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On weakly S-primary submodules

Ece Yetkin Celikel[a], Hani A. Khashan[b]

[a] *Department of Basic Sciences, Faculty of Engineering, Hasan Kalyoncu University, Gaziantep, Turkey.*
[b] *Department of Mathematics, Faculty of Science, Al al-Bayt University, Al Mafraq, Jordan.*

Abstract. Let R be a commutative ring with a non-zero identity, S be a multiplicatively closed subset of R and M be a unital R-module. In this paper, we define a submodule N of M with $(N :_R M) \cap S = \emptyset$ to be weakly S-primary if there exists $s \in S$ such that whenever $a \in R$ and $m \in M$ with $0 \neq am \in N$, then either $sa \in \sqrt{(N :_R M)}$ or $sm \in N$. We present various properties and characterizations of this concept (especially in faithful multiplication modules). Moreover, the behavior of this structure under module homomorphisms, localizations, quotient modules, cartesian product and idealizations is investigated. Finally, we determine some conditions under which two kinds of submodules of the amalgamation module along an ideal are weakly S-primary.

1. Introduction

Throughout this article, all rings are commutative with identity and all modules are unital. Let R be a ring and let M be an R-module. A non-empty subset S of a ring R is said to be a multiplicatively closed set if S is a subsemigroup of R under multiplication. For a submodule N of M, we will denote by $(N :_R M)$ the residual of N by M, that is, the set of all $r \in R$ such that $rM \subseteq N$. M is called a multiplication module if every submodule N of M has the form IM for some ideal I of R. Let N and K be submodules of a multiplication R-module M with $N = IM$ and $K = JM$ for some ideals I and J of R. The product of N and K denoted by NK is defined by $NK = IJM$. In particular, for $m_1, m_2 \in M$, by m_1m_2, we mean the product of Rm_1 and Rm_2, [4]. We call M faithful if it has a zero annihilator in R, that is $(0 :_R M) = 0$.

A proper submodule N of M is said to be prime (resp. primary) if whenever $r \in R$ and $m \in M$ such that $rm \in N$, then $r \in (N :_R M)$ (resp. $r \in \sqrt{(N :_R M)}$) or $m \in N$. For any submodule N of an R-module M the radical, $M - rad(N)$, of N is defined to be the intersection of all prime submodules of M containing N, [10]. It is shown in [20, Lemma 2.4] that if N is a proper submodule of a multiplication R-module M, then $M - rad(N) = \sqrt{(N :_R M)}M = \{m \in M \mid m^k \subseteq N \text{ for some } k \geq 0\}$. Moreover, we have $(M\text{-}rad(N) :_R M) = \sqrt{(N :_R M)}$ for any finitely generated multiplication module M. The concepts of prime and primary submodules have been generalized in several ways (see, for example, [3], [6], [16], [17], [20], [22] and [25]).

In [22], the authors introduced the concept of S-prime submodules and investigate many properties of this class of submodules. More generally, the concept of weakly S-prime submodules has been recently

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Email addresses: ece.celikel@hku.edu.tr, yetkinece@gmail.com (Ece Yetkin Celikel), hakhashan@aabu.edu.jo (Hani A. Khashan)

studied in [17]. Let S be a multiplicatively closed subset of a ring R and N be a submodule of an R-module M such that $(N :_R M) \cap S = \emptyset$. Then N is called an S-prime (resp. weakly S-prime) submodule if there exists $s \in S$ such that for $a \in R$ and $m \in M$, if $am \in N$ (resp. $0 \neq am \in N$), then $sa \in (N :_R M)$ or $sm \in N$. In 2021, Farshadifar, [14] defined a submodule N of M with $(N :_R M) \cap S = \emptyset$ to be an S-primary submodule if there exists a fixed $s \in S$ and whenever $am \in N$, then either $sa \in \sqrt{(N :_R M)}$ or $sm \in N$ for each $a \in R$ and $m \in M$. More recently, Ansari-Toroghy and Pourmortazavi, [7] studied many more properties of this class of submodules.

Motivated and inspired by the above works, the purpose of this article is to extend S-primary submodules to the context of weakly S-primary submodules. A submodule N of M satisfying $(N :_R M) \cap S = \emptyset$ is called a weakly S-primary submodule if there exists $s \in S$ such that for $a \in R$ and $m \in M$, whenever $0 \neq am \in N$, then either $sa \in \sqrt{(N :_R M)}$ or $sm \in N$.

