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On approximately biprojective and approximately biflat Banach algebras

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Abstract. In this paper, we study the approximate biprojectivity and the approximate biflatness of a Banach algebra $\mathcal{A}$ and find some relations between theses concepts with ϕ-amenability and ϕ-contractibility, where ϕ is a character on $\mathcal{A}$. Among other things, we show that θ-Lau product algebra $L^1(G) \times_\theta A(G)$ is approximately biprojective if and only if G is finite, where $L^1(G)$ and $A(G)$ are the group algebra and the Fourier algebra of a locally compact group G, respectively. We also characterize approximately biprojective and approximately biflat semigroup algebras associated with the inverse semigroups.

1. Introduction and preliminaries

Let $\mathcal{A}$ be a Banach algebra. We denote the first and second dual of $\mathcal{A}$ by $\mathcal{A}^*$ and $\mathcal{A}^{**}$, respectively. Consider the mapping $\pi : \mathcal{A} \otimes_p \mathcal{A} \longrightarrow \mathcal{A}$ given by $\pi_A(a \otimes b) = ab$, which is the canonical morphism (for emphasis, $\pi_{\mathcal{A}}$), where $\mathcal{A} \otimes_p \mathcal{A}$ is the projective tensor product $\mathcal{A}$ with itself. A Banach algebra $\mathcal{A}$ is called biprojective [resp., biflat] if there exists a bounded $\mathcal{A}$-bimodule morphism $\rho : \mathcal{A} \longrightarrow \mathcal{A} \otimes_p \mathcal{A}$ [resp., $\rho : \mathcal{A} \longrightarrow (\mathcal{A} \otimes_p \mathcal{A})^{**}$] such that $\pi_{\mathcal{A}} \circ \rho(a) = a$ [resp., $\pi^{**}_{\mathcal{A}} \circ \rho(a) = a$] for all $a \in \mathcal{A}$. These concepts have been introduced by Helemskii to study the structure of Banach algebras via Banach algebraic homology; the basic properties of biprojectivity and biflatness for Banach algebras are available in [6] and [18]. As for some known results about the group algebra $L^1(G)$, it is biprojective (resp. biflat) if and only if G is compact (resp. amenable). For some similar results as module versions of biprojectivity and biflatness for Banach algebras, we refer to [2].

Approximate notions in the homology were introduced for more observations on the structure of Banach algebras. Indeed, to study the nilpotent ideals of a Banach algebra, Zhang [23] defined the notion of approximate biprojectivity. In fact, a Banach algebra $\mathcal{A}$ is called *approximately biprojective* if there exists a net of $\mathcal{A}$-bimodule morphisms (ρ_α) from $\mathcal{A}$ into $\mathcal{A} \otimes_p \mathcal{A}$ such that $\pi_{\mathcal{A}} \circ \rho_\alpha(a) \xrightarrow{\|\cdot\|} a$, for all $a \in \mathcal{A}$. Next, Aghababa [15] introduced a new concept of (bounded) approximate biprojectivity and determine its relation to other notions of approximate biprojectivity defined in [23]. Some results about approximate homological notions of Banach homology can be found in [14] and [19].

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Samei et al. in [22] gave a concept of approximate biflatness and they studied some operator structures of Segal algebras and Fourier algebras via this notion. A Banach algebra $\mathcal{A}$ is called *approximately biflat* if there exists a net of $\mathcal{A}$-bimodule morphisms (ρ_α) from $(\mathcal{A}\otimes_p\mathcal{A})^*$ into $\mathcal{A}^*$ such that $W^*OT-\lim \rho_\alpha\circ\pi^*_{\mathcal{A}} = id_{\mathcal{A}^*}$, where W*OT denotes for the weak-star operator topology. Here, we remind that for Banach algebras $\mathcal{A}$ and $\mathcal{B}$ the *weak* operator topology* (W*OT) on $B(\mathcal{A},\mathcal{B}^*)$ (the set of all bounded linear operators from $\mathcal{A}$ into $\mathcal{B}^*$) is a topology determined by seminorms $\{p_{x,y} : x \in \mathcal{A}, y \in \mathcal{B}\}$ that $p_{x,y}(T) = |T(x)(y)|$, where $T \in B(\mathcal{A},\mathcal{B}^*)$. In other words, $T_\alpha \xrightarrow{W^*OT} T$ if and only if for every $x \in \mathcal{A}$; $T_\alpha(x) \xrightarrow{w^*} T(x)$. For a *SIN* group G, Samei et al. showed that the Segal algebra $S^1(G)$ is approximately biflat if and only if G is amenable. Recently, module approximately biflat and module approximately biprojective Banach algebras were studied in [4], applied to the weighted inverse semigroup algebra $l^1(S,\omega)$ and some results in [2] were improved as well.

