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Some geometric and physical properties of pseudo $\mathcal{M}^*$-projective symmetric manifolds

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Abstract. In this study we introduce a new tensor in a semi-Riemannian manifold, named the $\mathcal{M}^*$-projective curvature tensor which generalizes the m-projective curvature tensor. We start by deducing some fundamental geometric properties of the $\mathcal{M}^*$-projective curvature tensor. After that, we study pseudo $\mathcal{M}^*$-projective symmetric manifolds $(PM^*S)_n$. A non-trivial example has been used to show the existence of such a manifold. We introduce a series of interesting conclusions. We establish, among other things, that if the scalar curvature ρ is non-zero, the associated 1-form is closed for a $(PM^*S)_n$ with $\operatorname{div} M^* = 0$. We also deal with pseudo $\mathcal{M}^*$-projective symmetric spacetimes, $\mathcal{M}^*$-projectively flat perfect fluid spacetimes, and $\mathcal{M}^*$-projectively flat viscous fluid spacetimes. As a result, we establish some significant theorems.

1. Introduction

In Differential geometry, the investigation of curvature characteristics is the prime problem among others. In this context, S. S. Chern had uttered in [7] “A fundamental notion is curvature, in its different forms”. Hence, the discovery of the Riemann curvature tensor creates an extremely significant subject matter. In this paper, due to the above sense, we have introduced a new curvature tensor, called $\mathcal{M}^*$-projective curvature tensor which generalizes some known curvature tensors.

According to Chaki [3], for a non-vanishing 1-form D, a non-flat Riemannian or a semi-Riemannian manifold (M^n, g), $(n > 2)$ is named pseudosymmetric if its curvature tensor obeys

$$\begin{aligned}(\nabla_Z \mathcal{R})(G,H,J,K) = 2D(Z)\mathcal{R}(G,H,J,K) + D(G)\mathcal{R}(Z,H,J,K) + D(H)\mathcal{R}(G,Z,J,K)\\ + D(J)\mathcal{R}(G,H,Z,K) + D(K)\mathcal{R}(G,H,J,Z),\end{aligned}$$

∇ is the Levi-Civita connection and $\mathcal{R}(G,H,J,K) = g(R(G,H)J,K)$, R being the curvature tensor of type $(1,3)$. Let π be the associated vector field corresponding to the 1-form D, i.e.,

$$g(H,\pi) = D(H),$$

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for all H. Pseudosymmetric manifolds have been investigated by several authors ([18], [20], [21], [32], [33]) and many others.

In a Riemannian or a semi-Riemannian manifold the Ricci tensor $\mathcal{S}$ is said to be of Codazzi type [10] if the covariant derivative of Ricci tensor satisfies

$$(\nabla_G \mathcal{S})(H,J) = (\nabla_H \mathcal{S})(G,J)$$

and the Ricci tensor is said to be cyclic parallel [10] if

$$(\nabla_G \mathcal{S})(H,J) + (\nabla_H \mathcal{S})(J,G) + \left(\nabla_J \mathcal{S}\right)(G,H) = 0.$$

A non-flat semi-Riemannian manifold obeying the condition

$$(\nabla_G \mathcal{S})(H,J) = \gamma(G)\,\mathcal{S}(H,J) + \delta(G)\,g(H,J),$$

where γ and δ are non-zero 1-forms, is named a generalized Ricci recurrent manifold [8]. The manifold becomes a Ricci recurrent manifold when $\delta = 0$.

In a Riemannian or a semi-Riemannian manifold (M^n, g) $(n \geq 2)$, the m-projective curvature tensor M is defined as [25]

$$M(G,H)J = R(G,H)J - \frac{1}{2(n-1)}\left[\mathcal{S}(H,J)G - \mathcal{S}(G,J)H + g(H,J)QG - g(G,J)QH\right],$$

where R is the curvature tensor of type $(1,3)$, $\mathcal{S}$ is the Ricci tensor of type $(0,2)$ and Q is the Ricci operator defined by $g(QG,H) = \mathcal{S}(G,H)$.

We define the M^*-projective curvature tensor of type $(1,3)$ as

$$M^*(G,H)J = R(G,H)J - \frac{\varphi}{2(n-1)}\left[\mathcal{S}(H,J)G - \mathcal{S}(G,J)H + g(H,J)QG - g(G,J)QH\right], \tag{1}$$

φ being scalar. The M^*-projective curvature tensor reduces to the m-projective curvature tensor when $\varphi = 1$. The M^*-projective curvature tensor and the curvature tensor are identical if $\varphi = 0$. Equation (1) can be expressed as

$$\begin{aligned}\mathcal{M}^*(G,H,J,K) = \mathcal{R}(G,H,J,K) - \frac{\varphi}{2(n-1)}\big[&\mathcal{S}(H,J)g(G,K) - \mathcal{S}(G,J)g(H,K)\\ &+ g(H,J)\mathcal{S}(G,K) - g(G,J)\mathcal{S}(H,K)\big],\end{aligned} \tag{2}$$

where $\mathcal{M}^*(G,H,J,K) = g(M^*(G,H)J,K)$ and $\mathcal{R}(G,H,J,K) = g(R(G,H)J,K)$.

A non-flat Riemannian or a semi-Riemannian manifold $(M^n, g), (n > 2)$ is said to be a pseudo $\mathcal{M}^*$-projective symmetric manifold if the $\mathcal{M}^*$-projective curvature tensor of type $(0,4)$ satisfies the relation

$$\begin{aligned}(\nabla_Z \mathcal{M}^*)(G,H,J,K) = {} & 2D(Z)\mathcal{M}^*(G,H,J,K) + D(G)\mathcal{M}^*(Z,H,J,K) + D(H)\mathcal{M}^*(G,Z,J,K)\\ &+ D(J)\mathcal{M}^*(G,H,Z,K) + D(K)\mathcal{M}^*(G,H,J,Z),\end{aligned} \tag{3}$$

D being a non-vanishing 1-form and π is the associated vector field corresponding to the 1-form D, i.e.,

$$g(H,\pi) = D(H).$$

An n-dimensional pseudo $\mathcal{M}^*$-projective symmetric manifold is denoted by $(PM^*S)_n$, where P indicates pseudo, M^* is the $\mathcal{M}^*$-projective curvature tensor and S indicates symmetric. When $\varphi = 0$, the pseudo $\mathcal{M}^*$-projective symmetric manifold reduces to the pseudosymmetric manifold denoted by $(PS)_n$. Further, if $\varphi = 1$, pseudo $\mathcal{M}^*$-projective symmetric manifolds contain pseudo m-projective symmetric manifolds. Thus $(PM^*S)_n$ recovers some known geometric structures. We organized the paper as follows:

After preliminaries in section 3, we investigate the curvature property of $(PM^*S)_n$. The study of $(PM^*S)_n$ with Codazzi type of Ricci tensor is covered in section 4. In section 5, we analyze $(PM^*S)_n$ with div$M^* = 0$. After that, in section 6, we construct an example of $(PM^*S)_4$. Pseudo $\mathcal{M}^*$-projective symmetric spacetimes are discussed in section 7. Section 8 is devoted to study $\mathcal{M}^*$-projectively flat spacetimes. We focus at $\mathcal{M}^*$-projectively flat perfect fluid and viscous fluid spacetimes in the last two sections.

