1}(I - P_\sigma) \) (where \( |r| > \max|\lambda| \) be arbitrary), then the function \( f_\sigma(\lambda) = (\lambda + rg(\lambda))^{-1}(1 - g(\lambda)) \) has the required property. Moreover, we have \( \sigma(T^D_\sigma) = f_\sigma(\sigma(T)) = \{0\} \cup \{\lambda^{-1} : \lambda \in \sigma(T) \setminus \sigma\}. \) \( \square \)

According to [17], if \( \sigma \) is a spectral set of \( T \) then there exist two disjoint open sets \( \Omega_1 \) and \( \Omega_2 \) such that \( \sigma \subset \Omega_1 \) and \( \sigma(T) \setminus \sigma \subset \Omega_2 \). Choose a Cauchy domains \( S_1 \) and \( S_2 \) such that \( \sigma \subset S_1, \sigma(T) \setminus \sigma \subset S_2, \overline{S_1} \subset \Omega_1 \) and \( \overline{S_2} \subset \Omega_2 \). It follows that the spectral projection corresponding to \( \sigma \) is

\[
P_\sigma = \frac{1}{2\pi i} \int_{\partial S_1} (\lambda I - T)^{-1} d\lambda.
\]

Moreover, if \( 0 \in \sigma \) and \( \sigma \setminus \{0\} \subset \text{iso}(\sigma(T)) \), then from Theorem 4.22 we conclude that

\[
T^D_\sigma = \frac{1}{2\pi i} \int_{\partial S_2} \lambda^{-1} (\lambda I - T)^{-1} d\lambda.
\]

5. Weak SVEP and applications

As a continuation of some results proved in [19, 22], we begin this part by the next theorem which gives a new characterization of some Browder’s type theorems in terms of spectra introduced and studied in the preceding parts.

**Theorem 5.1.** For \( T \in L(X) \), we have

(i) \( T \in (B) \) if and only if \( \sigma_{B,w}(T) = \sigma_{g,w}(T) \).

(ii) \( T \in (B_1) \) if and only if \( \sigma_{B_1}(T) = \sigma_{g,w}(T) \).

(iii) \( T \in (aB) \) if and only if \( \sigma_{aB,w}(T) = \sigma_{g,w}(T) \).
Proof. (i) If \( \lambda \notin \sigma_{g.w}(T) \), then from Corollary 3.16 we have \( \lambda \notin \text{acc} \sigma_{phw}(T) \) [note that \( \text{acc} \sigma_{phw}(T - \lambda I) = \text{acc} \sigma_{phw}(T) - \lambda I \)]. Since \( T \in (B) \) then [22, Theorem 2.6] or [19, Theorem 2.8] implies that \( \lambda \notin \text{acc} \sigma_{phw}(T) \), and this implies from Theorem 4.11 that \( \lambda \notin \sigma_{phw}(T) \). As the inclusion \( \sigma_{g.w}(T) \subset \sigma_{phw}(T) \) is always true, it follows that \( \sigma_{g.w}(T) = \sigma_{phw}(T) \). Conversely, let \( \lambda \notin \sigma_{w}(T) \), then \( \lambda \notin \sigma_{g.w}(T) \). On the other hand, [5, Corollary 3.7] implies that there exists \( (M, N) \in \text{Red}(T) \) such that \( T_M - \lambda I \) is semi-regular and \( T_N - \lambda I \) is nilpotent. Since \( T - \lambda I \) is \( g_z \)-invertible then \( p(T_M - \lambda I) = p(T - \lambda I) = q(T - \lambda I) = q(T_M - \lambda I) = 0 \), and so \( T_M - \lambda I \) is invertible. Hence \( T - \lambda I \) is Browder and consequently \( T \in (B) \). Using [22, Corollary 2.10] or [19, Corollary 2.14], the point (ii) goes similarly with (i). And Using [22, Theorem 2.7], we obtain analogously the point (iii).