In the last decades, some homological notions for a Banach algebra $\mathcal{A}$ based on character space such as ϕ-amenability (character amenability) [9, 11], ϕ-contractibility (character contractibility) [13], ϕ-biprojectivity and ϕ-biflatness [20] have been studied by a number of authors, where ϕ is a character on $\mathcal{A}$. In [11], Monfared characterized the structure of (right) character amenable Banach algebras and proved that for any locally compact group G, (right) character amenability of $L^1(G)$ is equivalent to the amenability of G. Module character amenability of Banach algebras and application to inverse semigroup algebras can be found in [3]. As some results in [20], the authors showed that $L^1(G)$ is ϕ-biflat if and only if G is an amenable group and moreover $A(G)$ is ϕ-biprojective if and only if G is a discrete group. It is shown in [13] that $L^1(G)$ is left character contractible if and only if G is finite. The same result is valid for $A(G)$. Recently, approximate left ϕ-biflatness for Banach algebras was introduced and studied in [21].

A large class of Banach algebras (called F-algebras) equipped with $\theta-$Lau product has been introduced and investigated by Lau in [10] for certain class of Banach algebras, where θ is a character. This class includes group algebra, measure algebra and Fourier algebra of a locally compact group. This product is followed by Monfared in general [12].

Motivated by considerations above, we show that under which conditions, the approximate biprojectivity of a Banach algebra $\mathcal{A}$ or its second dual implies that $\mathcal{A}$ is left ϕ-contractible. The same results hold for approximate biflatness and left ϕ-amenability. We also study the approximate biprojectivity and the approximate biflatness of certain Banach algebras. In other words, we investigate the approximate biflatness and approximate biprojectivity of some θ-Lau product structures and semigroup algebras. More precisely, we prove that $L^1(G)\times_\theta A(G)$ is approximately biprojective if and only if G is finite.

2. Some properties of approximate biprojectivity and approximate biflatness

Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$-bimodule. Then, with the following actions X^* is also a Banach $\mathcal{A}$-bimodule:

$$a\cdot f(x) = f(x\cdot a), \quad f\cdot a(x) = f(x\cdot a) \qquad (a\in\mathcal{A}, x\in X, f\in X^*).$$

The projective tensor product $\mathcal{A}\otimes_p\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule with the following actions:

$$a\cdot(b\otimes c) = ab\otimes c, \quad (b\otimes c)\cdot a = b\otimes ca \qquad (a,b,c\in\mathcal{A}).$$

Throughout this paper, $\Delta(\mathcal{A})$ denotes the character space of $\mathcal{A}$, that is, all non-zero multiplicative linear functionals on $\mathcal{A}$. Let $\phi\in\Delta(\mathcal{A})$. Then, ϕ has a unique extension on $\mathcal{A}^{**}$ denoted by $\tilde{\phi}$ and defined via $\tilde{\phi}(F) = F(\phi)$ for every $F\in\mathcal{A}^{**}$. Clearly, this extension remains to be a character on $\mathcal{A}^{**}$.

Let $\mathcal{A}$ be a Banach algebra and $\phi\in\Delta(\mathcal{A})$. Then, $\mathcal{A}$ is called *left* (*right*) *ϕ-contractible* if there exists an element $m\in\mathcal{A}$ such that $am = \phi(a)m$ ($ma = \phi(a)m$) and $\phi(m) = 1$, for all $a\in\mathcal{A}$. Moreover, $\mathcal{A}$ is called *character contractible* if it is left ϕ-contractible for all $\phi\in\Delta(\mathcal{A})$ and posses a left identity [13].

Theorem 2.1. *Let $\mathcal{A}$ be a Banach algebra and $\phi\in\Delta(\mathcal{A})$. Suppose that I is a closed ideal of $\mathcal{A}$ which posses a left approximate identity such that $\phi|_I\neq 0$. If $\mathcal{A}$ approximately biprojective, then I is left ϕ-contractible. In particular, $\mathcal{A}$ is left ϕ-contractible.*

Proof. By our assumptions, there exists a net of $\mathcal{A}$-bimodule morphisms (ρ_α) from $\mathcal{A}$ into $\mathcal{A} \otimes_p \mathcal{A}$ such that $\pi_{\mathcal{A}} \circ \rho_\alpha(a) \to a$, for all $a \in \mathcal{A}$. Put $L = I \cap \ker \phi$. It is easy to see that L is a closed ideal of I. Consider the quotient map $q : I \longrightarrow \frac{I}{L}$. Pick $i_0 \in I$ such that $\phi(i_0) = 1$. Define the map $L_{i_0} : \mathcal{A} \longrightarrow I$ by $L_{i_0}(a) = i_0 a$ for all $a \in \mathcal{A}$. It is obvious that L_{i_0} is a continuous map. Now, set

$$\eta_\alpha := (id_{\mathcal{A}} \otimes q) \circ (id_{\mathcal{A}} \otimes L_{i_0}) \circ \rho_\alpha|_I : I \longrightarrow \mathcal{A} \otimes_p \frac{I}{L}.$$

It is easily verified that (η_α) is a net of I-bimodule morphisms. We claim that $\eta_\alpha(l) = 0$, for all $l \in L$. To see this, having a left approximate identity for I implies that $\overline{IL} = L$. For an arbitrary element l of L, there exist sequences (i_n) and (l_n) such that $l = \lim_n i_n l_n$. Since $q(L) = \{0\}$, we get

$$\eta_\alpha(l) = (id_{\mathcal{A}} \otimes q) \circ (id_{\mathcal{A}} \otimes L_{i_0}) \circ \rho_\alpha|_I(\lim_n i_n l_n) = \lim_n (id_{\mathcal{A}} \otimes q) \circ (id_{\mathcal{A}} \otimes L_{i_0})(\rho_\alpha|_I(i_n) \cdot l_n)) = 0.$$