2. Preliminaries

We can see from (1) the tensor M^* fulfills the following:

$$\begin{aligned} &(i) \quad M^*(G,H)J = -M^*(H,G)J, \\ &(ii) \quad M^*(G,H)J + M^*(H,J)G + M^*(J,G)H = 0. \end{aligned} \tag{4}$$

It is also clear from (2) that

$$\sum_{i=1}^{n} \varepsilon_i \mathcal{M}^*(G,H,e_i,e_i) = 0 = \sum_{i=1}^{n} \varepsilon_i \mathcal{M}^*(e_i,e_i,J,K) \tag{5}$$

and

$$\begin{aligned} \sum_{i=1}^{n} \varepsilon_i \mathcal{M}^*(e_i,G,H,e_i) &= \left[1 - \frac{(n-2)\varphi}{2(n-1)}\right] \mathcal{S}(G,H) - \frac{\varphi\rho}{2(n-1)} g(G,H) \\ &= \sum_{i=1}^{n} \varepsilon_i \mathcal{M}^*(G,e_i,e_i,H) \\ &= \overline{\mathcal{M}^*}(G,H), \quad \text{(say)} \end{aligned} \tag{6}$$

where at each point of the manifold $\{e_k\}$, $k = 1,2,\ldots,n$ be an orthonormal basis of the tangent space, $\rho = \sum_{i=1}^{n} \varepsilon_i \mathcal{S}(e_i,e_i)$ is the scalar curvature and $\varepsilon_i = g(e_i,e_i) = \pm 1$.

The followings are derived from (1) and (4):

$$\begin{aligned} &(i) \quad \mathcal{M}^*(G,H,J,K) = -\mathcal{M}^*(H,G,J,K), \\ &(ii) \quad \mathcal{M}^*(G,H,J,K) = -\mathcal{M}^*(G,H,K,J), \\ &(iii) \quad \mathcal{M}^*(G,H,J,K) = \mathcal{M}^*(J,K,G,H), \\ &(iv) \quad \mathcal{M}^*(G,H,J,K) + \mathcal{M}^*(H,J,G,K) + \mathcal{M}^*(J,G,H,K) = 0, \end{aligned} \tag{7}$$

where $\mathcal{M}^*(G,H,J,K) = g(M^*(G,H)J,K)$.

Proposition 2.1. *A $\mathcal{M}^*$-projectively flat semi-Riemannnian manifold is an Einstein manifold.*

Proof. The $\mathcal{M}^*$-projective curvature tensor is given by

$$\begin{aligned} \mathcal{M}^*(G,H,J,K) = \mathcal{R}(G,H,J,K) - \frac{\varphi}{2(n-1)} &[\mathcal{S}(H,J)g(G,K) - \mathcal{S}(G,J)g(H,K) \\ &+ g(H,J)\mathcal{S}(G,K) - g(G,J)\mathcal{S}(H,K)], \end{aligned}$$

φ being an arbitrary scalar. If $\mathcal{M}^*$-projective curvature tensor vanishes, then

$$\mathcal{R}(G,H,J,K) = \frac{\varphi}{2(n-1)} [\mathcal{S}(H,J)g(G,K) - \mathcal{S}(G,J)g(H,K) + g(H,J)\mathcal{S}(G,K) - g(G,J)\mathcal{S}(H,K)]. \tag{8}$$

In (8), we obtain by contracting H and J

$$\mathcal{S}(G,K) = \frac{\varphi\rho}{2(n-1) - (n-2)\varphi} g(G,K). \tag{9}$$

Thus, we complete the proof.

In (9), contracting G and K we have

$$\rho(n-1)(1-\varphi) = 0,$$

which implies either $\rho = 0$ or $\varphi = 1$. If $\rho \neq 0$, then the $\mathcal{M}^*$-projective curvature tensor is the same as the m-projective curvature tensor M for $\varphi = 1$. As a result, $\mathcal{M}^*$-projectively flatness and m-projectively flatness are the same.

The following corollary was established by the author in [31]:

Corollary 2.2. *m-projectively flat Riemannian manifold is an Einstein manifold.*

Proposition 2.3. *If $\mathcal{M}^*$-projective curvature tensor is parallel, then the manifold reduces to a generalized Ricci recurrent manifold.*

Proof. The M^*-projective curvature tensor is given by

$$M^*(G,H)J = R(G,H)J - \frac{\varphi}{2(n-1)}\left[\mathcal{S}(H,J)G - \mathcal{S}(G,J)H + g(H,J)QG - g(G,J)QH\right], \tag{10}$$

φ being an arbitrary scalar. Taking the covariant derivative of (10) gives us

$$\begin{aligned}(\nabla_Z M^*)(G,H)J &= (\nabla_Z R)(G,H)J - \frac{d\varphi(Z)}{2(n-1)}\left[\mathcal{S}(H,J)G - \mathcal{S}(G,J)H + g(H,J)QG - g(G,J)QH\right]\\ &\quad - \frac{\varphi}{2(n-1)}\left[(\nabla_Z\mathcal{S})(H,J)G - (\nabla_Z\mathcal{S})(G,J)H + g(H,J)(\nabla_Z Q)G - g(G,J)(\nabla_Z Q)H\right].\end{aligned}$$

By hypothesis, $\mathcal{M}^*$-projective curvature tensor is parallel. As a result of the previous equation,

$$\begin{aligned}(\nabla_Z R)(G,H)J &= \frac{d\varphi(Z)}{2(n-1)}\left[\mathcal{S}(H,J)G - \mathcal{S}(G,J)H + g(H,J)QG - g(G,J)QH\right]\\ &+ \frac{\varphi}{2(n-1)}\left[(\nabla_Z\mathcal{S})(H,J)G - (\nabla_Z\mathcal{S})(G,J)H + g(H,J)(\nabla_Z Q)G - g(G,J)(\nabla_Z Q)H\right].\end{aligned} \tag{11}$$

Contracting G in (11) we find

$$(\nabla_Z\mathcal{S})(H,J) = \frac{(n-2)}{2(n-1)-(n-2)\varphi}d\varphi(Z)\mathcal{S}(H,J) + \frac{1}{2(n-1)-(n-2)\varphi}\left[\rho d\varphi(Z) + \varphi d\rho(Z)\right]g(H,J). \tag{12}$$

Again, contracting H and J in (12) reveals that

$$(1-\varphi)\,d\rho(Z) = \rho d\varphi(Z). \tag{13}$$

This implies

$$d\varphi(Z) = (1-\varphi)(Z\log\rho). \tag{14}$$

From (12), (13) and (14) we obtain

$$(\nabla_Z\mathcal{S})(H,J) = \frac{(n-2)(1-\varphi)(Z\log\rho)}{2(n-1)-(n-2)\varphi}\mathcal{S}(H,J) + \frac{(Z\rho)}{2(n-1)-(n-2)\varphi}g(H,J).$$

So the proof is completed.

Proposition 2.4. *For a $\mathcal{M}^*$-projective curvature tensor with $divM^* = 0$, the curvature conditions $divM = 0$ and $divR = 0$ are equivalent, provided φ is constant.*

Proof. The M^*-projective curvature tensor is given by

$$M^*(G,H)J = R(G,H)J - \frac{\varphi}{2(n-1)}\left[\mathcal{S}(H,J)G - \mathcal{S}(G,J)H + g(H,J)QG - g(G,J)QH\right], \tag{15}$$

φ being an arbitrary scalar. Taking covariant derivative of the foregoing equation we find

$$(\nabla_Z M^*)(G,H)J = (\nabla_Z R)(G,H)J - \frac{d\varphi(Z)}{2(n-1)}[\mathcal{S}(H,J)G - \mathcal{S}(G,J)H + g(H,J)QG - g(G,J)QH] - \frac{\varphi}{2(n-1)}[(\nabla_Z \mathcal{S})(H,J)G - (\nabla_Z \mathcal{S})(G,J)H + g(H,J)(\nabla_Z Q)G - g(G,J)(\nabla_Z Q)H]. \quad (16)$$

Contracting Z in (16), we have

$$(\mathrm{div}M^*)(G,H)J = (\mathrm{div}R)(G,H)J - \frac{1}{2(n-1)}[\mathcal{S}(H,J)(G\varphi) - \mathcal{S}(G,J)(H\varphi) + g(H,J)(QG\varphi) - g(G,J)(QH\varphi)] - \frac{\varphi}{2(n-1)}\left[(\nabla_G \mathcal{S})(H,J) - (\nabla_H \mathcal{S})(G,J) + \frac{1}{2}g(H,J)\,d\rho(G) - \frac{1}{2}g(G,J)\,d\rho(H)\right].$$

Now if φ is constant, then the above equation reduces to

$$(\mathrm{div}M^*)(G,H)J = (1-\varphi)(\mathrm{div}R)(G,H)J + \varphi(\mathrm{div}M)(G,H)J.$$

Using $\mathrm{div}M^* = 0$, we have the following:

$$(\mathrm{div}M)(G,H)J = \left(\frac{\varphi-1}{\varphi}\right)(\mathrm{div}R)(G,H)J.$$

This completes the proof.