In section 2, many examples and characterizations of weakly S-primary submodules are given (see for example, Example 2.3, Theorems 2.4, 2.5). Moreover, several properties of weakly S-primary submodules are obtained (see for example, Theorem 2.7, Propositions 2.11, 2.13). We also investigate the behavior of this structure under module homomorphisms, localizations, quotient modules and Cartesian product of modules (see Propositions 2.14, 2.16, Theorem 2.19 and Corollary 2.17).

Let R be a ring and M be an R-module. The idealization ring $R \ltimes M$ of M in R is defined as the set $\{(r, m) : r \in R, m \in M\}$ with the usual componentwise addition and multiplication defined as $(r, m)(s, n) = (rs, rn + sm)$. It can be easily verified that $R \ltimes M$ is a commutative ring with identity $(1_R, 0_M)$. If I is an ideal of R and N is a submodule of M, then $I \ltimes N = \{(r, m) : r \in I, m \in N\}$ is an ideal of $R \ltimes M$ if and only if $IM \subseteq N$. In this case, $I \ltimes N$ is called a homogeneous ideal of $R \ltimes M$, see [5]. At the end of this section, for an R-module M, we clarify the relation between weakly S-primary submodules of M and weakly $S(M)$-primary ideals of the idealization ring $R(M)$ (Theorem 2.21). Let $f : R_1 \to R_2$ be a ring homomorphism, J be an ideal of R_2, M_1 be an R_1-module, M_2 be an R_2-module (which is an R_1-module induced naturally by f) and $\varphi : M_1 \to M_2$ be an R_1-module homomorphism. We conclude section 3 by investigating some kinds of weakly S-primary submodules in the amalgamation $(R_1 \bowtie^f J)$-module $M_1 \bowtie^\varphi JM_2$ of M_1 and M_2 along J with respect to φ (see Theorems 3.2, 3.4). Furthermore, we conclude some particular results for the duplication of a module along an ideal (see Corollaries 3.3 - 3.8).

As usual, $\mathbb{Z}$, $\mathbb{Z}_n$ and $\mathbb{Q}$ denotes the ring of integers, the ring of integers modulo n and the field of rational numbers, respectively. For more details and terminology, one may refer to [1], [2], [9], [15], [18].

2. Weakly S-primary Submodules

Our aim in this section is to study the weakly S-primary submodules in modules over commutative rings. We begin with our main definition.

Definition 2.1. *Let S be a multiplicatively closed subset of a ring R and N be a submodule of an R-module M with $(N :_R M) \cap S = \emptyset$. We call N a weakly S-primary submodule if there exists (a fixed) $s \in S$ such that for $a \in R$ and $m \in M$, whenever $0 \neq am \in N$ then either $sa \in \sqrt{(N :_R M)}$ or $sm \in N$. The fixed element $s \in S$ is said to be a weakly S-element of N.*

We observe several elementary relationships concerning weakly S-primary submodules in any R-module as follows:

Remark 2.2. *Let S be a multiplicatively closed subset of a ring R.*

1. An ideal I of R is a weakly S-primary ideal if and only if I is a weakly S-primary submodule of the R-module R.
2. Any S-primary submodule is a weakly S-primary submodule.
3. Any weakly primary submodule N of an R-module M satisfying $(N :_R M) \cap S = \emptyset$ is a weakly S-primary submodule of M. Moreover, the two concepts coincide if $S \subseteq U(R)$ where $U(R)$ denotes the set of units in R.

4. $\langle 0\rangle$ is a weakly S-primary submodule of any R-module M if and only if $(0:_R M)\cap S=\emptyset$

Example 2.3.