It follows that η_α induces a net of I-bimodule morphisms from $\frac{I}{L}$ into $\mathcal{A} \otimes_p \frac{I}{L}$, which we denote it again by (η_α). Fix α. Set $m := \eta_\alpha(i_0 + L) \in \mathcal{A} \otimes_p \frac{I}{L}$. From the fact $\frac{I}{L} \cong \mathbb{C}$, we find $\mathcal{A} \otimes_p \frac{I}{L} \cong \mathcal{A} \otimes_p \mathbb{C} \cong \mathcal{A}$. Hence, we may consider m as an element of $\mathcal{A}$. Here, we show that $im = \phi(i)m$ and $\phi(m) = 1$ for all $i \in I$. We have $ii_0 + L = \phi(i)i_0 + L$ and η_α is a bounded I-bimodule morphism. Thus

$$im = i\eta_\alpha(i_0 + L) = \eta_\alpha(ii_0 + L) = \eta_\alpha(\phi(i)i_0 + L) = \phi(i)\eta_\alpha(i_0 + L) = \phi(i)m,$$

and

$$\phi(m) = (\phi \otimes \overline{\phi}) \circ (id_{\mathcal{A}} \otimes q) \circ (id_{\mathcal{A}} \otimes L_{i_0}) \circ \rho_\alpha|_I(i_0 + L) = \phi \circ \pi_A \circ \rho_\alpha|_I(i_0) \to \phi(i_0) = 1,$$

for all $i \in I$. Replacing m with $\frac{mi_0}{\phi(m)}$, for a large enough α, we can assume that $m \in I$ and $\phi(m) = 1$. This shows that I is left ϕ-contractible. Moreover, Proposition 3.8 from [13] implies that $\mathcal{A}$ is left ϕ-contractible. □

The following corollaries are the direct consequences of Theorem 2.1.

Corollary 2.2. *Let $\mathcal{A}$ be a Banach algebra with a left approximate identity and $\phi \in \Delta(\mathcal{A})$. Suppose that $\mathcal{A}$ is a closed ideal of $\mathcal{A}^{**}$. If $\mathcal{A}^{**}$ is approximately biprojective, then $\mathcal{A}$ is left ϕ-contractible.*

Proof. It is known that if $\phi \in \Delta(\mathcal{A})$, then $\tilde{\phi} \in \Delta(\mathcal{A}^{**})$. The proof will be finished by Theorem 2.1. □

Corollary 2.3. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. Suppose that I is a closed ideal of $\mathcal{A}$ which posses an approximate identity such that $\phi|_I \neq 0$. If $\mathcal{A}$ is approximately biprojective, then there exists an element a_0 in $Z(\mathcal{A})$ (the center of $\mathcal{A}$) such that $\phi(a_0) = 1$.*

Proof. By Theorem 2.1, $\mathcal{A}$ is left and right ϕ-contractible. Then, there exist elements m_1 and m_2 in $\mathcal{A}$ such that $am_1 = \phi(a)m_1$, $m_2 a = \phi(a)m_2$ and $\phi(m_1) = \phi(m_2) = 1$ for all $a \in \mathcal{A}$. Put $M = m_1 m_2 \in \mathcal{A}$. Then

$$aM = am_1m_2 = \phi(a)m_1m_2 = m_1m_2\phi(a) = Ma, \quad \phi(M) = \phi(m_1)\phi(m_2) = 1,$$

for all $a \in \mathcal{A}$. □

Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. Recall from [20] that $\mathcal{A}$ is said to be *ϕ-biprojective*, if there exists a bounded $\mathcal{A}$-bimodule morphism $\rho : \mathcal{A} \longrightarrow \mathcal{A} \otimes_p \mathcal{A}$ such that $\phi \circ \pi_{\mathcal{A}} \circ \rho(a) = \phi(a)$, for all $a \in \mathcal{A}$. Furthermore, $\mathcal{A}$ is called *ϕ-biflat* if there exists a bounded $\mathcal{A}$-bimodule morphism $\rho : \mathcal{A} \longrightarrow (\mathcal{A} \otimes_p \mathcal{A})^{**}$ such that $\tilde{\phi} \circ \pi_{\mathcal{A}} \circ \rho(a) = \phi(a)$, for all $a \in A$ [20].

The proof of the next proposition is similar to the proof of Theorem 2.1, and so omitted.

Proposition 2.4. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. Suppose that I is a closed ideal of $\mathcal{A}$ which posses a left approximate identity such that $\phi|_I \neq 0$. If $\mathcal{A}$ is ϕ-biprojective, then I is left ϕ-contractible. Moreover, $\mathcal{A}$ is left ϕ-contractible.*

The upcoming lemmas are some fundamental tools in obtaining our results in this paper.

