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Drazin geometric quasi-mean for Lambert conditional operators

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Abstract. In this paper we introduce the Drazin geometric quasi-mean $A \circledS_\nu B = ||BA^d|^\nu A|^2$ for bounded conditional operator A and B in $L^2(\Sigma)$, where A has closed range and $\nu \geq 0$. In addition, we discuss some measure theoretic characterizations for conditional operators in some operator classes. Moreover, some practical examples are provided to illustrate the obtained results.

1. Introduction and Preliminaries

Let (X, Σ, μ) be a σ-finite measure space and let $\mathcal{A}$ be a σ-subalgebra of Σ such that $(X, \mathcal{A}, \mu)$ is also σ-finite. We denote the collection of σ-measurable complex-valued functions on X by $L^0(\Sigma) = L^0(X, \Sigma, \mu)$. We also adopt the convention that all comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. The support of a measurable function $f \in L^0(\Sigma)$ is defined by $\sigma(f) = \{f \neq 0\} = \{x \in X : f(x) \neq 0\}$. For $f \in L^0(\Sigma)$, by the Radon-Nikodym theorem, there exists a unique $\mathcal{A}$-measurable function $E^{\mathcal{A}}(f)$ such that

$$\int_A f d\mu = \int_A E_\mu^{\mathcal{A}}(f) d\mu, \quad (\forall A \in \mathcal{A})$$

for which $\int_A f d\mu$ exists. Note that $E_\mu^{\mathcal{A}}(f)$ depends both on μ and $\mathcal{A}$. Put $E = E_\mu^{\mathcal{A}}$. A real-valued measurable function $f = f^+ - f^-$ is said to be conditionable if $\mu(\{E(f^+) = \infty = E(f^-)\}) = 0$. If f is complex-valued, then $f \in \mathcal{D}(E) = \{f \in L^0(\Sigma) : f \text{ is conditionable}\}$ if the real and imaginary parts of f are conditionable and their respective expectations are not both infinite on the same set of positive measure. Note that for each $p \in [1, \infty]$, $L^p(\Sigma) = L^p(X, \Sigma, \mu) \subseteq \mathcal{D}(E)$. The mapping $E : L^p(\Sigma) \to L^p(\mathcal{A})$ defined by $f \mapsto E(f)$, is called the conditional expectation operator with respect to pair $(\mathcal{A}, \mu)$. Put $E = E^{\mathcal{A}}$. The mapping E is a linear orthogonal projection. Note that $\mathcal{D}(E)$, the domain of E, contains $L^2(\Sigma) \cup \{f \in L^0(\Sigma) : f \geq 0\}$. For more details on the properties of E see [7, 12, 13]. Those properties of E used in our discussion are summarized below.

$\diamond$ If f is an $\mathcal{A}$-measurable function, then $E(fg) = fE(g)$.
$\diamond$ If $f \geq 0$ then $E(f) \geq 0$; If $f > 0$ then $E(f) > 0$.
$\diamond$ $\sigma(E(|f|))$ is the smallest $\mathcal{A}$-measurable set containing $\sigma(f)$.
$\diamond$ $\sigma(f) \subsetneq \sigma(E(f))$, for each nonnegative $f \in L^2(\Sigma)$.
$\diamond$ $E(|f|^2) = |E(f)|^2$ if and only if $f \in L(\mathcal{A})$.

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Now suppose that $\{u, w, uw\} \subseteq \mathcal{D}(E)$, i.e., $E(u)$, $E(w)$ and $E(uw)$ are defined. Operators of the form M_wEM_u acting in $L^2(\Sigma)$ with $\mathcal{D}(M_wEM_u) = \{f \in L^2(\Sigma) : wE(uf) \in L^2(\Sigma)\}$ are called weighted conditional type operators. It is known that M_wEM_u is densely defined whenever K is finite-valued $\mathcal{A}$-measurable function. Also, by closed graph theorem, $M_wEM_u : \mathcal{D}(M_wEM_u) \to L^2(\Sigma)$ is continuous if and only if $\mathcal{D}(M_wEM_u) = L^2(\Sigma)$ (see [6]).

Let $B_C(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$ with closed range. For $T \in B_C(\mathcal{H})$, if there exists an operator $T^D \in B_C(\mathcal{H})$ satisfying the following three operator equations