3. Some curvature properties of $(PM^*S)_n$ $(n > 2)$

In this section, we show that the $\mathcal{M}^*$-projective curvature tensor satisfies Bianchi's second identity for a $(PM^*S)_n$ $(n > 2)$, i.e.,

$$(\nabla_Z \mathcal{M}^*)(G,H,J,K) + (\nabla_G \mathcal{M}^*)(H,Z,J,K) + (\nabla_H \mathcal{M}^*)(Z,G,J,K) = 0. \quad (17)$$

By virtue of (3) and (17) we acquire

$$\begin{aligned} &2D(Z)[\mathcal{M}^*(G,H,J,K) + \mathcal{M}^*(H,G,J,K)] + 2D(G)[\mathcal{M}^*(Z,H,J,K) + \mathcal{M}^*(H,Z,J,K)] \\ &+ 2D(H)[\mathcal{M}^*(G,Z,J,K) + \mathcal{M}^*(Z,G,J,K)] + D(J)[\mathcal{M}^*(G,H,Z,K) + \mathcal{M}^*(H,Z,G,K) \\ &+ \mathcal{M}^*(Z,G,H,K)] + D(K)[\mathcal{M}^*(G,H,J,Z) + \mathcal{M}^*(H,Z,J,G) + \mathcal{M}^*(Z,G,J,H)] \\ &= (\nabla_Z \mathcal{M}^*)(G,H,J,K) + (\nabla_G \mathcal{M}^*)(H,Z,J,K) + (\nabla_H \mathcal{M}^*)(Z,G,J,K). \end{aligned} \quad (18)$$

Using (7) in (18) we find

$$(\nabla_Z \mathcal{M}^*)(G,H,J,K) + (\nabla_G \mathcal{M}^*)(H,Z,J,K) + (\nabla_H \mathcal{M}^*)(Z,G,J,K) = 0. \quad (19)$$

This leads to the following theorem:

Theorem 3.1. *The $\mathcal{M}^*$-projective curvature tensor in $(PM^*S)_n$ $(n > 2)$ satisfies Bianchi's second identity.*

4. $(PM^*S)_n$ $(n > 2)$ with Codazzi type of Ricci tensor

Equation (2) provides us

$$\begin{aligned}
&-\frac{\varphi}{2(n-1)}\left[(\nabla_Z\mathcal{S})(H,J)\,g(G,K)-(\nabla_Z\mathcal{S})(G,J)\,g(H,K)+g(H,J)(\nabla_Z\mathcal{S})(G,K)-g(G,J)(\nabla_Z\mathcal{S})(H,K)\right.\\
&+(\nabla_G\mathcal{S})(Z,J)\,g(H,K)-(\nabla_G\mathcal{S})(H,J)\,g(Z,K)+g(Z,J)(\nabla_G\mathcal{S})(H,K)-g(H,J)(\nabla_G\mathcal{S})(Z,K)\\
&\left.+(\nabla_H\mathcal{S})(G,J)\,g(Z,K)-(\nabla_H\mathcal{S})(Z,J)\,g(G,K)+g(G,J)(\nabla_H\mathcal{S})(Z,K)-g(Z,J)(\nabla_H\mathcal{S})(G,K)\right]\\
&-\frac{(Z\varphi)}{2(n-1)}\left[\mathcal{S}(H,J)\,g(G,K)-\mathcal{S}(G,J)\,g(H,K)+g(H,J)\,\mathcal{S}(G,K)-g(G,J)\,\mathcal{S}(H,K)\right]\\
&-\frac{(G\varphi)}{2(n-1)}\left[\mathcal{S}(Z,J)\,g(H,K)-\mathcal{S}(H,J)\,g(Z,K)+g(Z,J)\,\mathcal{S}(H,K)-g(H,J)\,\mathcal{S}(Z,K)\right]\\
&-\frac{(H\varphi)}{2(n-1)}\left[\mathcal{S}(G,J)\,g(Z,K)-\mathcal{S}(Z,J)\,g(G,K)+g(G,J)\,\mathcal{S}(Z,K)-g(Z,J)\,\mathcal{S}(G,K)\right]\\
&=(\nabla_Z\mathcal{M}^*)(G,H,J,K)+(\nabla_G\mathcal{M}^*)(H,Z,J,K)+(\nabla_H\mathcal{M}^*)(Z,G,J,K).
\end{aligned}\tag{20}$$

If $(PM^*S)_n$ admits the Codazzi type of Ricci tensor, then (20) becomes

$$\begin{aligned}
&-\frac{(Z\varphi)}{2(n-1)}\left[\mathcal{S}(H,J)\,g(G,K)-\mathcal{S}(G,J)\,g(H,K)+g(H,J)\,\mathcal{S}(G,K)-g(G,J)\,\mathcal{S}(H,K)\right]\\
&-\frac{(G\varphi)}{2(n-1)}\left[\mathcal{S}(Z,J)\,g(H,K)-\mathcal{S}(H,J)\,g(Z,K)+g(Z,J)\,\mathcal{S}(H,K)-g(H,J)\,\mathcal{S}(Z,K)\right]\\
&-\frac{(H\varphi)}{2(n-1)}\left[\mathcal{S}(G,J)\,g(Z,K)-\mathcal{S}(Z,J)\,g(G,K)+g(G,J)\,\mathcal{S}(Z,K)-g(Z,J)\,\mathcal{S}(G,K)\right]\\
&\qquad=(\nabla_Z\mathcal{M}^*)(G,H,J,K)+(\nabla_G\mathcal{M}^*)(H,Z,J,K)+(\nabla_H\mathcal{M}^*)(Z,G,J,K).
\end{aligned}\tag{21}$$

Using (19) in (21), we obtain

$$\begin{aligned}
&\frac{(Z\varphi)}{2(n-1)}\left[\mathcal{S}(H,J)\,g(G,K)-\mathcal{S}(G,J)\,g(H,K)+g(H,J)\,\mathcal{S}(G,K)-g(G,J)\,\mathcal{S}(H,K)\right]\\
&+\frac{(G\varphi)}{2(n-1)}\left[\mathcal{S}(Z,J)\,g(H,K)-\mathcal{S}(H,J)\,g(Z,K)+g(Z,J)\,\mathcal{S}(H,K)-g(H,J)\,\mathcal{S}(Z,K)\right]\\
&+\frac{(H\varphi)}{2(n-1)}\left[\mathcal{S}(G,J)\,g(Z,K)-\mathcal{S}(Z,J)\,g(G,K)+g(G,J)\,\mathcal{S}(Z,K)-g(Z,J)\,\mathcal{S}(G,K)\right]=0.
\end{aligned}\tag{22}$$

