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Dual spherical elastica

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Abstract. The solutions of the dual bending energy functional on the dual unit sphere $\mathbb{S}^2 \subset \mathbb{D}^3$ are investigated. Dual spherical elastica is characterized by the dual Euler-Lagrange equation by using techniques of calculus of variation. The real and dual parts of the dual Euler-Lagrange differential equation are solved by different methods. Dual spherical elastica is shown to correspond to the elastic strip constituted by planar elastica in $\mathbb{R}^3$.

1. Introduction

Dual numbers are a combination of numbers called real and dual parts. The set that includes all of these numbers is a commutative ring under addition and multiplication. A dual vector is an ordered triple of dual numbers and the set of all dual vectors is a module called dual space $\mathbb{D}^3$ on the commutative ring. E. Study used dual numbers and dual vectors in his research on line geometry and kinematics. He devoted special interest to the representation of directed lines by dual unit vectors. This interest brought the following Study theorem to the literature: The oriented lines in Euclidean 3−space $\mathbb{R}^3$ are in one-to-one correspondence with the points of the dual unit sphere $\mathbb{S}^2$ in $\mathbb{D}^3$. Therefore, a smooth curve on the dual unit sphere represents a ruled surface in $\mathbb{R}^3$ (for details, see [4, 12]). This makes it attractive to study with a curve on the dual unit sphere.

There are two variational problems in which the curves arising from the mechanics and physical conditions of elastic rods are studied. The first of these problems is elastic curves formulated by D. Bernoulli in 1740 and characterized by L. Euler in 1744. The elastic curve, also known as elastica, is the solution of the variational problem that minimizes the bending energy of a thin, inextensible wire. Mathematically, an elastic curve is defined as one of critical points of the total squared curvature functional within the family of regular curve, whose starting and ending points and tangent vectors at these points are the same (see, [6, 11]). The elasticity problem and its various generalizations which examines the physical state of a thin elastic rod when subjected to bending has a rich history and is actively studied today. Another variational problem arising from the mechanics and physical conditions of elastic rods is elastic strips, introduced by M. Sadowsky in 1930 to find the formulation of a developable Möbius strip with minimum energy. An elastic strip is a ruled surface, called a rectifying strip, such that its directrix is a critical point of the Sadowsky functional (see, [3, 6]). Considering the differentiable curves on the dual unit sphere $\mathbb{S}^2$ correspond to ruled surfaces in $\mathbb{R}^3$, we investigate the answer to the following question in this paper: *Can a one-to-one*

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relationship be established between elastic curves on $\mathbb{S}^2$ and elastic strips in $\mathbb{R}^3$? To answer this question, the following path is followed: At first, geometric preliminaries concerning the structure of $\mathbb{D}^3$ is stated and a short information about dual elastica is given. In the main section of the paper, the dual Euler-Lagrange equation which characterizes critical dual points of the dual bending energy functional acting on suitable space of curves on $\mathbb{S}^2$ is derived. The dual Euler-Lagrange equation is solved; while the real part of the equation is solved by Jacobi elliptic functions, the dual part of the dual Euler-Lagrange equation is solved by integral factor method. Then, a classification for dual spherical elastica is obtained. In the last part of the paper, the solution of the above-mentioned research question is investigated. For this, first of all, the characterization of elastic strips in $\mathbb{R}^3$ is reminded. In which cases the ruled surface which a dual curve on $\mathbb{S}^2$ corresponds in $\mathbb{R}^3$ becomes a rectifying strip is examined. In such a case, it is concluded that the rectifying strip is a binormal surface. By proving that the directrix of the binormal surface is elastica, the answer to the research question is found: Any dual curve including elastica on $\mathbb{S}^2$ corresponds to elastic strips constituted by planar elastica in $\mathbb{R}^3$.

2. Preliminaries

In this section, the basic definitions on dual numbers theory and dual elastica are reminded.

2.1. Dual numbers and dual unit sphere

A dual number $\hat{a}$ is a combination of a real part a and a dual part a^* such that it is expressed by $\hat{a} = a + \varepsilon a^*$ where ε is the dual operator with the property $\varepsilon^2 = 0$, $\varepsilon \neq 0$. The set of all dual numbers is denoted by

$$\mathbb{D} = \left\{ \hat{a} = a + \varepsilon a^* | \, a, a^* \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0 \right\}$$

is a commutative ring with the following addition and multiplication operations:

$$\hat{a} + \hat{b} = (a + \varepsilon a^*) + (b + \varepsilon b^*) = (a + b) + \varepsilon\,(a^* + b^*)$$

and

$$\hat{a}.\hat{b} = (a + \varepsilon a^*).(b + \varepsilon b^*) = ab + \varepsilon(ab^* + a^* b),$$

for $\hat{a} = a + \varepsilon a^*$, $\hat{b} = b + \varepsilon b^*$. The division is also defined as follows

$$\frac{\hat{a}}{\hat{b}} = \frac{a}{b} + \varepsilon \frac{a^* b - ab^*}{b^2}, \; b \neq 0.$$