$$T^dTT^d = T^d, \;\; TT^d = T^dT, \;\; T^{k+1}T^d = T^k \tag{1}$$

then T^d is called a Drazin inverse of T. The smallest k such that (1) holds, is called the index of T, denoted by $ind(T)$. Notice also that $ind(T)$ (if it is finite) is the smallest non-negative integer k such that $R(T^{k+1}) = R(T^k)$ and $N(T^{k+1}) = N(T^k)$ hold. For other important properties of $T^\dagger$ and T^d see [1–3].
For $\nu \in [0, 1]$, the geometric mean $A\sharp_\nu B$ of positive invertible operator A and positive operator B is defined as $A\sharp_\nu B = \||B^{\frac{1}{2}}A^{\frac{-1}{2}}|^\nu A^{\frac{1}{2}}|^2$. Let $\mathbb{B}^{-1}(\mathcal{H})$ be the class of all bounded linear invertible operators on $\mathcal{H}$. Dragomir in [4] introduced the concept of quadratic weighted operator geometric mean of operators. For $\nu \geq 0$, the quadratic weighted operator geometric mean of the pair $(A, B) \in \mathbb{B}^{-1}(\mathcal{H}) \times \mathbb{B}(\mathcal{H})$ is defined by $A \circledS_\nu B = \||BA^{-1}|^\nu A|^2$. Also, for general case see [5]. When $A \in \mathbb{BC}(\mathcal{H})$ is not invertible, we introduce the Drazin-Dragomir geometric quasi-mean of the pair $(A, B) \in \mathbb{BC}(\mathcal{H}) \times \mathbb{B}(\mathcal{H})$ as $A \textcircled{d}_\nu B = \||BA^d|^\nu A|^2$. For $A \in \mathbb{B}^{-1}(\mathcal{H})$, $A \textcircled{d}_\nu B = A \circledS_\nu B$.
In the next section, first we review some basic results on $T = M_wEM_u$ and we introduce the Drazi-Dragomir geometric quasi-mean $A \textcircled{d}_\nu B = \||BA^d|^\nu A|^2$ for bounded conditional operator A and B in $L^2(\Sigma)$, where A has closed range and $\nu \geq 0$. Also, we obtained some operator equalities for the Drazin-Dragomir quasi-mean on $\mathbb{BC}(\mathcal{H}) \times \mathbb{B}(\mathcal{H})$. To explain the results, some examples are then presented. From now on, we assume that $C = \sigma(E(uw)), F = \sigma(E(rs))$.

2. Characterizations of Drazin geometric quasi-mean

Theorem 2.1. *[8, 9] Let $(u, w, uw) \in \mathcal{D}(E)$ and $T = M_wEM_u$ is a Lambert conditional type operator.*

(1) $T \in \mathbb{B}(L^2(\Sigma))$ if and only if $E(|w|^2)E(|u|^2) \in L^\infty(\Sigma)$, and in this case $\|T\|^2 = \|E(|w|^2)E(|u|^2)\|_\infty$.

(2) Let $T \in \mathbb{B}(L^2(\Sigma))$, $0 \leq u \in L^0(\Sigma)$ and $v = u(E(|w|^2))^{\frac{1}{2}}$. If $E(v) \geq \delta$ on $\sigma(v)$, then T has closed range.

(3) If $w = g\bar{u}$ for some $0 \leq g \in L^0(\mathcal{A})$, then $T = M_{g\bar{u}}EM_u \geq 0$ and for each $\beta > 0$, $T^\beta(f) = \left\{g^\beta E(|u|^2)^{\beta-1}\right\}\bar{u}E(uf)$.

Definition 2.2. *For $A \in \mathbb{BC}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{H})$, the Drazi-Dragomir geometric quasi-mean of (A, B) is defined by*

$$A \textcircled{d}_\nu B = \||BA^d|^\nu A|^2, \qquad \nu > 0.$$

Theorem 2.3. *Let $\nu \geq 0, A = M_wEM_u \in \mathbb{BC}(L^2(\Sigma))$ and $B = M_rEM_s \in \mathbb{B}(L^2(\Sigma))$. Then*

$$A \textcircled{d}_\nu B = M_{\frac{E(|r|^2)^\nu |E(sw)|^{2\nu} E(|u|^2)^{\nu-1} |E(uw)|^2 \chi_C}{E(uw)^{4\nu}}} M_{\bar{u}}EM_u.$$

Proof. Direct computations show that

$$\begin{aligned} B^*BA^d &= (M_{\bar{s}E(|r|^2)}EM_s)(M_{\frac{w\chi_C}{E(uw)^2}}EM_u) \\ &= M_{\frac{E(|r|^2)E(sw)\bar{s}\chi_C}{E(uw)^2}}EM_u, \end{aligned}$$

$$A^{d^*}B^*BA^d = (M_{\frac{\bar{u}\chi_C}{E(uw)^2}}EM_{\bar{w}})(M_{\frac{E(|r|^2)E(sw)s\chi_C}{E(uw)^2}}EM_u)$$
$$= M_{\frac{E(|r|^2)|E(sw)|^2\chi_C}{E(uw)^4}}M_{\bar{u}}EM_u,$$

$$(A^{d^*}B^*BA^d)^{\alpha} = M_{\frac{E(|r|^2)^{\alpha}|E(sw)|^{2\alpha}E(|u|^2)^{\alpha-1}\chi_C}{E(uw)^{4\alpha}}}M_{\bar{u}}EM_u,$$

$$A^*(A^{d^*}B^*BA^d)^{\alpha}A = M_{\frac{E(|r|^2)^{\alpha}|E(sw)|^{2\alpha}E(|u|^2)^{\alpha-1}|E(uw)|^2\chi_C}{E(uw)^{4\alpha}}}M_{\bar{u}}EM_u.$$