1. Unlike the case of weakly primary submodules, the zero submodule need not be weakly S-primary. For example, if we consider the multiplicatively closed subset $S=\{4^m : m\in\mathbb{N}\}$ of $\mathbb{Z}$, then $\bar{0}$ is not weakly S-primary in the $\mathbb{Z}$-module $\mathbb{Z}_4$ as clearly $(0:_{\mathbb{Z}}\mathbb{Z}_4)\cap S\neq\emptyset$.
2. For any multiplicatively closed subset S of a ring R and an R-module M, if $(0:_R M)\cap S=\emptyset$, then $\langle 0\rangle$ is a weakly $\{1\}$-primary submodule of R but not necessarily $\{1\}$-primary. Thus, the converse of (2) in Remark 2.2 is not true in general. For a non-trivial example consider the $\mathbb{Z}$-module $M=\mathbb{Z}\ltimes(\mathbb{Z}_2\times\mathbb{Z}_2)$ and the submodule $N=0\ltimes\langle(\bar{1},\bar{0})\rangle$ of M. For $S=\{3^m : m\in\mathbb{N}\}$, we observe that N is not an S-primary submodule since for example, $2\cdot(0,(\bar{1},\bar{1}))=(0,(\bar{0},\bar{0}))\in N$ but for every $s\in S$, $2s\notin\sqrt{(N:_R M)}=\{0\}$ and $s\cdot(0,(\bar{1},\bar{1}))\notin N$. On the other hand, N is weakly S-primary in M. Indeed, choose $s=1$ and let $r_1\in R$, $(r_2,(\bar{a},\bar{b}))\in M$ such that $(0,(\bar{0},\bar{0}))\neq r_1.(r_2,(\bar{a},\bar{b}))\in N$ and $sr_1\notin\sqrt{(N:_R M)}$. Then $r_1\neq 0$ and $(r_1r_2,r_1.(\bar{a},\bar{b}))\in N$. It follows that $r_2=0$ and $(\bar{0},\bar{0})\neq r_1.(\bar{a},\bar{b})\in\langle(\bar{1},\bar{0})\rangle$. If $s(\bar{a},\bar{b})=(\bar{1},\bar{1})$ or $(\bar{0},\bar{1})$, then $r_1.(\bar{a},\bar{b})\in\langle(\bar{1},\bar{0})\rangle$ if and only if $r_1\in\langle 2\rangle$ and so $r_1.(\bar{a},\bar{b})=(\bar{0},\bar{0})$, a contradiction. Thus, $s(\bar{a},\bar{b})\in\langle(\bar{1},\bar{0})\rangle$ and N is a weakly S-primary submodule of M.
3. While clearly every weakly S-prime submodule is weakly S-primary, the converse need not be true. For example the ideal $4\mathbb{Z}$ is a weakly $\{1\}$-primary submodule of $\mathbb{Z}$ but not weakly $\{1\}$-prime. For a non-trivial multiplicatively closed subset S, consider the $\mathbb{Z}$-module $M=\mathbb{Z}\times\mathbb{Z}$ and let $S=\{3^m : m\in\mathbb{N}\}$. Then $N=4\mathbb{Z}\times\mathbb{Z}$ is a (weakly) S-primary submodule of M. Indeed choose $s=1$ and let $r\in\mathbb{Z}$, $(a,b)\in\mathbb{Z}\times\mathbb{Z}$ such that $(0,0)\neq r.(a,b)\in N$ and $sr\notin\sqrt{(N:_R M)}=\langle 2\rangle$. Then clearly $a\in 4\mathbb{Z}$ and so $s(a,b)\in N$ as needed. On the other hand, N is not weakly S-prime since $2.(2,1)\in N$ but $2s\notin(N:_R M)=\langle 4\rangle$ and $s(2,1)\notin N$ for every $s\in S$.
4. Let p, q be distinct prime integers and $k<n$, $t<m$ be positive integers. Consider $S=\{q^n : n\in\mathbb{N}\}$ and the submodule $N=(\bar{p}^k\bar{q}^t)$ of the $\mathbb{Z}$-module $\mathbb{Z}_{p^nq^m}$. Then N is a weakly S-primary submodule of M associated with $s=q^t\in S$ that is not weakly primary.

Let N be a submodule of an R-module M and A be a subset of R. The residual of N by A is the set $(N:_M A)=\{m\in M : Am\subseteq N\}$ which is a submodule of M containing N. In the following theorem, we present some equivalent statements characterizing weakly S-primary submodules.

Theorem 2.4. *Let S be a multiplicatively closed subset of a ring R and N be a submodule of an R-module M with $(N:_R M)\cap S=\emptyset$. Then the following are equivalent.*

1. N is a weakly S-primary submodule of M.
2. There exists $s\in S$ such that for all $a\notin(\sqrt{(N:_R M)}:s)$, $(N:_M a)\subseteq(0:_M a)\cup(N:_M s)$.
3. There exists $s\in S$ such that for all $a\notin(\sqrt{(N:_R M)}:s)$, $(N:_M a)\subseteq(0:_M a)$ or $(N:_M a)\subseteq(N:_M s)$.
4. There exists $s\in S$ such that for any $a\in R$ and for any submodule K of M, if $0\neq aK\subseteq N$, then $sa\in\sqrt{(N:_R M)}$ or $sK\subseteq N$.
5. There exists $s\in S$ such that for any ideal I of R and a submodule K of M, if $0\neq IK\subseteq N$, then $sI\subseteq\sqrt{(N:_R M)}$ or $sK\subseteq N$.