Let $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ be a dual vector. The set of all dual vector is given by

$$\mathbb{D}^3 = \left\{ \hat{x} | \, \hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3) = \left(x_1 + \varepsilon x_1^*, x_2 + \varepsilon x_2^*, x_3 + \varepsilon x_3^* \right) = x + \varepsilon x^*, \; x, x^* \in \mathbb{R}^3 \right\}.$$

Scalar (or inner) and cross product of dual vector $\hat{x}$ and $\hat{y}$ are defined by

$$\langle \hat{x}, \hat{y} \rangle = \langle x, y \rangle + \varepsilon(\langle x, y^* \rangle + \langle x^*, y \rangle)$$

and

$$\hat{x} \times \hat{y} = x \times y + \varepsilon(x \times y^* + x^* \times y).$$

If $x \neq 0$, the norm $\|\hat{x}\|$ of $\hat{x}$ is defined by

$$\|\hat{x}\| = \sqrt{\langle \hat{x}, \hat{x} \rangle} = \|x\| + \varepsilon \frac{\langle x, x^* \rangle}{\|x\|}.$$

A dual vector $\hat{x}$ with norm 1 (or $(1,0)$) is called a dual unit vector. For a dual unit vector, we have $\langle x, x\rangle = 1$, $\langle x, x^*\rangle = 0$.

Let $\hat{\gamma}(t) = \gamma(t) + \varepsilon\gamma^*(t)$ be a dual curve with parameter $t \in I \subset \mathbb{R}$ in $\mathbb{D}^3$. The real curve $\gamma(t)$ is called the (real) indicatrix of $\hat{\gamma}(t)$. If every $\gamma_i(t)$ and $\gamma_i^*(t)$ are differentiable, then $\hat{\gamma}(t)$ is differentiable in $\mathbb{D}^3$. The dual arc length of the dual curve $\hat{\gamma}$ is defined as

$$\hat{s} = \int_0^s \left\|\dot{\hat{\gamma}}(t)\right\| dt = \int_0^s \left\|\dot{\gamma}(t)\right\| dt + \varepsilon \int_0^s \langle T, \dot{\gamma}^*(t)\rangle dt = s + \varepsilon s^*, \tag{1}$$

where s and T is arc length and the unit tangent vector to γ, respectively and "$\cdot$" expresses the derivative with respect to s.

Now we recall equations relative to derivatives of dual Frenet vectors along the dual curve $\hat{\gamma}$ in $\mathbb{D}^3$. Suppose that $\hat{\gamma}$ is a reparametrization curve with the parametrization s of the indicatrix. Then,

$$\hat{\gamma}' = \dot{\hat{\gamma}}\frac{ds}{d\hat{s}} = \hat{T} \tag{2}$$

is called the dual unit tangent vector to $\hat{\gamma}(s)$, where $\hat{\gamma}' = \frac{d\hat{\gamma}}{d\hat{s}}$ and $\dot{\hat{\gamma}} = \frac{d\gamma}{ds}$. Moreover we have $\frac{d\hat{s}}{ds} = 1 + \varepsilon\Delta$ from (1), where $\Delta = \langle T, \dot{\gamma}^*(t)\rangle$. The dual unit vectors $\hat{N}$ and $\hat{B}$ are called the dual principle normal and binormal of $\hat{\gamma}$ at the point $\hat{\gamma}(s)$, respectively. For the dual Frenet trihedron $\left\{\hat{T}, \hat{N}, \hat{B}\right\}$ along $\hat{\gamma}$, we have the formulas

$$\frac{d}{d\hat{s}}\begin{pmatrix}\hat{T}\\ \hat{N}\\ \hat{B}\end{pmatrix} = \begin{pmatrix}0 & \hat{\kappa} & 0\\ -\hat{\kappa} & 0 & \hat{\tau}\\ 0 & -\hat{\tau} & 0\end{pmatrix}\begin{pmatrix}\hat{T}\\ \hat{N}\\ \hat{B}\end{pmatrix}, \tag{3}$$

where $\hat{\kappa} = \kappa + \varepsilon\kappa^*$ and $\hat{\tau} = \tau + \varepsilon\tau^*$ are nowhere pure dual curvature and dual torsion functions of $\hat{\gamma}$ [8, 9, 13, 14].