**Definition 5.2.** Let \( A \) be a subset of \( \mathbb{C} \). We say that \( T \in L(X) \) has the Weak SVEP on \( A \) (\( T \) has the \( W_A \)-SVEP for brevity) if there exists a subset \( B \subset A \) such that \( T \) has the SVEP on \( B \) and \( T^* \) has the SVEP on \( A \setminus B \). If \( T \) has the \( W_C \)-SVEP, then \( T \) is said to have the Weak SVEP \( (T \) has the \( W \)-SVEP for brevity).

**Remark 5.3.** (i) Let \( A \) be a subset of \( \mathbb{C} \). Then \( T \in L(X) \) has the \( W_A \)-SVEP if and only if for every \( \lambda \in A \), at least \( T \) or \( T^* \) has the SVEP at \( \lambda \).
(ii) If \( T \) or \( T^* \) has the SVEP, then \( T \) has the \( W \)-SVEP. But the converse is not generally true. For this, the left shift operator \( L \in L(\ell^2(\mathbb{N})) \) defined by \( L(x_1, x_2, \ldots) = (x_2, x_3, \ldots) \) has the \( W \)-SVEP, but it does not have the SVEP.
(iii) The operator \( L \oplus L^* \) does not have the \( W \)-SVEP.

The next theorem gives a sufficient condition for an operator \( T \in L(X) \) to have the \( W \)-SVEP.

**Theorem 5.4.** Let \( T \in L(X) \). If
\[
X_T(\emptyset) \times X_{T^*}(\emptyset) \subset \{(x, 0) : x \in X\} \bigcup \{(0, f) : f \in X^*\},
\]
then \( T \) has the \( W \)-SVEP.

**Proof.** Let \( \lambda \in \mathbb{C} \) and let \( V, W \subset X \) two open neighborhood of \( \lambda \). Let \( f : V \to X \) and \( g : W \to X^* \) two analytic functions such that \( (T - \mu I)f(\mu) = 0 \) and \( (T^* - \nu I)g(\nu) = 0 \) for every \( (\mu, \nu) \in V \times W \). If we take \( U = V \cap W \), then [1, Theorem 2.9] implies that \( \sigma_T(f(\mu)) = \sigma_T(0) = \emptyset = \sigma_{T^*}(0) = \sigma_{T^*}(g(\mu)) \) for every \( \mu \in U \). Hence \( (f(\mu), g(\nu)) \in X_T(\emptyset) \times X_{T^*}(\emptyset) \) for every \( \mu, \nu \in U \). We discuss two cases. The first, there exists \( \mu \in U \) such that \( g(\mu) \neq 0 \). As \( (f(\nu), g(\mu)) \in X_T(\emptyset) \times X_{T^*}(\emptyset) \) for every \( \nu \in U \) then by hypotheses \( f \equiv 0 \) on \( U \). The identity theorem for analytic functions entails that \( T \) has the SVEP at \( \lambda \). The second, \( g(\mu) = 0 \) for every \( \mu \in U \). In the same way, we prove that \( T^* \) has the SVEP at \( \lambda \). Hence \( T \) has the \( W \)-SVEP.

**Question:** Similarly to [1, Theorem 2.14] which characterizes the SVEP of \( T \in L(X) \) in terms of its local spectral subspace \( X_T(\emptyset) \), we ask if the converse of Theorem 5.4 is true?

The next proposition characterizes the classes \((B)\) and \((aB)\) in terms of the Weak SVEP.