Lemma 2.5. *Let $\mathcal{A}$ be an approximately biprojective Banach algebra and $\phi \in \Delta(\mathcal{A})$. Suppose that $a_0 \in \mathcal{A}$ is an element satisfying $aa_0 = a_0a$ and $\phi(a_0) = 1$, for all $a \in \mathcal{A}$. Then, $\mathcal{A}$ is left ϕ-contractible.*

Proof. Our assumptions necessitate that there exists a net (ρ_α) of A-bimodule morphisms from $\mathcal{A}$ into $\mathcal{A}\otimes_p\mathcal{A}$ such that $\pi_{\mathcal{A}} \circ \rho_\alpha(a) \to a$ for all $a \in \mathcal{A}$. Set $m_\alpha := \rho_\alpha(a_0) \in A \otimes_p A$. We have

$$a \cdot m_\alpha = a \cdot \rho_\alpha(a_0) = \rho_\alpha(aa_0) = \rho_\alpha(a_0a) = \rho_\alpha(a_0) \cdot a = m_\alpha \cdot a,$$

and

$$\phi \circ \pi_{\mathcal{A}}(m_\alpha) = \phi \circ \pi_{\mathcal{A}} \circ \rho_\alpha(a_0) \to \phi(a_0) = 1,$$

for all $a \in \mathcal{A}$. Define the mapping $T : \mathcal{A} \otimes_p \mathcal{A} \to \mathcal{A}$ via $T(a \otimes b) = \phi(b)a$ for all $a, b \in \mathcal{A}$. Obviously, T is a bounded linear map which satisfies

$$aT(x) = T(a \cdot x), \quad T(x \cdot a) = \phi(a)T(x), \quad \phi \circ T(x) = \phi \circ \pi_{\mathcal{A}}(x),$$

for all $a \in \mathcal{A}$ and $x \in \mathcal{A} \otimes_p \mathcal{A}$. Put $n_\alpha = T(m_\alpha)$. Thus

$$an_\alpha = aT(m_\alpha) = T(a \cdot m_\alpha) = T(m_\alpha \cdot a) = \phi(a)T(m_\alpha) = \phi(a)n_\alpha,$$

and

$$\phi(n_\alpha) = \phi \circ T(m_\alpha) = \phi \circ \pi_{\mathcal{A}}(m_\alpha) \to 1,$$

for all $a \in \mathcal{A}$. Interchanging n_α into $\frac{n_\alpha}{\phi(n_\alpha)}$, we conclude that $an_\alpha = \phi(a)n_\alpha$ and $\phi(n_\alpha) = 1$ for all $a \in \mathcal{A}$. Therefore, $\mathcal{A}$ is left ϕ-contractible. □

Lemma 2.6. *Let $\mathcal{A}$ be an approximately biflat Banach algebra. Then, there exists a net of $\mathcal{A}$-bimodule morphisms from A^{**} into $(\mathcal{A} \otimes_p \mathcal{A})^{**}$ such that $\pi^{**}_{\mathcal{A}} \circ \rho_\alpha(\hat{a}) \xrightarrow{w^*} \hat{a}$, for all $a \in \mathcal{A}$, where $\hat{a}$ is denoted for the canonical embedding of a in $\mathcal{A}^{**}$.*

Proof. Our hypothesis implies that there exists a net of $\mathcal{A}$-bimodule morphisms (η_α) from $(\mathcal{A} \otimes_p \mathcal{A})^*$ into $\mathcal{A}^*$ such that $\eta_\alpha \circ \pi^*_{\mathcal{A}}(f) \xrightarrow{w^*} f$, for all $f \in (\mathcal{A} \otimes_p \mathcal{A})^*$. Take $\rho_\alpha = \eta^*_\alpha$. It is clear that (ρ_α) is a net of $\mathcal{A}$-bimodule morphisms from $\mathcal{A}^{**}$ into $(\mathcal{A} \otimes_p \mathcal{A})^{**}$. For each $a \in \mathcal{A}$, we obtain

$$(\pi^{**}_{\mathcal{A}} \circ \rho_\alpha(\hat{a}) - \hat{a})(f) = \pi^{**}_{\mathcal{A}} \circ \rho_\alpha(\hat{a})(f) - \hat{a}(f) = \hat{a}(\eta_\alpha \circ \pi^*_{\mathcal{A}}(f) - f) = \eta_\alpha \circ \pi^*_{\mathcal{A}}(f)(a) - f(a) \to 0.$$

This means that the result is valid. □

Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. We recall from [9] that $\mathcal{A}$ is *left* (*resp. right*) *ϕ-amenable* if there exists an element $m \in \mathcal{A}^{**}$ such that $am = \phi(a)m$ (resp. $ma = \phi(a)m$) and $\tilde{\phi}(m) = 1$, for all $a \in \mathcal{A}$. Moreover, $\mathcal{A}$ is said to be *character amenable* if it is left ϕ-amenable for all $\phi \in \Delta(\mathcal{A})$ and posses a bounded left approximate identity.

In analogues to Theorem 2.1, we have the following result for the left ϕ-amenability case.