Let x be the position vector with respect to the orthonormal frame of reference of a real point with coordinates (x_1, x_2, x_3) in $\mathbb{R}^3$. Then the set of all points x with $\langle x, x\rangle = 1$ gives the real unit sphere. Similarly, we suppose that the dual vector $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ is not real. The set of all dual points with $\langle \hat{x}, \hat{x}\rangle = 1$ is called the dual unit sphere and the set is denoted by

$$\mathbb{S}^2 = \left\{\hat{x} = x + \varepsilon x^* \in \mathbb{D}^3 \middle| \langle \hat{x}, \hat{x}\rangle = 1\right\}.$$

Assume that $\hat{\gamma}(t) = \gamma(t) + \varepsilon\gamma^*(t)$ be a dual curve on $\mathbb{S}^2$, that is $\langle \hat{\gamma}(t), \hat{\gamma}(t)\rangle = 1$ and the real curve γ on the real unit sphere. The trajectory of a differentiable dual curve on $\mathbb{S}^2$ is equivalent to a ruled surface which is a surface generated by the motion of a straight line (see for details [9, 13]). If a ruled surface parametrized by

$$\begin{array}{llll} R: & [0,\ell]\times[-\epsilon,\epsilon] & \rightarrow & \mathbb{R}^3 \\ & (t,\delta) & \rightarrow & R(t,\delta) = \gamma(t) + \delta B(t), \end{array} \tag{4}$$

where B is the binormal vector of directrix curve γ, then it is called binormal surface [7]. If $\delta = 0$ in (4), the normal vector of the surface is equivalent to the principle normal of γ. So γ'' is perpendicular to the tangent plane, that is, γ is a geodesic of R.

2.2. *Dual Elastica*

We denote a set of dual space curves by Ω satisfying the following conditions: $\hat{\gamma} : [0,\ell] \rightarrow \mathbb{D}^3$, $\hat{\gamma}(i\ell) = \hat{p}_i$, $\hat{\gamma}'(i\ell) = \hat{v}_i$ for $i = 0, 1$. The dual elastica minimizes the dual bending energy functional

$$\begin{array}{llll} \mathcal{E}: & \Omega \subset \mathbb{D}^3 & \rightarrow & \mathbb{D} \\ & \hat{\gamma} & \rightarrow & \mathcal{E}(\hat{\gamma}) = \int_{\hat{\gamma}} \left\|\hat{\gamma}''(t)\right\|^2 d\hat{t} \end{array}$$

and it is characterized by the following dual Euler-Lagrange equation

$$\hat{\kappa}'' + \frac{\hat{\kappa}^3}{2} - \frac{\hat{c}^2}{\hat{\kappa}^3} - \frac{\hat{\lambda}}{2}\hat{\kappa} = 0, \tag{5}$$

where $\hat{c} = c + \varepsilon c^*$ and $\hat{\lambda} = \lambda + \varepsilon\lambda^*$ are dual constants.

If $\hat{\kappa}$ is a non-zero constant real and dual part, then (3) is a system of linear ordinary differential equations with constant coefficients. Hence, the formula can be given directly.

If $\hat{\kappa}$ is not a constant dual and real part, then the solution of (5) is found as

$$\hat{\kappa} = \kappa_0\left(1 - \left(\frac{p}{h}\right)^2 sn^2\left(\frac{\kappa_0}{2h}s, p\right)\right)^{\frac{1}{2}} + \varepsilon\frac{1}{4\kappa\mu}\int \mu\frac{\kappa^2 aa^* + \kappa^2(\kappa^2-\lambda)\lambda^* + 4\kappa^2\dot{\kappa}^2\Delta - 4cc^*}{\kappa\dot{\kappa}}ds$$

and

$$\hat{\tau} = \frac{c}{\kappa^2} + \varepsilon\frac{\kappa c^* - 2c\kappa^*}{\kappa^3},$$

where sn is the elliptic sine function, and κ_0, h and p are real parameters related to λ, c and a as follows

$$2\lambda = \frac{\kappa_0^2}{h^2}\left(3h^2 - p^2 - 1\right),$$

$$4c^2 = \frac{\kappa_0^6}{h^4}\left(1 - h^2\right)\left(h^2 - p^2\right),$$

$$a^2 = (\frac{\kappa_0^2}{2h^2}(3h^2 - p^2 - 1))^2 + \frac{\kappa_0^4}{h^4}((1 - h^2)(2h^2 - p^2) + h^2(p^2 - h^2))$$

and

$$\mu = e^{\int \frac{2\kappa^4(\kappa^2-\lambda) - \left(2\kappa\dot{\kappa}\right)^2 - 4c^2}{4\dot{\kappa}\kappa^3}ds}.$$