**Proposition 5.5.** If \( T \in L(X) \), then
(a) For \( \sigma \in \{\sigma_w, \sigma_{bw}, \sigma_{g.w}\} \), the following statements are equivalent:
(i) \( T \in (B) \);
(ii) \( T \) has the Weak SVEP on \( \sigma(T)^C \);
(iii) For all \( \lambda \notin \sigma(T) \), \( T \oplus T^* \) has the SVEP at \( \lambda \);
(iv) For all \( \lambda \notin \sigma(T) \), \( T \) has the SVEP at \( \lambda \);
(v) For all \( \lambda \notin \sigma(T) \), \( T^* \) has the SVEP at \( \lambda \).
(b) For \( \sigma \in \{\sigma_{w}, \sigma_{bw}, \sigma_{g.w}\} \), the following statements are equivalent:
(i) \( T \in (B) \);
(ii) \( T \) has the Weak SVEP on \( \sigma(T)^C \);
(iii) \( T \) has the SVEP on \( \sigma(T) \);
(iv) For all \( \lambda \notin \sigma(T) \), \( T \) has the SVEP at \( \lambda \).
(c) For \( \sigma \in \{\sigma_{bw}, \sigma_{gbw}, \sigma_{ug.w}\} \), the following statements are equivalent:
(i) \( T \in (aB) \);
(ii) \( T \) has the Weak SVEP on \( \sigma(T)^C \);
(iii) For all \( \lambda \notin \sigma(T) \), \( T \) has the SVEP at \( \lambda \).
Proof. (a) For $\sigma_\ast = \sigma_{g,w}$, we have only to show (ii) $\Rightarrow$ (i), and the other implications are clear. Let $\lambda \notin \sigma_{g,w}(T)$, then there exists $(M,N) \in \text{Red}(T)$ such that $T_M - \lambda I$ is Weyl and $T_N - \lambda I$ is zeroloid. Hence $T$ or $T^\ast$ has the SVEP at $\lambda$ is equivalent to say that $T_M$ or $(T_M)^\ast$ has the SVEP at $\lambda$, and this is equivalent to $\min |p(T_M - \lambda I)q(T_M - \lambda I)| < \infty$. Therefore $T_M - \lambda I$ is Browder and then $\lambda \notin \sigma_{g,d}(T)$. From Theorem 5.1, it follows that $T \in (B)$. For $\sigma_\ast \in \{\sigma_{g,e}, \sigma_{g,w}\}$, the proof of (ii) $\Rightarrow$ (i) is similar, and the other implications are already done in [1]. The assertions (b) and (c) go similarly with (a). Note that some implications of assertions (b) and (c) are already done in [1, 6, 19, 22].

We end this part by the next result which extends [1, Theorem 5.6].

Theorem 5.6. If the $g_\ast$-Weyl spectrum of $T \in L(X)$ has empty interior that is, $\text{int} \sigma_{g,w}(T) = \emptyset$, then the following statements are equivalent:

(i) $T \in (B)$;
(ii) $T \in (B_\ast)$;
(iii) $T \in (aB)$;
(iv) $T$ has the SVEP;
(v) $T^\ast$ has the SVEP;
(vi) $T \oplus T^\ast$ has the SVEP;
(vii) $T$ has the W-SVEP.

Proof. (i) $\Rightarrow$ (vi) As $T \in (B)$ then by Proposition 5.5, $T \oplus T^\ast$ has the SVEP on $\sigma_{g,w}(T)^C$. Let $\lambda \in \sigma_{g,w}(T)$, $U \subset C$ be an open neighborhood of $\lambda$ and $f : U \rightarrow X$ be an analytic function which satisfies $(\mu - T)f(\mu) = 0$, for every $\mu \in U$. The hypothesis $\text{int} \sigma_{g,w}(T) = \emptyset$ implies that there exists $y \in U \cap (\sigma_{g,w}(T)^C$. Hence $f \equiv 0$ on $U$, since $T$ has the SVEP at $y$. It then follows that $T$ has the SVEP at $\lambda$. Analogously we prove that $T^\ast$ has the SVEP at $\lambda$, and consequently $T \oplus T^\ast$ has the SVEP. It is clear that the statement (vi) implies without condition on $T$ all other statements. Furthermore, all statements imply (i). This completes the proof.

References