It follows that

$$A ⓓ_{\nu} B = M_{\frac{E(|r|^2)^{\nu}|E(sw)|^{2\nu}E(|u|^2)^{\nu-1}|E(uw)|^2\chi_C}{E(uw)^{4\nu}}}M_{\bar{u}}EM_u.$$

This complete the proof. □

Theorem 2.4. *Let $\nu \geq 0, A = M_wEM_u \in \mathbb{BC}(L^2(\Sigma))$ and $B = M_rEM_s \in \mathbb{BC}(L^2(\Sigma))$. Then $(A ⓓ_{\nu} B)^d = A^d ⓓ_{\nu} B^d$ if and only if*

$$E(rs)E(uw) = \sqrt{E(|r|^2)E(|u|^2)}|E(sw)|.$$

Proof. We know that

$$A ⓓ_{\nu} B = M_{\frac{E(|r|^2)^{\nu}|E(sw)|^{2\nu}E(|u|^2)^{\nu-1}|E(uw)|^2\chi_C}{E(uw)^{4\nu}}}M_{\bar{u}}EM_u.$$

Thus, using the lemma..

$$(A ⓓ_{\nu} B)^d = M_{\frac{E(uw)^{4\nu}\chi_{C\cap\sigma(E(sw))}}{E(|r|^2)^{\nu}|E(sw)|^{2\nu}E(|u|^2)^{\nu+1}|E(uw)|^2}}M_{\bar{u}}EM_u.$$

Also,

$$A^d ⓓ_{\nu} B^d = M_{\frac{E(|r|^2)^{\nu}|E(sw)|^{2\nu}E(|u|^2)^{\nu-1}|E(uw)|^2\chi_{(C\cap F)}}{E(rs)^{4\nu}E(uw)^4}}M_{\bar{u}}EM_u.$$

Put $a = E(|u|^2)$ and $b = E(|r|^2)$. If $(A ⓓ_{\nu} B)^d = A^d ⓓ_{\nu} B^d$, then for each $f \in L^2(\Sigma)$, we have

$$\frac{E(uw)^{4\nu}\chi_{C\cap\sigma(E(sw))}\bar{u}E(uf)}{b^{\nu}|E(sw)|^{2\nu}a^{\nu+1}|E(uw)|^2} = \frac{b^{\nu}|E(sw)|^{2\nu}a^{\nu-1}|E(uw)|^2\chi_{(C\cap F)}\bar{u}E(uf)}{E(rs)^{4\nu}E(uw)^4}. \tag{2}$$

Take $f_n = \bar{u}\sqrt{E(|w|^2)}\chi_{A_n}$. Replacing f in (2) by f_n and by simplifying, we get that

$$E(rs)^{4\nu}E(uw)^{4\nu}\bar{u} = b^{2\nu}|E(sw)|^{4\nu}a^{2\nu}\bar{u}.$$

Now, by multiplying the sides of above by u and then taking E of both sides equation we obtain

$$E(rs)^{4\nu}E(uw)^{4\nu} = b^{2\nu}|E(sw)|^{4\nu}a^{2\nu}.$$

It follows that

$$E(rs)E(uw) = \sqrt{E(|r|^2)E(|u|^2)}|E(sw)|.$$

Conversely, if $E(rs)E(uw) = \sqrt{E(|r|^2)E(|u|^2)}|E(sw)|$, it is easy to check that $(A ⓓ_{\nu} B)^d = A^d ⓓ_{\nu} B^d$. This complete the proof. □

Theorem 2.5. *Let $\nu \geq 0, A = M_w E M_u \in \mathbb{BC}(L^2(\Sigma))$ and $B = M_r E M_s \in \mathbb{BC}(L^2(\Sigma))$. Then $(A ⓓ_\nu B)^n = A^n ⓓ_\nu B^n$ for all $n \in \mathbb{N}$ if and only if*

$$|E(rs)||E(uw)| = \sqrt{E(|r|^2)E(|u|^2)}|E(sw)|, \quad on\ C \cap F.$$

Proof. Since $A^n = M_{wE(uw)^{n-1}} E M_u, B^n = M_{rE(rs)^{n-1}} E M_s$. Then, we have

$$A^n ⓓ_\nu B^n = M_{\frac{E(|r|^2)^\nu |E(rs)|^{2\nu(n-1)} |E(sw)|^{2\nu} E(|u|^2)^{\nu-1} |E(uw)|^{(n-1)2\nu+2n} \chi_C}{E(uw)^{4\nu n}}} M_{\bar{u}} E M_u.$$

Also, by using the Theorem(2.1)

$$(A ⓓ_\nu B)^n = M_{\frac{E(|r|^2)^{\nu n} |E(sw)|^{2\nu n} E(|u|^2)^{n\nu-1} |E(uw)|^{2n} \chi_C}{E(uw)^{4\nu n}}} M_{\bar{u}} E M_u.$$

Put $a = E(|u|^2)$ and $b = E(|r|^2)$. It is easy to check that $(A ⓓ_\nu B)^n = A^n ⓓ_\nu B^n$ iff for each $f \in L^2(\Sigma)$,

$$\frac{b^{\nu n} |E(sw)|^{2\nu n} a^{n\nu-1} |E(uw)|^{2n} \bar{u} E(uf)}{E(uw)^{4\nu n}}$$

$$= \frac{b^\nu |E(rs)|^{2\nu(n-1)} |E(sw)|^{2\nu} a^{\nu-1} |E(uw)|^{(n-1)2\nu+2n} \bar{u} E(uf)}{E(uw)^{4\nu n}}.$$