The dual elastica has the following classification:

i) If $h = p$, then solution is found as follows

$$\hat{\kappa} = \kappa_0 cn\left(\frac{\kappa_0}{2p}s, p\right) + \varepsilon\frac{1}{4\kappa\mu}\int \mu\frac{\kappa aa^* + \kappa(\kappa^2-\lambda)\lambda^* + 4\kappa\dot{\kappa}^2\Delta}{\dot{\kappa}}ds,$$

where

$$\lambda = \frac{\kappa_0^2}{2p^2}\left(2p^2 - 1\right),$$

$$a^2 = \left(\frac{\kappa_0^2}{2p^2}\left(2p^2 - 1\right)\right)^2 + \frac{\kappa_0^4}{p^2}\left(1 - p^2\right)$$

and

$$\mu = e^{\int \frac{\kappa^2(\kappa^2-\lambda) - 2\left(\dot{\kappa}\right)^2}{2\dot{\kappa}\kappa}ds}.$$

At this case, the curve is called dual wavelike elastica.

ii) If $h = 1$, then we have

$$\hat{\kappa} = \kappa_0 dn\left(\frac{\kappa_0}{2p}s, p\right) + \varepsilon\frac{1}{4\kappa\mu}\int \mu\frac{\kappa aa^* + \kappa(\kappa^2-\lambda)\lambda^* + 4\kappa\dot{\kappa}^2\Delta}{\dot{\kappa}}ds,$$

where

$$\lambda = \frac{\kappa_0^2}{2}\left(2 - p^2\right),$$

$$a^2 = \left(\frac{\kappa_0^2}{2}\left(2 - p^2\right)\right)^2 + \kappa_0^4\left(p^2 - 1\right)$$

and

$$\mu = e^{\int \frac{\kappa^2(\kappa^2-\lambda)-2\left(\dot{\kappa}\right)^2}{2\dot{\kappa}\kappa} ds}.$$

At this case, the curve is called dual orbitlike elastica.

iii) If $h = p = 1$, then we get

$$\hat{\kappa} = \kappa_0 \sec h\left(\frac{\kappa_0}{2}s\right) + \varepsilon \frac{1}{4\kappa\mu} \int \mu \frac{\kappa\frac{\kappa_0^2}{2}a^* + \kappa(\kappa^2 - \frac{\kappa_0^2}{2})\lambda^* + 4\kappa\dot{\kappa}^2\Delta}{\dot{\kappa}} ds,$$

where

$$\mu = e^{\int \frac{\kappa^2(\kappa^2-\frac{\kappa_0^2}{2})-2\left(\dot{\kappa}\right)^2}{2\dot{\kappa}\kappa} ds}.$$

At this case, the curve is called dual borderlike elastica (for details, see [8]).

3. Dual Spherical Elastica

Let $\hat{\gamma}$ be a dual curve on the dual unit sphere $\mathbb{S}^2 \subset \mathbb{D}^3$. Then we have the orthonormal triple $\{\hat{\gamma}, \hat{T}, \hat{g}\}$ as called the dual geodesic trihedron of $\hat{\gamma}$, where

$$\hat{T} = \frac{d\hat{\gamma}}{d\hat{s}}$$

is the dual unit tangent vector to $\hat{\gamma}$ and

$$\hat{g} = \hat{\gamma} \times \hat{T}.$$

The dual geodesic trihedron of $\hat{\gamma}$ has the following fundamental relations

$$\begin{pmatrix} \hat{\gamma}' \\ \hat{T}' \\ \hat{g}' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & \hat{\kappa}_g \\ 0 & -\hat{\kappa}_g & 0 \end{pmatrix} \begin{pmatrix} \hat{\gamma} \\ \hat{T} \\ \hat{g} \end{pmatrix}, \tag{6}$$

where $\hat{\kappa}_g$ is the dual geodesic curvature of $\hat{\gamma}$. The dual curvature of $\hat{\gamma}$ has the identity

$$\hat{\kappa}^2 = 1 + \hat{\kappa}_g^2$$

[9, 13]. Therefore, we can define dual spherical elastica (or dual elastica on $\mathbb{S}^2$) as a critical point of the dual bending energy functional

$$\int_{\hat{\gamma}} \left(\hat{\kappa}_g^2 + \hat{\sigma}\right) d\hat{s} \tag{7}$$

in the space $\Phi = \left\{\hat{\gamma} : [0, \ell] \to \mathbb{S}^2 \subset \mathbb{D}^3,\ \hat{\gamma}(i\ell) = \hat{p}_i,\ \hat{\gamma}'(i\ell) = \hat{v}_i,\ i = 0, 1\right\}$ for fixed dual constant $\hat{\sigma}$.