Contracting G and K in (22), we infer that

$$\begin{aligned}
&(Z\varphi)\left[(n-2)\,\mathcal{S}(H,J)+\rho g(H,J)\right]+\mathcal{S}(Z,J)(H\varphi)-\mathcal{S}(H,J)(Z\varphi)\\
&+g(Z,J)\,g(QH,\mathrm{grad}\varphi)-g(H,J)\,g(QZ,\mathrm{grad}\varphi)+(H\varphi)\left[(2-n)\,\mathcal{S}(J,Z)-\rho g(J,Z)\right]=0.
\end{aligned}\tag{23}$$

Again contracting H and J in (23) yields

$$(Z\varphi)\,\rho=g(QZ,\mathrm{grad}\varphi)$$

which gives

$$\mathcal{S}(Z,\mathrm{grad}\varphi)=\rho g(Z,\mathrm{grad}\varphi).$$

Thus we conclude the following theorem:

Theorem 4.1. *For a $(PM^*S)_n$ admitting Codazzi type of Ricci tensor, ρ is an eigenvalue of the Ricci tensor $\mathcal{S}$ corresponding to the eigenvector gradφ.*

If φ is constant, we can deduce from (20) and Bianchi's second identity

$$\begin{aligned}&(\nabla_Z \mathcal{S})(H,J)\,g(G,K) - (\nabla_Z \mathcal{S})(G,J)\,g(H,K) + g(H,J)(\nabla_Z \mathcal{S})(G,K) - g(G,J)(\nabla_Z \mathcal{S})(H,K)\\&+ (\nabla_G \mathcal{S})(Z,J)\,g(H,K) - (\nabla_G \mathcal{S})(H,J)\,g(Z,K) + g(Z,J)(\nabla_G \mathcal{S})(H,K) - g(H,J)(\nabla_G \mathcal{S})(Z,K)\\&+ (\nabla_H \mathcal{S})(G,J)\,g(Z,K) - (\nabla_H \mathcal{S})(Z,J)\,g(G,K) + g(G,J)(\nabla_H \mathcal{S})(Z,K) - g(Z,J)(\nabla_H \mathcal{S})(G,K) = 0.\end{aligned} \tag{24}$$

Contracting G and K in (24) reveals that

$$(n-3)(\nabla_Z \mathcal{S})(H,J) + \frac{1}{2}g(H,J)(Z\rho) - \frac{1}{2}g(Z,J)(H\rho) - (n-3)(\nabla_H \mathcal{S})(Z,J) = 0. \tag{25}$$

Further, if ρ remains constant, (25) becomes

$$(\nabla_Z \mathcal{S})(H,J) = (\nabla_H \mathcal{S})(Z,J).$$

As a result, we can deduce the following corollary:

Corollary 4.2. *In a $(PM^*S)_n$ the Ricci tensor is of Codazzi type provided φ and the scalar curvature ρ are constants.*

Using (6) once again,

$$\overline{\mathcal{M}^*}(G,H) = \left[1 - \frac{(n-2)\varphi}{2(n-1)}\right]\mathcal{S}(G,H) - \frac{\varphi\rho}{2(n-1)}g(G,H).$$

Contracting G and H gives

$$\overline{m^*} = (1-\varphi)\,\rho. \tag{26}$$

The $\mathcal{M}^*$-projective curvature tensor satisfies the relation for a $(PM^*S)_n$:

$$\begin{aligned}(\nabla_Z \mathcal{M}^*)(G,H,J,K) = 2D(Z)\,\mathcal{M}^*(G,H,J,K) + D(G)\,\mathcal{M}^*(Z,H,J,K) + D(H)\,\mathcal{M}^*(G,Z,J,K)\\+ D(J)\,\mathcal{M}^*(G,H,Z,K) + D(K)\,\mathcal{M}^*(G,H,J,Z),\end{aligned} \tag{27}$$

D being a non-vanishing 1-form and π is the associated vector field corresponding to the 1-form D, i.e.,

$$g(H,\pi) = D(H).$$

Contracting G and K in (27) we get

$$\left(\nabla_Z \overline{\mathcal{M}^*}\right)(H,J) = 2D(Z)\,\overline{\mathcal{M}^*}(H,J) + \mathcal{M}^*(Z,H,J,\pi) + D(H)\,\overline{\mathcal{M}^*}(Z,J) + D(J)\,\overline{\mathcal{M}^*}(H,Z) + \mathcal{M}^*(\pi,H,J,Z). \tag{28}$$

Again contracting H and J in (28) we find

$$\nabla_Z \overline{m^*} = 2D(Z)\,\overline{m^*} + 4\overline{\mathcal{M}^*}(Z,\pi). \tag{29}$$

If we use (26) in (29), we obtain

$$(1-\varphi)\,d\rho(Z) - d\varphi(Z)\,\rho = 2D(Z)(1-\varphi)\,\rho + 4\overline{\mathcal{M}^*}(Z,\pi). \tag{30}$$

Between (6) and (30), we have

$$(1-\varphi)\,d\rho(Z) - d\varphi(Z)\,\rho = \left[2(1-\varphi)\rho - \frac{2\varphi\rho}{(n-1)}\right]D(Z) + 4\left[1 - \frac{(n-2)\varphi}{2(n-1)}\right]D(\mathcal{Q}Z).$$

Thus we can state the following theorem:

Theorem 4.3. *For a* $(PM^{*}S)_{n}$ $(n > 2)$ *the following identity holds:*

$$(1-\varphi)\, d\rho\,(Z) - d\varphi\,(Z)\,\rho = \left[2\,(1-\varphi)\,\rho - \frac{2\varphi\rho}{(n-1)}\right] D\,(Z) + 4\left[1 - \frac{(n-2)\,\varphi}{2\,(n-1)}\right] D\,(QZ)\,.$$

In particular, let us consider $\varphi = 0$, then from Theorem 4.2. we get

$$d\rho\,(Z) = 2D\,(Z)\,\rho + 4D\,(QZ)\,.$$

Chaki established the following corollary in [3]:

Corollary 4.4. *For a* $(PS)_{n}$ *the following identity holds:*

$$d\rho\,(Z) = 2D\,(Z)\,\rho + 4D\,(QZ)\,.$$

5. $(PM^{*}S)_{n}$ $(n > 2)$ with $\mathrm{div}M^{*} = 0$

We know that for a $(PM^{*}S)_{n}$ $(n > 2)$,

$$\begin{aligned}(\nabla_{Z}M^{*})\,(G,H)\,J = 2D\,(Z)\,M^{*}\,(G,H)\,J + D\,(G)\,M^{*}\,(Z,H)\,J + D\,(H)\,M^{*}\,(G,Z)\,J \\ + D\,(J)\,M^{*}\,(G,H)\,Z + g\,(M^{*}\,(G,H)\,J,Z)\,\pi,\end{aligned}$$

D being a non-vanishing 1-form and π is the associated vector field corresponding to the 1-form D, i.e.,

$$g\,(H,\pi) = D\,(H)\,.$$

Hence,

$$\begin{aligned}(\mathrm{div}M^{*})\,(G,H)\,J &= \sum_{i=1}^{n} \varepsilon_{i} g\,((\nabla_{e_i}M^{*})\,(G,H)\,J,e_{i}) \\ &= \sum_{i=1}^{n} \varepsilon_{i}\,[2D\,(e_{i})\,g\,(M^{*}\,(G,H)\,J,e_{i}) + D\,(G)\,g\,(M^{*}\,(e_{i},H)\,J,e_{i}) + D\,(H)\,g\,(M^{*}\,(G,e_{i})\,J,e_{i}) \\ &\qquad + D\,(J)\,g\,(M^{*}\,(G,H)\,e_{i},e_{i}) + g\,(M^{*}\,(G,H)\,J,e_{i})\,g\,(\pi,e_{i})] \\ &= 3D\,(M^{*}\,(G,H)\,J) + D\,(G)\,\overline{\mathcal{M}^{*}}\,(H,J) - D\,(H)\,\overline{\mathcal{M}^{*}}\,(G,J)\,.\end{aligned}$$