Put $f_n = \bar{u}\sqrt{E(w^2)}\chi_{A_n}$. After substituting f_n in above and using the similar argument in Theorem 2.2, we obtain

$$b^{\nu n} |E(sw)|^{2\nu n} a^{n\nu-1} \bar{u} = b^\nu |E(rs)|^{2\nu(n-1)} |E(sw)|^{2\nu} a^{\nu-1} |E(uw)|^{(n-1)2\nu} \bar{u}, \quad on\ \ C.$$

Now, by multiplying the sides of above by u and then taking E of both sides equation we obtain

$$b^{\nu(n-1)} |E(sw)|^{2\nu(n-1)} a^{(n-1)\nu} = |E(rs)|^{2\nu(n-1)} |E(uw)|^{(n-1)2\nu}, \quad on\ C \cap F.$$

It is equivalent to

$$|E(rs)||E(uw)| = \sqrt{E(|r|^2)E(|u|^2)}|E(sw)|, \quad on\ C \cap F.$$

This complete the proof. □

Corollary 2.6. *Let $\nu \geq 0, A = M_w E M_u \in \mathbb{BC}(L^2(\Sigma))$ and $B = M_r E M_s \in \mathbb{BC}(L^2(\Sigma))$. Then $(A ⓓ_\nu B)^n = A^n ⓓ_\nu B^n$ for all $n \in \mathbb{N}$ if and only if $(A ⓓ_\nu B)^d = A^d ⓓ_\nu B^d$*

Definition 2.7. *For $0 \leq A \in \mathbb{BC}(\mathcal{H})$ and $0 \leq B \in \mathbb{B}(\mathcal{H})$, the Drazin-Ando geometric quasi-mean of (A, B) is defined by*

$$A \sharp_\nu^d B = ||B^{\frac{1}{2}} (A^d)^{\frac{1}{2}}|^\nu A^{\frac{1}{2}}|^2, \qquad \nu > 0.$$

Not that $A \sharp_\nu^d B = A^{\frac{1}{2}} |B^{\frac{1}{2}} (A^d)^{\frac{1}{2}}|^{2\nu} A^{\frac{1}{2}} = A^{\frac{1}{2}} [(A^d)^{\frac{1}{2}} B (A^d)^{\frac{1}{2}}]^\nu A^{\frac{1}{2}}$. For $\nu = \frac{1}{2}$, we denote the above means by $A ⓓ B$ and $A \sharp^d B$.

Theorem 2.8. *Let $\nu \geq 0, 0 \leq A = M_w E M_u \in \mathbb{BC}(L^2(\Sigma))$ and $0 \leq B = M_r E M_s \in \mathbb{B}(L^2(\Sigma))$. Then*

$$A \sharp_\nu^d B = M_{\frac{E(ur)^\nu E(s\bar{u})^\nu}{g^{\nu-1} E(|u|^2)^{2\nu}}} \chi_{S \cap \sigma(g)} M_{\bar{u}} E M_u$$

for some $0 \leq g \in L^0(\mathcal{A})$.

Proof. First, note that A is positive. Then, by Theorem 2.1 we have

$$A^{\frac{1}{2}} = M_{\bar{u}\sqrt{\frac{g}{E(|u|^2)}}}\chi_S EM_u,$$

$$(A^d)^{\frac{1}{2}} = M_{\frac{\bar{u}}{\sqrt{g}E(|u|^2)^{\frac{3}{2}}}}\chi_{S\cap\sigma(g)}EM_u,$$

where $0 \leq g \in L^0(\mathcal{A})$. Thus, direct computations show that

$$(A^d)^{\frac{1}{2}}B(A^d)^{\frac{1}{2}} = M_{\frac{\bar{u}E(ur)E(s\bar{u})}{gE(|u|^2)^3}}\chi_{S\cap\sigma(g)}EM_u,$$

$$[(A^d)^{\frac{1}{2}}B(A^d)^{\frac{1}{2}}]^\nu = M_{\frac{\bar{u}E(ur)^\nu E(s\bar{u})^\nu}{g^\nu E(|u|^2)^{2\nu+1}}}\chi_{S\cap\sigma(g)}EM_u,$$

$$A^{\frac{1}{2}}[(A^d)^{\frac{1}{2}}B(A^d)^{\frac{1}{2}}]^\nu A^{\frac{1}{2}} = M_{\frac{E(ur)^\nu E(s\bar{u})^\nu}{g^{\nu-1}E(|u|^2)^{2\nu}}}\chi_{S\cap\sigma(g)}M_{\bar{u}}EM_u.$$

It follows that

$$A\sharp_\nu^d B = M_{\frac{E(ur)^\nu E(s\bar{u})^\nu}{g^{\nu-1}E(|u|^2)^{2\nu}}}\chi_{S\cap\sigma(g)}M_{\bar{u}}EM_u.$$