Theorem 2.7. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. Suppose that I is a closed ideal of $\mathcal{A}$ which posses a left approximate identity such that $\phi|_I \neq 0$. If $\mathcal{A}^{**}$ is approximately biflat, then I is left ϕ-amenable. In addition, $\mathcal{A}$ is left ϕ-amenable.*

Proof. By Lemma 2.6, there exists a net of $\mathcal{A}^{**}$-bimodule morphisms, say Γ_α, from $\mathcal{A}^{****}$ into $(\mathcal{A}^{**} \otimes_p \mathcal{A}^{**})^{**}$ such that $\pi^{**}_{\mathcal{A}^{**}} \circ \Gamma_\alpha(\hat{a}) \xrightarrow{w^*} \hat{a}$, for all $a \in \mathcal{A}^{**}$. On the other hand, by [5, Lemma 1.7], there exists a bounded linear map $\psi : \mathcal{A}^{**} \otimes_p \mathcal{A}^{**} \longrightarrow (\mathcal{A} \otimes_p \mathcal{A})^{**}$ such that for $a, b \in \mathcal{A}$ and $m \in \mathcal{A}^{**} \otimes_p \mathcal{A}^{**}$, the following holds:

(i) $\psi(a \otimes b) = a \otimes b$;
(ii) $\psi(m) \cdot a = \psi(m \cdot a)$;
(iii) $a \cdot \psi(m) = \psi(a \cdot m)$;
(iv) $\pi^{**}_{\mathcal{A}}(\psi(m)) = \pi_{\mathcal{A}^{**}}(m)$.

Consider the mapping $\rho_\alpha : \psi^{**} \circ \Gamma_\alpha|_{\mathcal{A}} : \mathcal{A} \longrightarrow (\mathcal{A} \otimes_p \mathcal{A})^{****}$. Since ψ^{**} and $\pi_{\mathcal{A}}^{****}$ are w^*-continuous maps, we have

$$\pi_{\mathcal{A}}^{****} \circ \rho_\alpha(a) - a = \pi_{\mathcal{A}}^{****} \circ \psi^{**} \circ \Gamma_\alpha(a) - a = \pi_{\mathcal{A}^{**}}^{**} \circ \Gamma_\alpha(a) - a \xrightarrow{w^*} 0,$$

where $a \in \mathcal{A}$. Following the notation and the arguments in the proof of Theorem 2.1, set

$$\eta_\alpha := (id_A \otimes q)^{**} \circ (id_{\mathcal{A}} \otimes L_{i_0})^{**} \circ \rho_\alpha|_I : I \longrightarrow \left(\mathcal{A} \otimes_p \frac{I}{L}\right)^{****}.$$

Put $m_\alpha = \eta_\alpha(i_0 + L) \in \mathcal{A}^{****}$. One can readily see that $im_\alpha = \phi(i)m_\alpha$ and $\tilde{\tilde{\phi}}(m_\alpha) = 1$ for all $i \in I$. Using Mazur's lemma and replacing m_α with $m_\alpha i_0$, we may assume that $m_\alpha \in I^{**}$. Hence, I is left ϕ-amenable. It also concludes that $\mathcal{A}$ is left ϕ-amenable. □

The next corollary has a similar proof to Theorem 2.1. We include it witout the proof.

Corollary 2.8. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. Suppose that I is a closed ideal of $\mathcal{A}$ which posses a left approximate identity such that $\phi|_I \neq 0$. Under one of the following conditions, I is left ϕ-amenable. In particular, $\mathcal{A}$ is left ϕ-amenable.*

(i) *$\mathcal{A}$ is approximately biflat;*
(ii) *$\mathcal{A}$ is ϕ-biflat.*

3. Applications for known Banach algebras

For two normed Banach algebras $\mathcal{A}$ and $\mathcal{B}$ such that $\theta \in \Delta(\mathcal{B})$, the Cartesian product $\mathcal{A} \times \mathcal{B}$ with the multiplication

$$(a, b)(a', b') = (aa' + \theta(b')a + \theta(b)a', bb'),$$

and norm $\|(a, b)\| = \|a\| + \|b\|$, is a Banach algebra, for all $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$. The Cartesian product $\mathcal{A} \times \mathcal{B}$ with the above properties is called the $\theta-$Lau product of $\mathcal{A}$ and $\mathcal{B}$ which is denoted by $\mathcal{A} \times_\theta B$. From [12], we identify $\mathcal{A} \times \{0\}$ with $\mathcal{A}$, and $\{0\} \times \mathcal{B}$ with $\mathcal{B}$. It is clear that $\mathcal{A}$ is a closed two-sided ideal while $\mathcal{B}$ is a closed subalgebra of $\mathcal{A} \times_\theta \mathcal{B}$, and $(\mathcal{A} \times_\theta \mathcal{B})/\mathcal{A}$ is isometrically isomorphic to $\mathcal{B}$. If $\theta = 0$, then we obtain the usual direct product of $\mathcal{A}$ and $\mathcal{B}$. Since the direct products often exhibit different properties, we have excluded the possibility that $\theta = 0$. Moreover, if $\mathcal{B} = \mathbb{C}$, the complex numbers, and θ is the identity map on $\mathbb{C}$, then $\mathcal{A} \times_\theta \mathcal{B}$ is the unitization $\mathcal{A}^\sharp$ of $\mathcal{A}$. Note that, by [12, Proposition 2.4], the character space $\Delta(\mathcal{A} \times_\theta \mathcal{B})$ of $\mathcal{A} \times_\theta \mathcal{B}$ is equal to

$$\{(\phi, \theta) : \ \phi \in \Delta(\mathcal{A})\} \bigcup \{(0, \psi) : \ \psi \in \sigma(\mathcal{B})\}.$$