Now $(\mathrm{div}M^{*})\,(G,H)\,J = 0$ implies

$$3D\,(M^{*}\,(G,H)\,J) + D\,(G)\,\overline{\mathcal{M}^{*}}\,(H,J) - D\,(H)\,\overline{\mathcal{M}^{*}}\,(G,J) = 0. \tag{31}$$

In (31), by contracting H and J,

$$2\overline{\mathcal{M}^{*}}\,(G,\pi) + \rho\,(1-\varphi)\,D\,(G) = 0. \tag{32}$$

Using (6) in (32) we deduce that

$$\mathcal{S}\,(G,\pi) = \frac{\rho\,(\varphi n - n + 1)}{[(n-1)\,(2-\varphi) + \varphi]} g\,(G,\pi)\,.$$

This implies

$$\mathcal{S}\,(G,\pi) = \mu g\,(G,\pi)\,, \tag{33}$$

where $\mu = \dfrac{\rho\,(\varphi n - n + 1)}{[(n-1)\,(2-\varphi) + \varphi]}$ is a scalar. Thus we can say that:

Theorem 5.1. *In a $(PM^*S)_n$ $(n > 2)$ with $divM^* = 0$, μ is an eigenvalue of the Ricci tensor $\mathcal{S}$ corresponding to the eigenvector π.*

Now if we take covariant derivative of (29), then we find

$$\nabla_W \nabla_Z \overline{m^*} = 2 (\nabla_W D)(Z) \overline{m^*} + 2D(Z) \left(\nabla_W \overline{m^*}\right) + 4 \left(\nabla_W \overline{\mathcal{M}^*}\right)(Z, \pi). \tag{34}$$

Using (28) and (29) in (34) we reach

$$\begin{aligned}\nabla_W \nabla_Z \overline{m^*} &= 2 (\nabla_W D)(Z) \overline{m^*} + 4D(Z) D(W) \overline{m^*} + 8D(Z) \overline{\mathcal{M}^*}(W, \pi) + 8D(W) \overline{\mathcal{M}^*}(Z, \pi) \\ &\quad + 4\mathcal{M}^*(W, Z, \pi, \pi) + 4D(Z) \overline{\mathcal{M}^*}(W, \pi) + 4D(\pi) \overline{\mathcal{M}^*}(Z, W) + 4\mathcal{M}^*(\pi, Z, \pi, W).\end{aligned} \tag{35}$$

Changing Z and W in (35) and subtracting these two equations, we obtain from (6) and (7) that

$$2\left[1 - \frac{(n-2)\varphi}{2(n-1)}\right] [D(W) \mathcal{S}(Z, \pi) - D(Z) \mathcal{S}(W, \pi)] + (\nabla_Z D)(W) \overline{m^*} - (\nabla_W D)(Z) \overline{m^*} = 0. \tag{36}$$

Assume that the scalar curvature ρ is non-zero, then from (26), (33) and (36) we can derive

$$(\nabla_Z D)(W) = (\nabla_W D)(Z).$$

As a result, we may say the following:

Theorem 5.2. *The associated 1-form of a $(PM^*S)_n$ $(n > 2)$ with $divM^* = 0$ is closed provided the scalar curvature ρ is non-zero.*

6. Example of a $(PM^*S)_4$

Let us consider a Lorentzian metric g on $\mathbb{R}^4$ by [15]

$$ds^2 = g_{ij} dx^i dx^j = \left(dx^1\right)^2 + \left(x^1\right)^2 \left(dx^2\right)^2 + \left(x^2\right)^2 \left(dx^3\right)^2 - \left(dx^4\right)^2,$$

where $i, j = 1, 2, 3, 4$. We calculate the non-vanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are

$$\Gamma^1_{22} = -x^1, \quad \Gamma^2_{33} = -\frac{x^2}{(x^1)^2}, \quad \Gamma^2_{12} = \frac{1}{x^1}, \quad \Gamma^3_{23} = \frac{1}{x^2}, \quad \mathcal{R}_{1332} = -\frac{x^2}{x^1}, \quad \mathcal{S}_{12} = -\frac{1}{x^1 x^2}$$

and using the symmetry properties, the other components are obtained.
Let us consider the scalar φ as follows:

$$\varphi = 6x^1. \tag{37}$$

The only non-vanishing $\mathcal{M}^*$-projective curvature tensor and its covariant derivatives are written by

$$\mathcal{M}^*_{1332} = -\frac{x^2}{x^1} + x^2, \quad \mathcal{M}^*_{1332,1} = \frac{x^2}{(x^1)^2}, \quad \mathcal{M}^*_{1332,2} = -\frac{1}{x^1} + 1. \tag{38}$$

The 1-form is chosen as follows:

$$D_i(x) = \begin{cases} \dfrac{1}{3x^1(x^1 - 1)}, & \text{when } i = 1 \\ \dfrac{1}{3x^2}, & \text{when } i = 2 \\ 0, & \text{otherwise} \end{cases} \tag{39}$$

Now, using these 1-forms, the equation (3) may be reduced to the following equations:

$$\mathcal{M}^*_{1332,1} = 3D_1\mathcal{M}^*_{1332} \tag{40}$$

and

$$\mathcal{M}^*_{1332,2} = 3D_2\mathcal{M}^*_{1332}. \tag{41}$$

From the equations (38), (39) and (41) we get

$$\begin{aligned}\text{Right hand side of (41)} &= 3D_2\mathcal{M}^*_{1332}\\ &= 3\cdot\frac{1}{3x^2}\cdot\left(-\frac{x^2}{x^1}+x^2\right)\\ &= -\frac{1}{x^1}+1 = \mathcal{M}^*_{1332,2}.\end{aligned}$$

It may be proved that (40) is also true using similar argument.
So, the manifold $\left(\mathbb{R}^4, g\right)$ under consideration is a $(PM^*S)_4$.

7. Pseudo $\mathcal{M}^*$-projective symmetric spacetimes

The spacetime of general relativity is a connected four-dimensional semi-Riemannian manifold $\left(M^4, g\right)$ with Lorentzian metric g whose signature $(-,+,+,+)$. The study of the casual character of vector of the manifold is the first step in the Lorentzian geometry. The Lorentzian manifold is a useful choice for studying general relativity because of this coincidence. Spacetimes have been studied by several authors in different ways such as ([2], [6], [9], [11], [12], [13], [16], [17], [19], [22], [26], [29], [34]).

Lorentzian manifolds with the Ricci tensor

$$\mathcal{S}(G,H) = \alpha g(G,H) + \beta D(G)D(H), \tag{42}$$

where α and β are scalars and π is a unit timelike vector field that corresponds to the 1-form D, are called perfect fluid spacetimes.

If the matter content of the spacetime is perfect fluid with velocity vector field π, the above form (42) of the Ricci tensor is derived from Einstein's equation. The energy momentum tensor T represents the matter content of spacetime, which is considered to be fluid. The energy momentum tensor for a perfect fluid spacetime has the form [24]

$$T(G,H) = (\sigma+p)D(G)D(H) + pg(G,H), \tag{43}$$

where σ represents the energy density and p represents the isotropic pressure. The velocity vector field π is a time-like vector that is metrically equal to the non-zero 1-form D. Because there are no heat conduction terms and stress factors corresponding to viscosity, the fluid is called perfect [14]. Furthermore, an equation of state governing the type of perfect fluid under consideration connects p and σ. In general, this is an equation of the form $p = p(\sigma, T_0)$, where T_0 denotes absolute temperature. We will just look at cases where T_0 is effectively constant and the state equation becomes $p = p(\sigma)$. The perfect fluid in this situation is known as isentropic [14]. In addition, if $p = \sigma$, the perfect fluid is referred to as stiff matter ([27], p. 66).