This complete the proof. □

Lemma 2.9. *[11] Let $u, w \in \mathcal{D}(E)$. $|E(uw)|^2 = E(|u|^2)E(|w|^2)$ if and only if $w = g\bar{u}$ for some $g \in L^0(\mathcal{A})$.*

Theorem 2.10. *Let $A = M_w EM_u \in \mathbb{BC}(L^2(\Sigma))$ and $B = M_r EM_s \in \mathbb{BC}(L^2(\Sigma))$.Then the following assertions hold on Q.*

(1) If $A\textcircled{d}B = B\textcircled{d}A$, then $\frac{|E(sw)|^2}{E(|s|^2)E(|w|^2)} = \frac{|E(ru)|^2}{E(|r|^2)E(|u|^2)}$.

(2) If $s = g_1\bar{w}$ and $r = g_2\bar{u}$ for some $g_1, g_2 \in L^0(\mathcal{A})$, then $A\textcircled{d}B = B\textcircled{d}A$.

Where $Q = \sigma(E(|u|^2)) \cap \sigma(E(|w|^2)) \cap \sigma(E(|r|^2)) \cap \sigma(E(|s|^2)) \cap \sigma(E(s\bar{u}))$.

Proof. (1) We know that

$$A\textcircled{d}B = M_{\sqrt{\frac{E(|r|^2)}{E(|u|^2)}}|E(sw)|}M_{\bar{u}}EM_u,$$

$$B\textcircled{d}A = M_{\sqrt{\frac{E(|w|^2)}{E(|s|^2)}}|E(ru)|}M_{\bar{s}}EM_s.$$

Put $a = E(|u|^2), b = E(|w|^2), c = E(|r|^2)$ and $d = E(|s|^2)$. If $A\textcircled{d}B = B\textcircled{d}A$, then for each $f \in L^2(\Sigma)$, we have

$$M_{\sqrt{\frac{c}{a}}|E(sw)|}\bar{u}E(uf) = M_{\sqrt{\frac{b}{d}}|E(ru)|}\bar{s}E(sf). \tag{3}$$

Take $f_n = \bar{u}\sqrt{b}\chi_{A_n}$. Replacing f in (3) by f_n and so, we obtain

$$M_{\sqrt{\frac{c}{a}}|E(sw)|}\bar{u}a\sqrt{b}\chi_{A_n} = M_{\sqrt{\frac{b}{d}}|E(ru)|}\bar{s}E(s\bar{u})\sqrt{b}\chi_{A_n}.$$

As $n \to \infty$. It follows that

$$\sqrt{ac}|E(sw)|\bar{u} = \sqrt{\frac{b}{d}}|E(ru)|\bar{s}E(s\bar{u}).$$

By multiplying the sides of above by s and then taking E of both sides equation we obtain

$$\sqrt{ac}|E(sw)|E(s\bar{u}) = \sqrt{bd}|E(ru)|E(s\bar{u}),$$

and so $\frac{|E(sw)|^2\chi_{\sigma(b)\cap\sigma(d)}}{E(|s|^2)E(|w|^2)} = \frac{|E(ru)|^2\chi_{\sigma(a)\cap\sigma(c)}}{E(|r|^2)E(|u|^2)}, \quad on \quad \sigma(E(s\bar{u}))$.

(2) if $s = g_1\bar{w}$ and $r = g_2\bar{u}$ for some $g_1, g_2 \in L^0(\mathcal{A})$, by Lemma 2.9, it is easy to check that $A\textcircled{d}B = B\textcircled{d}A$. This complete the proof. $\square$

Theorem 2.11. *Let* $0 \leq A = M_wEM_u \in \mathbb{BC}(L^2(\Sigma))$ *and* $0 \leq B = M_rEM_s \in \mathbb{BC}(L^2(\Sigma))$. *Then* $A\textcircled{d}B = |A|^2\sharp^d|B|^2$ *if and only if*

$$\frac{|E(sw)|}{|E(s\bar{u})|} = \sqrt{\frac{E(|w|^2)}{E(|u|^2)}}, \ on \ \sigma(E(|u|^2)) \cap \sigma(E(|r|^2)).$$

Proof. First, we recall that if $A = M_wEM_u, B = M_rEM_s$ then

$$|A| = M_{\sqrt{\frac{E(|w|^2)}{E(|u|^2)}}\chi_{\sigma(E(|u|^2))}} M_{\bar{u}}EM_u,$$

$$\sqrt{|A|} = M_{\sqrt[4]{\frac{E(|w|^2)}{E(|u|^2)^3}}\chi_{\sigma(E(|u|^2))}} M_{\bar{u}}EM_u,$$

$$\sqrt{|B|} = M_{\sqrt[4]{\frac{E(|r|^2)}{E(|s|^2)^3}}\chi_{\sigma(E(|s|^2))}} M_{\bar{s}}EM_s,$$

$$|A|^d = M_{\frac{\chi_{\sigma(E(|u|^2))\cap\sigma(E(|w|^2))}}{\sqrt{E(|u|^2)^3E(|w|^2)}}} M_{\bar{u}}EM_u.$$