Furthermore, the dual space $(\mathcal{A} \times_\theta \mathcal{B})^*$ of $\mathcal{A} \times_\theta \mathcal{B}$ is identified with $\mathcal{A}^* \times \mathcal{B}^*$ such that for each $(a, b) \in \mathcal{A} \times_\theta \mathcal{B}$, $\phi \in \Delta(\mathcal{A})$ and $\psi \in \Delta(\mathcal{B})$ we have

$$\langle(\phi, \psi), (a, b)\rangle = \phi(a) + \psi(b).$$

Now, assume that $\mathcal{A}^{**}$, $\mathcal{B}^{**}$ and $(\mathcal{A} \times_\theta \mathcal{B})^{**}$ are equipped with their first Arens product $\square$. Then, $(\mathcal{A} \times_\theta \mathcal{B})^{**}$ is isometrically isomorphic with $\mathcal{A}^{**} \times_\theta \mathcal{B}^{**}$. In addition, for all $(m, n), (p, q) \in (\mathcal{A} \times_\theta \mathcal{B})^{**}$ the first Arens product is defined by

$$(m, n)\square(p, q) = (m\square p + n(\theta)p + q(\theta)m, n\square q);$$

for more details, we refer to [12, Proposition 2.12].

Let G be a locally compact group. We denote $A(G)$ and $L^1(G)$ for the Fourier algebra and group algebra, respectively.

Theorem 3.1. *Let G be a locally compact group G. Then, $L^1(G) \times_\theta A(G)$ is approximately biprojective if and only if G is finite.*

Proof. Suppose that $L^1(G) \times_\theta A(G)$ is approximately biprojective. It is well-known that $L^1(G)$ has a bounded approximate identity and also $L^1(G)$ is a closed ideal of $L^1(G) \times_\theta A(G)$. Applying Theorem 2.1, we find that $L^1(G)$ is left ϕ-contractible for all $\phi \in \Delta(L^1(G))$. Using [13, Theorem 6.1], we see that G is compact. On the other hand, the element $(0, a)$ commutes with each elements of $L^1(G) \times_\theta A(G)$. Pick $a_0 \in A(G)$ which $\psi(a_0) = 1$, where ψ is a character on $A(G)$ with $\psi \neq \theta$. Since $A(G)$ is a commutative Banach algebra, the element $(0, a_0)$ commutes with each element of $L^1(G) \times_\theta A(G)$ and $(0, \psi)(0, a_0) = \psi(a_0) = 1$. By Lemma 2.5, $L^1(G) \times_\theta A(G)$ is left $(0, \psi)$-contractible and so $A(G)$ is left ψ-contractible. Now, Theorem 3.5 from [1] can be applied to show that G is discrete, and therefore G must be finite.
The converse is clear. □

Let G be a locally compact group. A linear subspace $S^1(G)$ of $L^1(G)$ is said to be a *Segal algebra* on G if it satisfies the following conditions:

(i) $S^1(G)$ is dense in $L^1(G)$,

(ii) $S(G)$ with a norm $\|\cdot\|_{S^1(G)}$ is a Banach space and $\|f\|_{L^1(G)} \leq \|f\|_{S^1(G)}$ for all $f \in S^1(G)$,

(iii) for $f \in S^1(G)$ and $y \in G$, we have $L_y(f) \in S^1(G)$ the map $y \mapsto L_y(f)$ from G into $S^1(G)$ is continuous, where $L_y(f)(x) = f(y^{-1}x)$,

(iv) $\|L_y(f)\|_{S^1(G)} = \|f\|_{S^1(G)}$ for all $f \in S^1(G)$ and $y \in G$.

It is well-known that $S(G)$ always has a left approximate identity; for more information refer to [17].

It has been shown in [1, Lemma 2.2] that for a Segal algebra $S^1(G)$

$$\Delta(S^1(G)) = \left\{\phi|_{S^1(G)} | \phi \in \Delta(L^1(G))\right\}.$$

Besides, it was proved in [1, Corollary 3.4] that for a locally compact group G, $S^1(G)$ is left ϕ-amenable if and only if G is amenable. Using the above facts, we have the next result.

Proposition 3.2. *Let G be a locally compact group. If $S^1(G) \times_\theta S^1(G)$ is approximately biflat, then G is amenable.*

Proof. Since $S^1(G)$ has an approximate identity and is a closed ideal of $S^1(G) \times_\theta S^1(G)$, it is left ϕ-amenable, and so by [1, Corollary 3.4], G is amenable. □

Theorem 3.3. *Let G be a locally compact group G. If $(S^1(G) \times_\theta \mathcal{A})^{**}$ is approximately biflat, then G is amenable, where $\mathcal{A}$ is any Banach algebra with a non-empty character space.*

Proof. Suppose that $(S^1(G) \times_\theta \mathcal{A})^{**}$ is approximately biflat. It is known that $S^1(G)$ is a closed ideal of $S^1(G) \times_\theta A(G)$ which has a left approximate identity. Applying Theorem 2.7, we arrive to the left ϕ-amenability of $S^1(G)$ and hence by Corollary 3.4 from [1], G is amenable. □

One should remember that a Banach algebra $\mathcal{A}$ is amenable if and only if there exists an element $M \in (\mathcal{A} \otimes_p \mathcal{A})^{**}$ such that $a \cdot M = M \cdot a$ and $\pi^{**}_{\mathcal{A}}(M)a = a$, for all $a \in \mathcal{A}$. Furthermore, $\mathcal{A}$ is amenable if and only if $\mathcal{A}$ is biflat with a bounded approximate identity (see chapter 4 of [18]).