Proof. (1)⇒(2). Suppose that $s\in S$ is a weakly S-element of N and let $a\notin(\sqrt{(N:_R M)}:s)$. Let $m\in(N:_M a)$. If $am=0$, then $m\in(0:_M a)$. If $0\neq am\in N$, then $sa\notin\sqrt{(N:_R M)}$ implies $sm\in N$. Thus, $m\in(N:_M s)$ and so $(N:_M a)\subseteq(0:_M a)\cup(N:_M s)$.

(2)⇒(3). Suppose $m_1\in(N:_M a)\backslash(0:_M a)$ and $m_2\in(N:_M a)\backslash(N:_M s)$. Then $m=m_1+m_2\in(N:_M a)\subseteq(0:_M a)\cup(N:_M s)$. If $m\in(0:_M a)$ or $m\in(N:_M s)$, we get a contradiction. Thus, $(N:_M a)\subseteq(0:_M a)$ or $(N:_M a)\subseteq(N:_M s)$ as required.

(3)⇒(4). Let $s\in S$ be an element satisfying the statement (3). Let $a\in R$ such that $a\notin(N:_R M):s)$ and K be a submodule of M with $\{0\}\neq aK\subseteq N$. Then, $K\subseteq(N:_M a)$. By hypothesis, $(N:_M a)\subseteq(0:_M a)$ or

$(N :_M a) \subseteq (N :_M s)$. Since $K \subseteq (N :_M a)$ and $K \nsubseteq (0 :_M a)$, we get that $(N :_M a) \subseteq (N :_M s)$, and so $K \subseteq (N :_M s)$, as desired.

(4)⇒(5). Choose $s \in S$ as in (4). Let I be an ideal of R and K a submodule of M with $0 \neq IK \subseteq N$. Assume that $sI \nsubseteq \sqrt{(N :_R M)}$. Then there exists $a \in I$ with $sa \notin \sqrt{(N :_R M)}$. If $aK \neq 0$, then we have $sK \subseteq N$ by our assumption (4). Now, suppose that $aK = 0$. Since $IK \neq 0$, there is some $b \in I$ with $bK \neq 0$. If $sb \notin \sqrt{(N :_R M)}$, then we have $sK \subseteq N$ by (4). If $sb \in \sqrt{(N :_R M)}$, then as $sa \notin \sqrt{(N :_R M)}$, we conclude $s(a + b) \notin \sqrt{(N :_R M)}$. Thus, $0 \neq (a + b)K \subseteq N$ implies $sK \subseteq N$ again by (4), as required.

(5)⇒(1). Take $I = aR$ and $K = Rm$ in (5). □

Next, we give a characterization for weakly S-primary submodule of faithful multiplication modules.

Theorem 2.5. *Let S be a multiplicatively closed subset of a ring R and M be a faithful multiplication R-module. Then N is a weakly S-primary submodule of M if and only if $(N :_R M) \cap S = \emptyset$ and there exists $s \in S$ such that whenever K, L are submodules of M and $0 \neq KL \subseteq N$, then $sK \subseteq M - rad(N)$ or $sL \subseteq N$.*

Proof. (⇒) Let s be a weakly S-element of N and suppose that $0 \neq KL \subseteq N$ for some submodules K, L of M. Since M is multiplication, we may write $K = IM$ for some ideal I of R. Hence, $0 \neq IL \subseteq N$ and so by Theorem 2.4, $sI \subseteq \sqrt{(N :_R M)}$ or $sL \subseteq N$. It follows that $sK = sIM \subseteq \sqrt{(N :_R M)}M = M - rad(N)$ or $sL \subseteq N$.