Thus, direct computations show that

$$A|^2\sharp^d|B^2 = |A|[|A|^d|B|^2|A|^d]^{\frac{1}{2}}|A|$$

$$= M_{\frac{\sqrt{E(|w|^2)E(|r|^2)}|E(s\bar{u})|}{E(|u|^2)}\chi_{\sigma(E(|u|^2))}} M_{\bar{u}}EM_u.$$

Also, we have

$$A\textcircled{d}B = M_{\sqrt{\frac{E(|r|^2)}{E(|u|^2)}}|E(sw)|\chi_{\sigma(E(|u|^2))}} M_{\bar{u}}EM_u.$$

Put $a = E(|u|^2), b = E(|w|^2), c = E(|r|^2)$ and $d = E(|s|^2)$. If $A\textcircled{d}B = |A|^2\sharp^d|B^2$, then for each $f \in L^2(\Sigma)$ we have

$$M_{\sqrt{\frac{c}{a}}|E(sw)|}\bar{u}E(uf) = M_{\frac{\sqrt{bc}|E(s\bar{u})|}{a}}\bar{u}E(uf). \tag{4}$$

Take $f_n = \bar{u}\sqrt{b}\chi_{A_n}$. Replacing f in (4) by f_n and so, we obtain

$$M_{\sqrt{\frac{c}{a}}|E(sw)|}\bar{u}a\sqrt{b}\chi_{A_n} = M_{\frac{\sqrt{bc}|E(s\bar{u})|}{a}}\bar{u}a\sqrt{b}\chi_{A_n}.$$

As $n \to \infty$. It follows that

$$\sqrt{\frac{c}{a}}|E(sw)|\bar{u} = \frac{\sqrt{bc}|E(s\bar{u})|}{a}\bar{u}.$$

By multiplying the sides of above by u and then taking E of both sides equation we obtain

$$\sqrt{\frac{c}{a}}|E(sw)| = \frac{\sqrt{bc}|E(s\bar{u})|}{a}, \text{ on } \sigma(a)$$

and so $\frac{|E(sw)|}{|E(s\bar{u})|} = \sqrt{\frac{E(|w|^2)}{E(|u|^2)}}$, *on* $\sigma(a) \cap \sigma(c)$. Conversely, if $\frac{|E(sw)|}{|E(s\bar{u})|} = \sqrt{\frac{E(|w|^2)}{E(|u|^2)}}$ it is easy to check that $A \textcircled{d} B = |A|^2 \sharp^d |B|^2$. This complete the proof. □

If $A = M_w E M_u$, then $\widetilde{A} = M_{\frac{E(uw)}{E(|u|^2)}} \bar{u} E M_u$. It is easy to check that

$$\widetilde{A \textcircled{d}_\nu B} = M_{\frac{E(|r|^2)^\nu |E(sw)|^{2\nu} E(|u|^2)^{\nu-1} |E(uw)|^2 \chi_C}{E(uw)^{4\nu}}} M_{\bar{u}} E M_u.$$

Hence we have the following corollary.

Corollary 2.12. *Let* $\nu \geq 0, A = M_w E M_u \in \mathbb{BC}(L^2(\Sigma))$ *and* $B = M_r E M_s \in \mathbb{BC}(L^2(\Sigma))$. *Then* $\widetilde{A \textcircled{d}_\nu B} = A \textcircled{d}_\nu B$.

In a special case, we have

$$\widetilde{A \textcircled{d}_\nu A} = M_{\frac{E(|u|^2)^{\nu-1} E(|w|^2)^\nu \chi_C}{E(uw)^{2\nu-2}}} M_{\bar{u}} E M_u,$$

$$\widetilde{A} \textcircled{d}_\nu \widetilde{A} = M_{\frac{E(uw)^2 \chi_{\sigma(E(|u|^2))}}{E(|u|^2)}} M_{\bar{u}} E M_u.$$

Then, we have the following corollary.

Corollary 2.13. *Let* $\nu \geq 0, A = M_w E M_u \in \mathbb{BC}(L^2(\Sigma))$. *Then* $\widetilde{A \textcircled{d}_\nu A} = \widetilde{A} \textcircled{d}_\nu \widetilde{A}$. *if and only if* $E(uw) = \sqrt{E(|u|^2)E(|w|^2)}$.

For $T \in \mathbb{B}(H)$, the spectrum of T is denoted by $\sigma(T)$ and $r(T)$ its spectral radius. In [10] it was proved that the spectrum of $T = M_w E M_u \in \mathbb{B}(L^2(\Sigma))$ is the essential range of $E(uw)$.

Theorem 2.14. *Let* $A = M_w E M_u \in \mathbb{BC}(L^2(\Sigma))$ *and* $B = M_r E M_s \in \mathbb{BC}(L^2(\Sigma))$. *Then*

(1) $r(A \textcircled{d} B) = \|E(|r|^2)E(|u|^2)|E(sw)|\|_\infty^{\frac{1}{2}}$.

(2) $A \textcircled{d} B$ *is quasinilpotent iff* $E(|r|^2)E(|u|^2)|E(sw)| = 0$.