We can easily see from (6) the equality

$$\left\|\hat{T}'\right\|^2 = 1 + \hat{\kappa}_g^2.$$

Hence, we want to minimize the dual functional

$$\int_{\hat{\gamma}} \left(\left\|\hat{T}'\right\|^2 + \hat{\rho}\right) d\hat{s},$$

where $\hat{\rho} = \hat{\sigma} - 1$ under the restrictions

$$\left\|\hat{T}\right\|^2 = 1,\ \hat{T} = \hat{\gamma}',\ <\hat{\gamma}, \hat{\gamma}>= 1.$$

So we can apply the dual Euler-Lagrange equations to the dual functional

$$\hat{F} = \left\|\hat{T}'\right\|^2 + \hat{\rho} + \hat{\lambda}\left(\left\|\hat{T}\right\|^2 - 1\right) + \hat{\mu}\left(\left\|\hat{\gamma}\right\|^2 - 1\right) - 2 < \hat{\Lambda}, \hat{\gamma}' - \hat{T} > .$$

If $\hat{\gamma}$ is an extremal for $\hat{F}$, then the following equations hold;

$$\frac{\partial \hat{F}}{\partial \hat{\gamma}} - \frac{d}{d\hat{s}}\left(\frac{\partial \hat{F}}{\partial \hat{\gamma}'}\right) = 0, \quad \frac{\partial \hat{F}}{\partial \hat{T}} - \frac{d}{d\hat{s}}\left(\frac{\partial \hat{F}}{\partial \hat{T}'}\right) = 0.$$

Then we obtain

$$\hat{\mu}\hat{\gamma} - \hat{\Lambda}' = 0 \tag{8}$$

and

$$\hat{\lambda}\hat{T} - \hat{T}'' = \hat{\Lambda}. \tag{9}$$

Combining (8) and (9), we get

$$\hat{\lambda}'\hat{T} + \hat{\lambda}\hat{T}' - \hat{T}''' = \hat{\mu}\hat{\gamma}. \tag{10}$$

From (6), we have the following derivatives:

$$\hat{T}' = -\hat{\gamma} + \hat{\kappa}_g\hat{g}, \tag{11}$$

$$\hat{T}'' = \hat{\kappa}_g'\hat{g} - \left(\hat{\kappa}_g + 1\right)\hat{T} \tag{12}$$

and

$$\hat{T}''' = \left(\hat{\kappa}_g^2 + 1\right)\hat{\gamma} - 3\hat{\kappa}_g\hat{\kappa}_g'\hat{T} + \left(\hat{\kappa}_g'' - \hat{\kappa}_g^3 - \hat{\kappa}_g\right)\hat{g}. \tag{13}$$

Substituting (11), (12) and (13) into (10), we calculate

$$-\left(\hat{\lambda} + \hat{\mu} + \hat{\kappa}_g^2 + 1\right)\hat{\gamma} + \left(\hat{\lambda}' + 3\hat{\kappa}_g\hat{\kappa}_g'\right)\hat{T} + \left(\hat{\lambda}\hat{\kappa}_g - \hat{\kappa}_g'' + \hat{\kappa}_g^3 + \hat{\kappa}_g\right)\hat{g} = 0.$$

Since the dual vectors $\hat{\gamma}$, $\hat{T}$ and $\hat{g}$ are orthonormal, we have

$$\hat{\lambda} + \hat{\mu} + \hat{\kappa}_g^2 + 1 = 0, \tag{14}$$

$$\hat{\lambda} = -\frac{3}{2}\hat{\kappa}_g^2 + \hat{C} \tag{15}$$

and

$$\hat{\lambda}\hat{\kappa}_g - \hat{\kappa}_g'' + \hat{\kappa}_g^3 + \hat{\kappa}_g = 0. \tag{16}$$

Substituting (15) into (16), we get

$$\hat{\kappa}_g'' + \frac{1}{2}\hat{\kappa}_g^3 - \left(1 + \hat{C}\right)\hat{\kappa}_g = 0. \tag{17}$$

In order to determine the dual constant $\hat{C}$ with respect to the dual constant $\hat{\sigma}$, we consider the boundary condition

$$\hat{F}(\ell) - \frac{\partial \hat{F}}{\partial \hat{\gamma}'}(\ell)\,\hat{\gamma}'(\ell) - \frac{\partial \hat{F}}{\partial \hat{T}'}(\ell)\,\hat{T}'(\ell) = 0$$

for $\hat{\gamma}$. Then we have

$$-\hat{\kappa}_g^2(\ell) - 1 + \hat{\rho} - 2 < \hat{\Lambda}(\ell), \hat{\gamma}'(\ell) >= 0. \tag{18}$$