The Einstein's field equations (briefly, $\mathcal{EFE}$) without cosmological constant is as follows:

$$\mathcal{S}(G,H) - \frac{\rho}{2}g(G,H) = \kappa T(G,H), \tag{44}$$

where the Ricci tensor and scalar curvature are denoted by $\mathcal{S}$ and ρ, respectively. The gravitational constant is κ, whereas the energy momentum tensor is T. According to $\mathcal{EFE}$, matter controls the geometry of spacetime, while the motion of matter is dictated by the non-flat metric of space.

In this paper, we look at a special type of spacetime known as pseudo $\mathcal{M}^*$-projective symmetric spacetime. The results obtained for the pseudo $\mathcal{M}^*$-projective symmetric manifolds also holds for the Lorentzian setting. A 4-dimensional Lorentzian manifold (M, g) is said to be pseudo $\mathcal{M}^*$-projective symmetric spacetime if the $\mathcal{M}^*$-projective curvature tensor satisfies the relation (3). Here we consider the associated vector corresponding to the 1-form D is a unit timelike vector field, i.e., $g(\pi,\pi) = -1$. Thus equation (3) can be represented for a pseudo $\mathcal{M}^*$-projective symmetric spacetime as

$$\begin{aligned}
&(\nabla_Z\mathcal{R})(G,H,J,K) - \frac{\varphi}{6}\left[(\nabla_Z\mathcal{S})(H,J)\,g(G,K) - (\nabla_Z\mathcal{S})(G,J)\,g(H,K) + g(H,J)(\nabla_Z\mathcal{S})(G,K)\right.\\
&\left.-g(G,J)(\nabla_Z\mathcal{S})(H,K)\right] - \frac{d\varphi(Z)}{6}\left[\mathcal{S}(H,J)\,g(G,K) - \mathcal{S}(G,J)\,g(H,K) + g(H,J)\,\mathcal{S}(G,K) - g(G,J)\,\mathcal{S}(H,K)\right]\\
&= 2D(Z)\Big[\mathcal{R}(G,H,J,K) - \frac{\varphi}{6}\{\mathcal{S}(H,J)\,g(G,K) - \mathcal{S}(G,J)\,g(H,K) + g(H,J)\,\mathcal{S}(G,K)\\
&-g(G,J)\,\mathcal{S}(H,K)\}\Big] + D(G)\Big[\mathcal{R}(Z,H,J,K) - \frac{\varphi}{6}\{\mathcal{S}(H,J)\,g(Z,K) - \mathcal{S}(Z,J)\,g(H,K)\\
&+g(H,J)\,\mathcal{S}(Z,K) - g(Z,J)\,\mathcal{S}(H,K)\}\Big] + D(H)\Big[\mathcal{R}(G,Z,J,K) - \frac{\varphi}{6}\{\mathcal{S}(Z,J)\,g(G,K)\\
&-\mathcal{S}(G,J)\,g(Z,K) + g(Z,J)\,\mathcal{S}(G,K) - g(G,J)\,\mathcal{S}(Z,K)\}\Big] + D(J)\Big[\mathcal{R}(G,H,Z,K)\\
&-\frac{\varphi}{6}\{\mathcal{S}(H,Z)\,g(G,K) - \mathcal{S}(G,Z)\,g(H,K) + g(H,Z)\,\mathcal{S}(G,K) - g(G,Z)\,\mathcal{S}(H,K)\}\Big]\\
&+ D(K)\Big[\mathcal{R}(G,H,J,Z) - \frac{\varphi}{6}\{\mathcal{S}(H,J)\,g(G,Z) - \mathcal{S}(G,J)\,g(H,Z) + g(H,J)\,\mathcal{S}(G,Z) - g(G,J)\,\mathcal{S}(H,Z)\}\Big]. \qquad (45)
\end{aligned}$$

Taking a frame field and contracting G and K in (45) we get

$$\begin{aligned}
&2D(Z)\left[\left(1-\frac{\varphi}{3}\right)\mathcal{S}(H,J) - \frac{\varphi\rho}{6}g(H,J)\right] + D(H)\left[\left(1-\frac{\varphi}{3}\right)\mathcal{S}(Z,J) - \frac{\varphi\rho}{6}g(Z,J)\right]\\
&+ D(J)\left[\left(1-\frac{\varphi}{3}\right)\mathcal{S}(H,Z) - \frac{\varphi\rho}{6}g(H,Z)\right] + D(R(Z,H)J) - \frac{\varphi}{6}\left[D(Z)\,\mathcal{S}(H,J)\right.\\
&\left.-D(H)\,\mathcal{S}(Z,J) + D(QZ)\,g(H,J) - D(QH)\,g(Z,J)\right] + D(R(Z,J)H)\\
&- \frac{\varphi}{6}\left[D(Z)\,\mathcal{S}(H,J) - D(QJ)\,g(H,Z) + D(QZ)\,g(H,J) - D(J)\,\mathcal{S}(H,Z)\right]\\
&= \left(1-\frac{\varphi}{3}\right)(\nabla_Z\mathcal{S})(H,J) - \frac{\varphi}{6}d\rho(Z)\,g(H,J) - \frac{d\varphi(Z)}{6}\left[2\mathcal{S}(H,J) + \rho g(H,J)\right]. \qquad (46)
\end{aligned}$$

In $(PM^*S)_4$ spacetimes, we take the associated vector field π to be a parallel vector field. Then

$$\nabla_G\pi = 0, \qquad (47)$$

for every vector field G.
Using Ricci identity, we can now deduce

$$R(G,H)\,\pi = 0. \qquad (48)$$

It is clear from (48) that

$$\mathcal{R}(G,H,J,\pi) = 0, \qquad (49)$$

where $\mathcal{R}(G,H,J,\pi) = g(R(G,H)J,\pi)$.
Hence,

$$D(R(G,H)J) = 0. \qquad (50)$$

Contracting H in (48) we infer that

$$\mathcal{S}(G,\pi) = 0. \tag{51}$$

Now,

$$(\nabla_Z \mathcal{S})(G,\pi) = \nabla_Z \mathcal{S}(G,\pi) - \mathcal{S}(\nabla_Z G,\pi) - \mathcal{S}(G,\nabla_Z \pi).$$

As a result, if we use (47) and (51) in the previous equation, we find

$$(\nabla_Z \mathcal{S})(G,\pi) = 0. \tag{52}$$

Using (50), (51) and (52) in (46) we reach by putting $H = \pi$,

$$\left(1 - \frac{\varphi}{6}\right)\mathcal{S}(Z,J) = \frac{\varphi\rho}{6} g(Z,J) - \frac{\varphi\rho}{2} D(Z) D(J) + \frac{1}{6}\left[\varphi d\rho(Z) + \rho d\varphi(Z)\right] D(J). \tag{53}$$

Using $J = \pi$, in the preceding equation once more, we get

$$\frac{1}{6}\left[\varphi d\rho(Z) + \rho d\varphi(Z)\right] = \frac{2\varphi\rho}{3} D(Z). \tag{54}$$

By combining the equations (53) and (54), we arrive to the following result:

$$\mathcal{S}(Z,J) = \alpha g(Z,J) + \beta D(Z) D(J),$$

where $\alpha = \beta = \dfrac{\varphi\rho}{6-\varphi}$.
As a result, we may say the following:

Theorem 7.1. *A pseudo $\mathcal{M}^*$-projective symmetric spacetime with associated vector field as a parallel vector field is a perfect fluid spacetime.*

According to $\mathcal{EFE}$ without cosmological constant, the Ricci tensor takes the form

$$\mathcal{S}(G,H) = \kappa\left(\frac{p-\sigma}{-2}\right) g(G,H) + \kappa(p+\sigma) D(G) D(H).$$