Proof. (1) We know that

$$A \textcircled{d} B = M_{\sqrt{\frac{E(|r|^2)}{E(|u|^2)}|E(sw)|}} M_{\bar{u}} E M_u.$$

Thus,

$$(A \textcircled{d} B)^n = M_{E(|r|^2)^{\frac{n}{2}} |E(sw)|^{\frac{n}{2}} E(|u|^2)^{\frac{n}{2}-1}} M_{\bar{u}} E M_u.$$

Then by Theorem 2.1 (i), we get that

$$\|(A\textcircled{d}B)^n\| = \|E(|r|^2)^{\frac{n}{2}}|E(sw)|^{\frac{n}{2}}E(|u|^2)^{\frac{n}{2}-1}(E(|u|^2))^{\frac{1}{2}}(E(|u|^2))^{\frac{1}{2}}\|_\infty, \ \ n \in \mathbb{N}.$$

Hence,

$$\|(A\textcircled{d}B)^n\| = \|E(|r|^2)^{\frac{n}{2}}|E(sw)|^{\frac{n}{2}}E(|u|^2)^{\frac{n}{2}}\|_\infty, \ \ n \in \mathbb{N}.$$

It follows that

$$r(A\textcircled{d}B) = \lim_{n\to\infty} \|(A\textcircled{d}B)^n\|^{\frac{1}{n}} = \|E(|r|^2)E(|u|^2)|E(sw)|\|_\infty^{\frac{1}{2}}$$

(2) Since

$$r(A\textcircled{d}B) = \lim_{n\to\infty} \|(A\textcircled{d}B)^n\|^{\frac{1}{n}} = \|E(|r|^2)E(|u|^2)|E(sw)|\|_\infty^{\frac{1}{2}}.$$

It follows that $r(A\textcircled{d}B) = 0$, whenever $E(|r|^2)E(|u|^2)|E(sw)| = 0$.
Conversely, suppose $A\textcircled{d}B$ is quasinilpotent. It is easy to check that $E(|r|^2)E(|u|^2)|E(sw)| = 0$. This complete the proof. □

Corollary 2.15. *Let $A = M_wEM_u \in \mathbb{BC}(L^2(\Sigma))$ and $B = M_rEM_s \in \mathbb{BC}(L^2(\Sigma))$. Then*

$$\sigma(A\textcircled{d}_\nu B) = ess\ range\left(E(|r|^2)E(|u|^2)|E(sw)|\right) \setminus \{0\}.$$

For $u, w \in L^2(\Sigma)\backslash\{0\}$ the rank-one operator $u \otimes w$ on $L^2(\Sigma)$ is defined by $(u \otimes w)f = \langle f, w\rangle u$ for all $f \in L^2(\Sigma)$. Let $\mu(X) = 1$ and $\mathcal{A}_0 = \{\emptyset, X\}$. Put $E^{\mathcal{A}_0} = E$. Then we have $\int_X f d\mu = \int_X E(f)d\mu$. Since X is an $\mathcal{A}_0$-atom, then the $\mathcal{A}_0$-measurable function $E(f)$ is constant on X. It follows that $E(f) = \int_X f d\mu$, for all $f \in L^2(\Sigma)$.
In [10], Jabbarzadeh and Emamalipour show that if $T = M_wEM_u \in \mathbb{BC}(L^2(\Sigma))$ be nonzero elements, then $T = w \otimes \bar{u}$ is a rank-one operator and

$$T^d = M_{\frac{\chi_C}{E(uw)^2}}(w \otimes \bar{u}), \quad T^\dagger = \bar{u} \otimes \frac{w}{\|u\|_2^2\|w\|_2^2}.$$

Like this, we have the following proposition.

Proposition 2.16. *Let $A = M_wEM_u \in \mathbb{BC}(L^2(\Sigma))$, $B = M_rEM_s \in \mathbb{BC}(L^2(\Sigma))$ and A, B be the nonzero elements. Then*

$$A\textcircled{d}_\nu B = \frac{\|u\|_2^{2\nu-2}\|r\|_2^{2\nu}\chi_C}{E(uw)^{4\nu}}\langle\bar{s}, w\rangle^\nu\langle w, \bar{s}\rangle^\nu\langle w, \bar{u}\rangle\langle\bar{u}, w\rangle(\bar{u} \otimes \bar{u}).$$

Proof. Since $A = w \otimes \bar{u}, B = r \otimes \bar{s}$. Then we have

$$B^*B = (\bar{s} \otimes r)(r \otimes \bar{s}) = \|r\|_2^2(\bar{s} \otimes \bar{s}),$$

$$B^*BA^d = \|wr\|_2^2(\bar{s} \otimes \bar{s})\frac{\chi_C}{E(uw)^2}(w \otimes \bar{u}) = \frac{\|r\|_2^2\chi_C}{E(uw)^2}(\bar{s} \otimes \bar{s})(w \otimes \bar{u}),$$

$$A^{d^*}(B^*B)A^d = \frac{\|r\|_2^2\chi_C}{E(uw)^4}(\bar{u} \otimes w)(\bar{s} \otimes \bar{s})(w \otimes \bar{u})$$