By using (9), we calculate

$$< \hat{\Lambda}(\ell), \hat{\gamma}'(\ell) >= \hat{\lambda} + \hat{\kappa}_g^2(\ell) + 1.$$

From (14), we obtain

$$< \hat{\Lambda}(\ell), \hat{\gamma}'(\ell) >= -\frac{1}{2}\hat{\kappa}_g^2(\ell) + 1 + \hat{C}. \tag{19}$$

Substituting (19) into (18), we have

$$\hat{C} + 1 = \frac{1}{2}(\hat{\rho} - 1) = \frac{1}{2}(\hat{\sigma} - 2).$$

So we can rearrange (17) as follows

$$\hat{\kappa}_g'' + \frac{1}{2}\hat{\kappa}_g^3 + \left(1 - \frac{\hat{\sigma}}{2}\right)\hat{\kappa}_g = 0. \tag{20}$$

Then we can give the following theorem.

Theorem 1. A dual elastica on the dual unit sphere $\mathbb{S}^2 \subset \mathbb{D}^3$ can be determined by the dual Euler-Lagrange equation (20).

A solution of (20) is the case of constant dual geodesic curvature. Assume that $\hat{\kappa}_g$ has a non dual constant value. Then, (20) can be integrated to

$$\left(\hat{\kappa}_g'\right)^2 = \hat{C}_1 - \frac{1}{4}\hat{\kappa}_g^4 - \left(1 - \frac{\hat{\sigma}}{2}\right)\hat{\kappa}_g^2. \tag{21}$$

(21) can be written in terms of the squared of the dual geodesic curvature, $\hat{u} = \hat{\kappa}_g^2$, as follows

$$(\hat{u}')^2 + \hat{u}^3 + 4\left(1 - \frac{\hat{\sigma}}{2}\right)\hat{u}^2 - 4\hat{u}\hat{C}_1 = 0, \tag{22}$$

where the real and dual parts of (22) are respectively given by

$$\left(\dot{u}\right)^2 + u^3 + 4\left(1 - \frac{\sigma}{2}\right)u^2 - 4uC_1 = 0$$

and

$$2u\dot{u}^* + 3u^2u^* + 8(1 - \frac{\sigma}{2})uu^* + 2\sigma^* u^2 - 4C_1u^* - 4uC_1^* = 0, \tag{23}$$

where $\hat{u} = u + \varepsilon u^*$, $\sigma = \sigma + \varepsilon\sigma^*$ and $\hat{C}_1 = C_1 + \varepsilon C_1^*$. In order to solve the dual Euler-Lagrange equation (22), we solve both of the real and dual parts.

Since the real part of (22) is of the form $\left(\dot{u}\right)^2 = P\left(u\right)$, where P is the cubic polynomial, it can be solved by using Jacobi elliptic functions as follows

$$\kappa_g^2 = \kappa_{g_0}^2 dn^2\left(\frac{\kappa_{g_0}}{2}s, p\right),$$

where κ_{g_0} is the maximal geodesic curvature and p is real parameter related to σ and C_1 as follows

$$\sigma = \frac{\kappa_{g_0}^2}{2}(2 - p^2) + 2$$

and

$$C_1 = \frac{1}{4}\kappa_{g_0}^4\left(p^2 - 1\right)$$

(see, [1, 10]). Now we focus on the solution of (23) which is the dual part of (22). (23) can be revised as follows

$$\dot{u}^* + (\frac{3}{2}u + 4(1 - \frac{\sigma}{2}) - \frac{2C_1}{u})u^* = 2C_1^* - \sigma^* u. \tag{24}$$

One can see that (24) can be solved by integral factor method. The integral factor is found as follows

$$\mu = e^{\int(\frac{3}{2}u+4(1-\frac{\sigma}{2})-\frac{2C_1}{u})ds} = e^{(\frac{3}{2}\kappa_{g_0}^2 - \frac{2C_1}{\kappa_{g_0}^2(1-p^2)})E(\frac{\kappa_{g_0}^2}{2}s) + \frac{2C_1p^2 sn(\frac{\kappa_{g_0}}{2}s,p)cn(\frac{\kappa_{g_0}}{2}s,p)}{\kappa_{g_0}^2(1-p^2)} + 4(1-\frac{\sigma}{2})s + A},$$

where $E(\frac{\kappa_{g_0}}{2}s)$ is Legender's elliptic integral of second kind (see, [2, 10]) and A is integration constant. If we multiply both of two side of (24) by μ, we obtain