In contrast to equation (42) we notice $\alpha = \dfrac{\kappa}{2}(\sigma - p)$ and $\beta = \kappa(p+\sigma)$.
Now $\alpha = \beta$ gives $p = -\dfrac{1}{3}\sigma$.
In view of this observation, we can conclude:

Theorem 7.2. *A pseudo $\mathcal{M}^*$-projective symmetric spacetime with associated vector field as a parallel vector field represents the limiting case of dark energy and the limiting case of violating the strong energy condition.*

Now, we consider the pseudo $\mathcal{M}^*$-projective symmetric spacetime with cyclic parallel Ricci tensor. Then

$$(\nabla_Z \mathcal{S})(H,J) + (\nabla_H \mathcal{S})(J,Z) + \left(\nabla_J \mathcal{S}\right)(Z,H) = 0. \tag{55}$$

Since the scalar curvature ρ is constant in a spacetime with cyclic parallel Ricci tensor, $d\rho(G) = 0$, for all G.
If we consider that the scalar φ is constant, then $d\varphi(G) = 0$, for all G.
Using (46) in (55), we now have

$$\begin{aligned}
&4D(Z)\left[\left(1-\frac{\varphi}{3}\right)\mathcal{S}(H,J) - \frac{\varphi\rho}{6} g(H,J)\right] + 4D(H)\left[\left(1-\frac{\varphi}{3}\right)\mathcal{S}(Z,J) - \frac{\varphi\rho}{6} g(Z,J)\right] \\
&+ 4D(J)\left[\left(1-\frac{\varphi}{3}\right)\mathcal{S}(H,Z) - \frac{\varphi\rho}{6} g(H,Z)\right] + D(R(Z,H)J) + D(R(Z,J)H) \\
&+ D(R(H,J)Z) + D(R(H,Z)J) + D(R(J,Z)H) + D(R(J,H)Z) = 0.
\end{aligned}$$

Following some computations, we arrive at

$$D(Z)\,E(H,J)+D(H)\,E(Z,J)+D(J)\,E(H,Z)=0,$$

where $E(G,H)=(6-2\varphi)\,\mathcal{S}(G,H)-\varphi\rho g(G,H)$. The above equation can be expressed in local coordinates as

$$D_iE_{jk}+D_jE_{ki}+D_kE_{ij}=0.$$

Walker's Lemma [30] is now listed as follows:

Lemma 7.3. *If α_{ij}, β_i are numbers satisfying $\alpha_{ij}=\alpha_{ji}$, $\alpha_{ij}\beta_k+\alpha_{jk}\beta_i+\alpha_{ki}\beta_j=0$ for $i,j,k=1,2,3,\ldots,n$, then either all α_{ij} are zero or all β_i are zero.*

As $D(G)\neq 0$, according to Walker's Lemma, we have $E(H,J)=0$, i.e.,

$$\mathcal{S}(H,J)=\left(\frac{\varphi\rho}{6-2\varphi}\right)g(H,J).$$

As a result, we arrive at the following theorem:

Theorem 7.4. *If the scalar φ is constant, then a pseudo $\mathcal{M}^*$-projective symmetric spacetime satisfying the cyclic parallel Ricci tensor is an Einstein spacetime.*

8. $\mathcal{M}^*$-projectively flat spacetimes

In this section we consider $\mathcal{M}^*$-projectively flat spacetimes, which are 4-dimensional Lorentzian manifolds with a timelike vector field. Hence from (2) we have

$$\mathcal{R}(G,H,J,K)=\frac{\varphi}{6}\left[\mathcal{S}(H,J)\,g(G,K)-\mathcal{S}(G,J)\,g(H,K)+g(H,J)\,\mathcal{S}(G,K)-g(G,J)\,\mathcal{S}(H,K)\right]. \tag{56}$$

Contracting H and J we can get

$$\mathcal{S}(G,K)=\frac{\varphi\rho}{2(3-\varphi)}g(G,K). \tag{57}$$

Again contracting G and K we reach

$$(1-\varphi)\,\rho=0.$$

This means $\varphi=1$, if $\rho\neq 0$.
Using this in (57) we obtain

$$\mathcal{S}(G,K)=\frac{\rho}{4}g(G,K). \tag{58}$$

As a consequence, $\mathcal{M}^*$-projectively flatness implies

$$R(G,H)\,J=\frac{\rho}{12}\left[g(H,J)\,G-g(G,J)\,H\right],$$

which reflects that the spacetime is of constant curvature. It is well known that space of constant curvature implies the spacetime is conformally flat and hence the spacetime is of Petrov type O.
Thus we get to the following conclusion:

Theorem 8.1. *A $\mathcal{M}^*$-projectively flat spacetime with non-zero scalar curvature is a space of constant curvature and of Petrov type O.*

Using (58) in (44) we find

$$T(G,H) = -\frac{\rho}{4\kappa} g(G,H). \tag{59}$$

Covariant differentiation of (59) yields

$$\left(\nabla_J T\right)(G,H) = -\frac{1}{4\kappa} d\rho(J)\, g(G,H). \tag{60}$$

The scalar curvature ρ is constant because the $\mathcal{M}^*$-projectively flat spacetime is Einstein. Thus

$$d\rho(J) = 0, \tag{61}$$

for all J.
Equations (60) and (61) implies

$$\left(\nabla_J T\right)(G,H) = 0.$$

Thus we can say that:

Theorem 8.2. *For a $\mathcal{M}^*$-projectively flat spacetime obeying $\mathcal{EFE}$ without cosmological constant the energy-momentum tensor is covariant constant.*

Remark 8.3. *It may be mentioned that Chaki and Ray [4] proved that a general relativistic spacetime with covariant constant energy-momentum tensor is Ricci symmetric.*

The matter collineation is defined by the energy momentum tensor T

$$\left(£_\eta T\right)(G,H) = 0, \tag{62}$$

where η is the symmetry-generating vector field and $£_\eta$ is the Lie derivative operator along the η vector field.
Let η be a Killing vector field with vanishing $\mathcal{M}^*$-projective curvature tensor on the spacetime. Then

$$\left(£_\eta g\right)(G,H) = 0. \tag{63}$$

Taking the Lie derivatives on both sides of (59) with respect to η we reach

$$\left(£_\eta T\right)(G,H) = -\frac{\rho}{4\kappa}\left(£_\eta g\right)(G,H). \tag{64}$$

We have from (63) and (64)

$$\left(£_\eta T\right)(G,H) = 0.$$

This means that matter collineation is possible in the spacetime.
If $\left(£_\eta T\right)(G,H) = 0$, on the other hand, we get from (64) that

$$\left(£_\eta g\right)(G,H) = 0.$$

As a consequence, the following theorem can be formulated:

Theorem 8.4. *For a $\mathcal{M}^*$-projectively flat spacetime obeying $\mathcal{EFE}$ without cosmological constant, the spacetime admits matter collineation with respect to a vector field η if and only if η is a Killing vector field.*

η is taken to be a conformal Killing vector field. Then we have

$$\left(\pounds_{\eta} g\right)(G, H) = 2\psi g(G, H), \tag{65}$$

where ψ being scalar.
Then from (64) and (65) we get

$$\left(\pounds_{\eta} T\right)(G, H) = -\frac{\rho}{4\kappa} 2\psi g(G, H). \tag{66}$$

Using (59) in (66) we deduce that

$$\left(\pounds_{\eta} T\right)(G, H) = 2\psi T(G, H). \tag{67}$$

According to (67), the energy-momentum tensor possesses the Lie inheritance property along η. If (67) holds, then (65) holds as well, indicating that η is a conformal Killing vector field. As a result, we can say that:

Theorem 8.5. *If a $\mathcal{M}^*$-projectively flat spacetime obeying $\mathcal{EFE}$ without cosmological constant, then a vector field η on the spacetime is a conformal Killing vector field if and only if the energy-momentum tensor has the Lie inheritance property along η.*