Theorem 3.4. *Let G be a locally compact group. Then, $L^1(G) \times_\theta L^1(G)$ is approximately biflat if and only if G is amenable.*

Proof. Assume that $L^1(G) \times_\theta L^1(G)$ is approximately biflat. Since $L^1(G)$ is a closed ideal of $L^1(G) \times_\theta L^1(G)$, Corollary 2.8 gives that $L^1(G)$ is left ϕ-amenable. Now, by [1, Corollary 3.4], G is amenable.

Conversely, suppose that G is amenable group. The celebrated Johnson's theorem [8] implies that $L^1(G)$ is amenable. Hence, by [12, pp. 285], $L^1(G) \times_\theta L^1(G)$ is amenable and thus it is biflat. This implies that $L^1(G) \times_\theta L^1(G)$ is approximately biflat. □

Suppose that $\mathcal{A}$ is a Banach algebra and I is a totally ordered set. We denote $UP(I, \mathcal{A})$ for the set of all $I \times I$ upper triangular matrices which its entries come from $\mathcal{A}$ and

$$\|(a_{i,j})_{i,j \in I}\| = \sum_{i,j \in I} \|a_{i,j}\| < \infty.$$

With matrix operations and $\|\cdot\|$ as a norm, $UP(I, \mathcal{A})$ becomes a Banach algebra.

Example 3.5. The Banach algebra $UP(\mathbb{N},\mathbb{C})$ is not approximately biprojective. To see this, we go toward a contradiction and assume that $UP(\mathbb{N},\mathbb{C})$ is approximately biprojective. Define $\phi((a_{i,j})_{i,j\in\mathbb{N}}) = a_{1,1}$, for every $(a_{i,j})_{i,j\in\mathbb{C}} \in UP(\mathbb{N},\mathbb{C})$. It is easy to see that ϕ is a character on $UP(\mathbb{N},\mathbb{C})$. One can show that $UP(\mathbb{N},\mathbb{C})$ has an approximate identity. Consider $UP(\mathbb{N},\mathbb{C})$ as its closed ideal, by Theorem 2.1, $UP(\mathbb{N},\mathbb{C})$ is right ϕ-contractible. Put

$$J = \{(a_{i,j})_{i,j\in I} \in UP(\mathbb{N},\mathbb{C}) | a_{i,j} = 0 \quad \text{for} \quad i \neq 1\}.$$

It is easily checked that J is a closed ideal of $UP(\mathbb{N},\mathbb{C})$ and $\phi|_J \neq 0$ and thus by [13, Proposition 3.8], J is right ϕ-contractible. Therefore, there exists an element j_0 in J such that

$$j_0 j = \phi(j) j_0, \quad \phi(j_0) = 1 \quad (j \in J). \tag{1}$$

Set

$$j = \begin{pmatrix} 0 & 1 & \cdots & 1 & \cdots \\ 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}_{\mathbb{N}\times\mathbb{N}} \quad \text{and } j_0 = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & \cdots \\ 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}_{\mathbb{N}\times\mathbb{N}},$$

for some $(a_{i,j})$ in $\mathbb{C}$. Put these facts in (1), gives that $a_{1,1} = 0$. But $\phi(j_0) = a_{1,1} = 1$ which is a contradiction.

An inverse semigroup is a semigroup S such that for each $s \in S$, there exists a unique element $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. The set $E(S)$ of idempotents of S is a commutative subsemigroup; it is ordered by $e \leq f$ if and only if $ef = e$. With this ordering $E(S)$ is a meet semilattice (every element is idempotenet)with the meet given by the product; see [7, Theorem 5.1.1]. The order on E extends to S as so-called natural partial order by

$$s \leq t \Leftrightarrow s = ss^*t \quad (s,t \in S).$$

Suppose that $(S, \leq)$ is an inverse semigroup. For an arbitrary element $s \in S$, put $(x] = \{y \in S | \, y \leq x\}$. We say that S is *uniformly locally finite* if $\sup\{|(x]| : x \in S\} < \infty$. With respect to $e \in E(S)$, $G_e = \{s \in S | ss^* = s^*s = e\}$ is denoted for a maximal subgroup of S. An inverse semigroup S is called *Clifford semigroup* if for each $s \in S$ there exists $s^* \in S$ such that $ss^* = s^*s$; for more details see [7].