$$(\dot{\mu u^*}) = \mu(2C_1^* - \sigma^* u).$$

So, we get

$$u^* = 2\kappa_g\kappa_g^* = \frac{1}{\mu}\int \mu(2C_1^* - \sigma^*\kappa_g^2)ds$$

since $\hat{u} = u + \varepsilon u^* = \hat{\kappa}_g^2 = \kappa_g^2 + \varepsilon 2\kappa_g\kappa_g^*$. Therefore, we find

$$\kappa_g^* = \frac{1}{2\kappa_g\mu}\int \mu(2C_1^* - \sigma^*\kappa_g^2)ds.$$

Then we can give the following classification according to parameter p :
i) If $p = 0$, then $\sigma = \kappa_{g_0}^2 + 2$ has maximal value and solutions are

$$\hat{\kappa}_g = \kappa_{g_0} + \varepsilon\frac{1}{2\kappa_{g_0}}\int(2C_1^* - \sigma^*\kappa_{g_0}^2)ds.$$

ii) If $p^2 < 1$ and $\sigma < \frac{1}{2}\kappa_{g_0}^2 + 2$, one uses the formula

$$\hat{\kappa}_g = \kappa_{g_0} cn\left(\sqrt{\kappa_{g_0}^2 + 2 - \frac{\sigma}{2}}s, \frac{1}{p}\right) + \varepsilon \frac{1}{2\kappa_g \mu} \int \mu(2C_1^* - \sigma^* \kappa_g^2) ds.$$

Especially, if $\sigma = 2$, then the dual geodesic curvature is given by

$$\hat{\kappa}_g = \kappa_{g_0} coslemn\left(\frac{\kappa_{g_0}}{2}s\right) + \varepsilon \frac{1}{2\kappa_g \mu} \int \mu(2C_1^* - \sigma^* \kappa_g^2) ds.$$

iii) If $p^2 = 1$, then $\sigma = \frac{1}{2}\kappa_{g_0}^2 + 2$ and we have

$$\hat{\kappa}_g = \kappa_{g_0} sech\left(\frac{\kappa_{g_0}}{2}s\right) + \varepsilon \frac{1}{2\kappa_g \mu} \int \mu(2C_1^* - \sigma^* \kappa_g^2) ds.$$

iv) If $p^2 > 1$, then we have

$$\hat{\kappa}_g = \pm\kappa_{g_0} dn\left(\frac{\kappa_{g_0}}{2}s, p\right) + \varepsilon \frac{1}{2\kappa_g \mu} \int \mu(2C_1^* - \sigma^* \kappa_g^2) ds.$$

On the other hand, (21) can also be expressed by the dual curvature $\hat{\kappa}$ and the dual torsion $\hat{\tau}$ using the following relation

$$\left(\hat{\kappa}_g'\right)^2 = \hat{\tau}^2 \hat{\kappa}^4. \tag{25}$$

Substituting (25) into (21), we obtain

$$\hat{\tau}^2 \hat{\kappa}^4 = -\frac{1}{4}\hat{\kappa}^4 - \frac{1}{2}(1 - \hat{\sigma})\hat{\kappa}^2 + \frac{1}{2}\left(\frac{3}{2} - \hat{\sigma}\right) + \hat{C}_1.$$

4. Conclusions

According to E. Study mapping, a differentiable dual curve on $\mathbb{S}^2$ corresponds to a ruled surface in $\mathbb{R}^3$. It is natural to inquire about the relationship between dual spherical elastica and elastic strips in $\mathbb{R}^3$. The answer to this question will be investigated in the following section of the paper.

Now we recall some relations about elastic strips in $\mathbb{R}^3$. From [3], we known that an elastic strip is a developable ruled surface (or rectifying strip) described by

$$\begin{array}{llll} R_\gamma : & [0,\ell] \times [-\epsilon, \epsilon] & \to & \mathbb{R}^3 \\ & (t, \delta) & \to & R_\gamma(t,\delta) = \gamma(t) + \delta(\omega(t) T(t) + B(t)) \end{array} \tag{26}$$

if γ is a critical point of the modified Sadowsky functional

$$S_\mu(\gamma) = \int_0^\ell (\kappa^2(1 + \omega^2)^2 - \eta) v dt,$$

where η is Lagrange multiplier, standing for the length constraint. Also T is the unit tangent vector, B is the unit binormal of γ and $\omega = \frac{\tau}{\kappa}$ is the modified torsion of γ such that κ is the curvature and τ is the torsion of γ. Therefore, an elastic strip is are characterized by the Euler-Lagrange equations