Thus, direct computations show that

$$[A^{d^*}(B^*B)A^d]^\nu = \frac{\|r\|_2^{2\nu}\chi_C}{E(uw)^{4\nu}}\{(\bar{u}\otimes w)(\bar{s}\otimes\bar{s})(w\otimes\bar{u})\}^\nu$$

$$= \frac{\|r\|_2^{2\nu}\chi_C}{E(uw)^{4\nu}}\langle\bar{s},w\rangle^\nu\langle w,\bar{s}\rangle^\nu\langle\bar{u},\bar{u}\rangle^{\nu-1}(\bar{u}\otimes\bar{u}).$$

Then,

$$A^*[A^{d^*}(B^*B)A^d]^\nu A =$$

$$= \frac{\|r\|_2^{2\nu}\chi_C}{E(uw)^{4\nu}}\langle\bar{s},w\rangle^\nu\langle w,\bar{s}\rangle^\nu\langle\bar{u},\bar{u}\rangle^{\nu-1}\langle w,\bar{u}\rangle\langle\bar{u},w\rangle(\bar{u}\otimes\bar{u}).$$

It follows that

$$A\textcircled{d}_\nu B = \frac{\|u\|_2^{2\nu-2}\|r\|_2^{2\nu}\chi_C}{E(uw)^{4\nu}}\langle\bar{s},w\rangle^\nu\langle w,\bar{s}\rangle^\nu\langle w,\bar{u}\rangle\langle\bar{u},w\rangle(\bar{u}\otimes\bar{u}).$$

This complete the proof. □

Example 2.17. *(i) Let* $X = [-1,1]$, $d\mu = dx$, Σ *be the Lebesgue sets, and let* $\mathcal{A}\subseteq\Sigma$ *be the* σ*-algebra generated by the symmetric sets about the origin. Then for each* $f\in\mathcal{D}(E)$, $2E(f)(x) = f(x)+f(-x)$*. Put* $u(x) = 1+x$, $w(x) = x^2+x^3$, $r(x) = \cos^{(} x)$, $s(x) = \cos(x)$ *and* $A = M_wEM_u$, $B = M_rEM_s$*. Then* $E(u) = 1, E(|u|^2) = 1+x^2, E(uw) = x^2+x^4$, $E(|w|^2) = x^4+x^6$, $E(|r|^2) = \cos^4(x), E(|s|^2) = \cos^2(x), E(sw) = x^2\cos(x), E(rs) = \cos^3(x), E(ru) = \cos^2(x)$ *and* $E(s\overline{u}) = \cos^2(x)$*. For* $\nu > 0$*, direct computations show that*

$$\widetilde{A\textcircled{d}_\nu A} = M_{x^4(1+x)(1+x^2)}\ ;$$
$$\widetilde{A}\textcircled{d}_\nu\widetilde{A} = M_{x^4(1+x)(1+x^2)}\ .$$

So, by Corollary 2.13, $\widetilde{A\textcircled{d}_\nu A} = \widetilde{A}\textcircled{d}_\nu\widetilde{A}$*. In this case, by Theorem 2.11, it is easy to check that* $A\textcircled{d}B = |A|^2\sharp^d|B|^2$*. Also, we obtain*

$$A\textcircled{d}_\nu B = M_{\frac{x^2(1+x)\cos^3(x)}{\sqrt{1+x^2}}}\ ;$$
$$B\textcircled{d}_\nu A = M_{x^2\cos^3(x)\sqrt{1+x^2}}\ .$$

Thus, $A\textcircled{d}_\nu B \neq B\textcircled{d}_\nu A$*. In addition, in this case we can show that Theorems 2.4 and 2,5 are not hold.*
Now, put $u(x) = x^2$, $w(x) = \cos(x)$, $r(x) = x^4$, $s(x) = x^2\cos(x)$, $A = M_wEM_u$ *and* $B = M_rEM_s$*. Then* $E(u) = x^2$, $E(|u|^2) = x^4$, $E(uw) = x^2\cos(x)$, $E(w) = \cos(x)$, $E(|w|^2) = \cos^2(x)$, $E(r) = x^4$, $E(|r|^2) = x^8$, $E(|s|^2) = x^4\cos^2(x)$, $E(s\overline{u}) = x^4\cos(x)$ *and* $E(ru) = x^6$*. In this case we get that*

$$A\textcircled{d}_\nu B = M_{x^8\cos^2(x)}\ ;$$
$$B\textcircled{d}_\nu A = M_{x^8\cos^2(x)}\ .$$

So, by Theorem 2.10, $A\textcircled{d}_\nu B = B\textcircled{d}_\nu A$*. Also, direct computations show that*

$$(A\textcircled{d}_\nu B)^d = M_{\frac{1}{x^{8\nu+4}\cos^2(x)}}\ ;$$
$$A^d\textcircled{d}_\nu B^d = M_{\frac{1}{x^{8\nu+4}\cos^2(x)}}\ .$$

Thus, by Theorem 2.4, $(A\textcircled{d}_\nu B)^d = A^d\textcircled{d}_\nu B^d$*.*

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