Proposition 3.6. *Let $S = \cup_{e\in E(S)} G_e$ be a Clifford semigroup such that $E(S)$ is uniformly locally finite. Then, $l^1(S)$ is approximately biprojective if and only if each maximal subgroup G_e is finite.*

Proof. Let $l^1(S)$ be approximately biprojective. Using [16, Theorem 2.16], we have $l^1(S) \cong \ell^1 - \oplus_{e\in E(S)} l^1(G_e)$. It is obvious that $l^1(G_e)$ is a closed ideal of $l^1(S)$ which posses an identity. Furthermore, each character on $l^1(G_e)$ can be extended to whole $l^1(S)$ (for instance the augmentation character of $l^1(G_e)$). Applying Theorem 2.1 follows that $l^1(G_e)$ is left ϕ-contractible, where ϕ is the augmentation character on $l^1(G_e)$. By [1, Theorem 3.3] the discrete group G_e is compact. Then G_e is finite.
The converse is clear by [16, Theorem 3.7]. □

Proposition 3.7. *Let $S = \cup_{e\in E(S)} G_e$ be a Clifford semigroup such that $E(S)$ is uniformly locally finite. Then, $l^1(S)$ is approximately biflat if and only if G_e is amenable.*

Proof. Suppose that $l^1(S)$ is approximately biflat. Theorem 2.16 from [16] implies that $l^1(S) \cong \ell^1 - \oplus_{e\in E(S)} l^1(G_e)$. Since $l^1(G_e)$ is a closed ideal of $l^1(S)$, it has an identity. It now follows from Theorem 2.7 that $l^1(G_e)$ is left ϕ-amenable, where ϕ is the augmentation character on $l^1(G_e)$. By [11, Corollary 2.4], G_e is amenable.
Conversely, let G_e be amenable. Now, Theorem 3.7 of [16] shows that $l^1(S)$ is biflat, and so is approximately biflat. □

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References

[1] M. Alaghmandan, R. Nasr Isfahani and M. Nemati, Character amenability and contractibility of abstract Segal algebras, Bull. Aust. Math. Soc. 82 (2010), 274–281.
[2] A. Bodaghi and M. Amini, Module biprojective and module biflat Banach algaebras, U.P.B. Sci. Bull. Series A. 75 (2013), Iss. 3, 25–36.
[3] A. Bodaghi and M. Amini, Module character amenability of Banach algebras, Arch. Math (Basel). 99 (2012), 353–365.
[4] A. Bodaghi and S. Grailoo Tanha, Module approximate biprojectivity and module approximate bifatness of Banach algebras, Rend. del Cir. Mat. di Palermo Series 2. 70 (2021), 409–425.
[5] F. Ghahramani, R. J. Loy and G. A. Willis, Amenability and weak amenability of second conjugate Banach algebras, Proc. Amer. Math. Soc. 124 (1996), 1489–1497.
[6] A. Ya. Helemskii, The homology of Banach and topological algebras, Kluwer, Academic Press, Dordrecht, 1989.
[7] J. Howie, Fundamental of Semigroup Theory, London Math. Soc Monographs, vol. 12, Clarendon Press, Oxford, 1995.
[8] B. E. Johnson, Cohomology in Banach algebras, Memoirs Amer. Math. Soc. 127, Providence, 1972.
[9] E. Kaniuth, A. T. Lau and J. Pym, On ϕ-amenability of Banach algebras, Math. Proc. Cambridge Philos. Soc. 144 (2008), 85–96.
[10] A. T. Lau, Analysis on a class of Banach algebras with application to harmonic analysis on locally compact groups and semigroups, Fund. Math. 118 (1983), 161–175.
[11] M. S. Monfared, Character amenability of Banach algebras, Math. Proc. Camb. Philos. Soc. 144 (2008), 697–706.
[12] M. S. Monfared, On certain products of Banach algebras with applications to harmonic analysis, Studia Math. 178 (2007), 277–294.
[13] R. Nasr Isfahani and S. Soltani Renani, Character contractibility of Banach algebras and homological properties of Banach modules, Studia Math. 202 (3) (2011), 205–225.
[14] M. Nemati, Some properties of Banach algebras associated with locally compact groups, Colleq. Math. 139 (2) (2015), 259–271.
[15] H. Pourmahmood-Aghababa, Approximately biprojective Banach algebras and nilpotent ideals, Bull. Aust. Math. Soc. 87 (2013), 158–173.
[16] P. Ramsden, Biflatness of semigroup algebras, Semigroup Forum. 79 (2009), 515–530.
[17] H. Reiter, L^1-algebras and Segal Algebras, Lecture Notes in Mathematics. 231, Springer, 1971.
[18] V. Runde, Lectures on Amenability, Springer, New York, 2002.
[19] A. Sahami, Approximate biflatness and approximate biprojectivity of some Banach algebras, Quaestiones Math. 43 (2020), No. 9, 1273–1284.
[20] A. Sahami and A. Pourabbas, On ϕ-biflat and ϕ-biprojective Banach algebras, Bull. Belg. Math. Soc. Simon Stevin. 20 (5) (2013), 789–801.
[21] A. Sahami, M. Rostami and A. Bodaghi, A notion of approximate biflatness for Banach algebras based on character space, Rend. del Cir. Mat. di Palermo Series 2. 72 (2023), 483-492.
[22] E. Samei, N. Spronk and R. Stokke, Biflatness and Pseudo-Amenability of Segal algebras, Canad. J. Math. 62(4) (2010), 845–869.
[23] Y. Zhang, Nilpotent ideals in a class of Banach algebras, Proc. Amer. Math. Soc. 127 (1999), 3237–3242.