9. $\mathcal{M}^*$-projectively flat perfect fluid spacetimes

Now imagine a matter distribution in a perfect fluid whose velocity vector field is the vector field π, which is identical with 1-form D of the spacetime. As a result, T is given by [24]:

$$T(G, H) = (\sigma + p) D(G) D(H) + p g(G, H). \tag{68}$$

The energy density and isotropic pressure are represented by σ and p, respectively.
As a result, we get from $\mathcal{EFE}$ without cosmological constant

$$\mathcal{S}(G, H) - \frac{\rho}{2} g(G, H) = \kappa \left[(\sigma + p) D(G) D(H) + p g(G, H)\right]. \tag{69}$$

Contracting the previous equation, we acquire

$$\rho = \kappa(\sigma - 3p). \tag{70}$$

Using (57) in (69) we infer that

$$\left[\frac{\varphi\rho}{2(3-\varphi)} - \frac{\rho}{2}\right] g(G, H) = \kappa \left[(\sigma + p) D(G) D(H) + p g(G, H)\right]. \tag{71}$$

Setting $H = \pi$ in (71) and using $D(G) \neq 0$, we acquire

$$\frac{\rho(3-2\varphi)}{(6-2\varphi)} = \kappa\sigma. \tag{72}$$

Equations (70) and (72) also produce

$$\sigma = (2\varphi - 3) p. \tag{73}$$

As a result, based on the above, we can conclude:

Theorem 9.1. *The energy density and the isotropic pressure are related by* (73) *for a $\mathcal{M}^*$-projectively flat perfect fluid spacetime obeying $\mathcal{EFE}$ without the cosmological constant.*

Remark 9.2. $p = \dfrac{\sigma}{(2\varphi - 3)}$, *i.e.,* $p = p(\sigma)$, *in this situation. So we can conclude that the fluid is isentropic [14].*

From (73) we obtain $\dfrac{p}{\sigma} > -1$, when $\varphi < 1$. In most cases, the dark energy is characterised by an "equation-of-state" parameter $\omega \equiv \dfrac{p}{\sigma}$, the ratio of the spatially homogenous dark-energy pressure p and its energy density σ. Now $\omega > -1$ denotes that the model describes the evaluation in Quintessence region.
We can conclude the following from the preceding discussion:

Theorem 9.3. *A* $\mathcal{M}^*$*-projectively flat perfect fluid spacetime obeying* $\mathcal{EFE}$ *without the cosmological constant describes the model of the evaluation in Quintessence region, provided the scalar* $\varphi < 1$.

Now we consider the radiation era in $\mathcal{M}^*$-projectively flat perfect fluid spacetimes. In a perfect fluid spacetime, $\dfrac{p}{\sigma} = \dfrac{1}{3}$ defines the radiation era. In such instances, the energy–momentum tensor is of the form

$$T(G,H) = pg(G,H) + 4pD(G)D(H). \tag{74}$$

As a result, the $\mathcal{EFE}$ without the cosmological constant yields

$$\mathcal{S}(G,H) - \frac{\rho}{2}g(G,H) = \kappa\left[pg(G,H) + 4pD(G)D(H)\right]. \tag{75}$$

Using (57) in (75) reveals that

$$\left(\frac{\varphi\rho}{6-2\varphi} - \frac{\rho}{2}\right)g(G,H) = \kappa\left[pg(G,H) + 4pD(G)D(H)\right]. \tag{76}$$

Contracting the foregoing equation by taking a frame field, we obtain

$$4\left(\frac{\varphi\rho}{6-2\varphi} - \frac{\rho}{2}\right) = 0. \tag{77}$$

Setting $H = \pi$ in (76) we find

$$\left(\frac{\varphi\rho}{6-2\varphi} - \frac{\rho}{2}\right) = -3\kappa p. \tag{78}$$

By combining the equations (77) and (78), we arrive to the following result:

$$p = 0. \tag{79}$$

Thus, we can conclude from (74) and (79) that

$$T(G,H) = 0.$$

This indicates that the spacetime is devoid of matter. Thus we can state the following:

Theorem 9.4. *A radiation era in* $\mathcal{M}^*$*-projectively flat perfect fluid spacetime satisfying* $\mathcal{EFE}$ *without cosmological constant is vacuum.*

10. $\mathcal{M}^*$-projectively flat viscous fluid spacetimes

In a viscous fluid spacetime, the T is given by [23, 24]:

$$T(G,H) = pg(G,H) + (\sigma + p)D(G)D(H) + N(G,H), \tag{80}$$

where $N(G,H)$ is the fluid's anisotropic pressure. Also trace of $N = 0$ and $N(G,\pi) = 0$, where π is a velocity vector field.
Using (44) and (57) in (80) we get

$$\left(\frac{\varphi\rho}{6-2\varphi}-\frac{\rho}{2}\right)g(G,H)=\kappa\left[pg(G,H)+(\sigma+p)D(G)D(H)+N(G,H)\right]. \tag{81}$$

Putting $G = H = \pi$ in (81), yields

$$\sigma=-\frac{\rho(2\varphi-3)}{\kappa(6-2\varphi)}. \tag{82}$$

Again contracting (81) over G and H, we infer that

$$4\left(\frac{\varphi\rho}{6-2\varphi}-\frac{\rho}{2}\right)=\kappa\left[4p-(\sigma+p)\right],$$

implying

$$p=\frac{\rho(2\varphi-3)}{\kappa(6-2\varphi)}. \tag{83}$$

By combining the equations (82) and (83), we arrive to the following result:

$$\sigma+p=0.$$

We know the scalar curvature ρ of a $\mathcal{M}^*$-projectively flat spacetime is constant. If the scalar φ is constant, then $\sigma =$ constant follows from (82) and therefore $\sigma + p = 0$ gives $p =$ constant. Now, $\sigma + p = 0$ indicates that the fluid behaves as a cosmological constant [28]. It is also known as a phantom barrier [5]. In cosmology, such a choice $\sigma = -p$ entails rapid expansion of spacetime, which is known as inflation [1].
In view of this observation, we can conclude:

Theorem 10.1. *If a $\mathcal{M}^*$-projectively flat viscous fluid spacetime obeying $\mathcal{EFE}$ without cosmological constant, then the spacetime has constant energy density and isotropic pressure and the spacetime represents inflation and also the fluid behaves as a cosmological constant provided the scalar φ is constant.*

We will now consider whether or not a $\mathcal{M}^*$-projectively flat viscous fluid spacetime can accept heat flux. Assume T has the following shape: [23, 24]:

$$T(G,H)=pg(G,H)+(\sigma+p)D(G)D(H)+D(G)B(H)+D(H)B(G), \tag{84}$$

where $B(G) = g(G,\nu)$ for all vector fields G, ν being the heat flux vector field. Thus we have $g(\pi,\nu) = 0$, i.e., $B(\pi) = 0$.
In virtue of (44) and (57), equation (84) takes the form

$$\left(\frac{\varphi\rho}{6-2\varphi}-\frac{\rho}{2}\right)g(G,H)=\kappa\left[pg(G,H)+(\sigma+p)D(G)D(H)+D(G)B(H)+D(H)B(G)\right]. \tag{85}$$

Setting $H = \pi$ in (85), we notice that

$$B(G)=-\frac{1}{\kappa}\left[\frac{\rho(2\varphi-3)}{(6-2\varphi)}+\kappa\sigma\right]D(G).$$

As a result, we arrive at the following theorem:

Theorem 10.2. *A $\mathcal{M}^*$-projectively flat viscous fluid spacetime obeying $\mathcal{EFE}$ without cosmological constant admits heat flux, provided* $\left[\frac{\rho(2\varphi-3)}{(6-2\varphi)}+\kappa\sigma\right]\neq 0.$

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