$$r_1 = r_2 = 0, \tag{27}$$

where

$$\begin{aligned} r_1 := & \frac{d\left(\frac{d\kappa}{ds}\left(1+\omega^2\right)^2+2\kappa\left(1+\omega^2\right)\omega\frac{d\omega}{ds}\right)}{ds} \\ & + \frac{\kappa}{2}\left(\kappa^2\left(1+\omega^2\right)^2+\eta\right)+\omega\kappa(\kappa^2\left(1+\omega^2\right)^2\omega \\ & + \frac{d\left(\frac{2}{\kappa}\frac{d\kappa}{ds}\left(1+\omega^2\right)\omega\right)}{ds}+\frac{d^2\left(2\left(1+\omega^2\right)\omega\right)}{ds^2}) \end{aligned}$$

and

$$\begin{aligned} r_2 := & -\frac{d\left(\kappa^2\left(1+\omega^2\right)^2\omega+\frac{d\left(\frac{2}{\kappa}\frac{d\kappa}{ds}\left(1+\omega^2\right)\omega\right)}{ds}+\frac{d^2\left(2\left(1+\omega^2\right)\omega\right)}{ds^2}\right)}{ds} \\ & + \omega\kappa(\frac{d\kappa}{ds}\left(1+\omega^2\right)^2+2\kappa\left(1+\omega^2\right)\omega\frac{d\omega}{ds}) \end{aligned}$$

[3].

From E. Study mapping, we know that the dual curve $\hat{\gamma} = \gamma + \varepsilon\gamma^*$ on $\mathbb{S}^2$ corresponds to the ruled surface in $\mathbb{R}^3$ parametrized by

$$R(t,\delta) = \gamma(t) \times \gamma^*(t) + \delta\gamma(t),$$

where $\gamma \times \gamma^*$ is the directrix and γ is the director curve of the ruled surface $R(t,\delta)$. In the following result, we give the necessary condition that the ruled surface which a dual curve corresponds in $\mathbb{R}^3$ to be rectifying strip defined in (26).

Conclusion 1. A dual curve $\hat{\gamma} = \gamma + \varepsilon\gamma^*$ on $\mathbb{S}^2 \subset \mathbb{D}^3$ corresponds to the rectifying strip with planar directrix $\gamma \times \gamma^*$ such that it is the binormal surface of $\gamma \times \gamma^*$.

Proof. Let $\hat{\gamma} = \gamma + \varepsilon\gamma^*$ be a dual curve on $\mathbb{S}^2$. Suppose that the ruled surface where the dual curve $\hat{\gamma}$ corresponds to a rectifying strip in $\mathbb{R}^3$. Then the rectifying strip must be in the form of (26), that is, the rectifying strip is given by

$$R_\gamma(t,\delta) = \gamma(t) \times \gamma^*(t) + \delta\left(\frac{\tau(t)}{\kappa(t)}T(t) + B(t)\right),$$

where $\frac{\tau(t)}{\kappa(t)}T(t) + B(t) = \gamma(t)$, τ, κ, T and B are the torsion, curvature, tangent vector and binormal vector of $\gamma \times \gamma^*$ at the point $(\gamma \times \gamma^*)(t)$, respectively. Since $\hat{\gamma}$ is a dual curve on $\mathbb{S}^2$, we have $\tau = 0$ for all $t \in \mathbb{R}$.

We recall that an elastica on a regular surface characterized by the Euler-Lagrange equation

$$\dot{\kappa}_g + \frac{\dot{(\kappa_n^2\tau_g)}}{\kappa_n} + \kappa_g\left(\frac{\kappa_g^2+\kappa_n^2}{2} - \tau_g^2 - C\right) = 0 \tag{28}$$

[5]. The best examples of elastica have often been geodesics. Then, we can give the following result.

Conclusion 2. Let $\hat{\gamma} = \gamma + \varepsilon\gamma^*$ be a dual curve on $\mathbb{S}^2$ and R_γ corresponding binormal surface. Then the directrix $\gamma \times \gamma^*$ of R_γ is an elastica.

Proof. Suppose that R_γ is the binormal surface corresponding to a dual curve $\hat{\gamma} = \gamma + \varepsilon\gamma^*$ on $\mathbb{S}^2$. Since the directrix of a binormal surface is a geodesic, we have from Conclusion 1 and (28), the directrix $\gamma \times \gamma^*$ of R_γ is an elastica.

A planar elastica satisfies the Euler-Lagrange equations (27), that is a rectifying strip formed by a planar elastica is an elastic strip. In this case, we arrive at the following conclusion, the proof of which is obvious.

Conclusion 3. A dual curve on $\mathbb{S}^2$ corresponds to elastic strip formed by planar elastica in $\mathbb{R}^3$. So a dual spherical elastica correspond to elastic strips constituted by planar elastica in $\mathbb{R}^3$, too.

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