(⇐) Let $s \in S$ such that whenever K, L are submodules of M and $0 \neq KL \subseteq N$, then $sK \subseteq M - rad(N)$ or $sL \subseteq N$. Suppose that $0 \neq IL \subseteq N$ for some ideal I of R and submodule L of M and $sL \nsubseteq N$. Then as M is faithful, we have $0 \neq (IM)L \subseteq N$. By assumption, we have $sIM \subseteq M - rad(N)$ and so $sI \subseteq (M - rad(N) :_R M) = \sqrt{(N :_R M)}$. Thus, s is a weakly S-element of N and by Theorem 2.4, we are done . □

Lemma 2.6. *[23] Let N be a submodule of a faithful multiplication R-module M. For an ideal I of R, $(IN : M) = I(N : M)$, and in particular, $(IM : M) = I$.*

Let I be a proper ideal of a ring R. In the following proposition, the notation $Z_I(R)$ denotes the set $\{r \in R : rs \in I \text{ for some } s \in R\backslash I\}$.

Theorem 2.7. *Let S be a multiplicatively closed subset of a ring R and N be a submodule of an R-module M. The following statements hold.*

1. If N is a weakly S-primary submodule of M, then for every submodule K with $(N :_R K) \cap S = \emptyset$ and $Ann(K) = 0$, $(N :_R K)$ is a weakly S-primary ideal of R. In particular, if M is faithful, then $(N :_R M)$ is a weakly S-primary ideal of R.
2. If M is multiplication and $(N :_R M)$ is a weakly S-primary ideal of R, then N is a weakly S-primary submodule of M.
3. Let M be a faithful multiplication module and I be an ideal of R. Then I is a weakly S-primary ideal of R if and only if IM is a weakly S-primary submodule of M.
4. If N is a weakly S-primary submodule of M and T is a subset of R such that $(0 :_M T) = 0$ and $Z_{(N:_RM)}(R) \cap T = \emptyset$, then $(N :_M T)$ is a weakly S-primary submodule of M.

Proof. (1) Let s be a weakly S-element of N and suppose that $0 \neq ab \in (N :_R K)$ for some $a, b \in R$. Then, $0 \neq abK \subseteq N$ as $Ann(K) = 0$ and Theorem 2.4 implies $sa \in \sqrt{(N :_R M)} \subseteq \sqrt{(N :_R K)}$ or $sbK \subseteq N$. Thus, $sa \in \sqrt{(N :_R K)}$ or $sb \in (N :_R K)$, as needed.

(2) Since $(N :_R M)$ is a weakly S-primary ideal of R, $(N :_R M) \cap S = \emptyset$.. Let $s \in S$ be an associate element to $(N :_R M)$. Let $a \in R$ and $m \in M$ such that $0 \neq am \in N$ and $sa \notin \sqrt{(N :_R M)}$. Set $Rm = IM$ for some ideal I of M. We have $aIM = Ram \subseteq N$. Then, $aI \subseteq (N :_R M)$. Let $x_0 \in I$ such that $ax_0 \neq 0$. Such element exists, otherwise $am = 0$. Let $x \in I$. If $0 \neq ax$, then $sx \in (N :_R M)$. In particular, $sx_0 \in (N :_R M)$. Now, if $ax = 0$,then $a(x + x_0) \neq 0$, and so $s(x + x_0) \in (N :_R M)$. Thus, $sx \in (N :_R M)$. We conclude that $sI \subseteq (N :_R M)$. Hence, $sm \in sIM \subseteq N$.

(3) Let I be a weakly S-primary ideal of R. Since by Lemma 2.6, $(IM :_R M) = I$, we conclude that IM is a weakly S-primary submodule of M by (2). Conversely, suppose that IM is a weakly S-primary submodule. Then, $I = (IM :_R M)$ is a weakly S-primary ideal of R by (1) and again by Lemma 2.6.

(4) First, we show that $((N :_M T) :_R M) \cap S = \emptyset$. Assume $s \in ((N :_M T) :_R M) \cap S$, then $sT \subseteq (N :_R M)$ and since $Z_{(N:_RM)}(R) \cap T = \emptyset$, we conclude that $s \in (N :_R M)$ which contradicts $(N :_R M) \cap S = \emptyset$. Let s be a weakly S-element of N and suppose that $0 \neq am \in (N :_M T)$ for some $a \in R$ and $m \in M$. Since $(0 :_M T) = 0$, we have $0 \neq a(Tm) \subseteq N$ which yields $sa \in \sqrt{(N :_R M)} \subseteq \sqrt{((N :_M T) :_R M)}$ or $sTm \subseteq N$. Thus, $sa \in \sqrt{((N :_M T) :_R M)}$ or $sm \in (N :_M T)$ and $(N :_M T)$ is weakly S-primary in M. □

We show by the following example that the condition "faithful module" in (1) of Theorem 2.7 is crucial.

Example 2.8. *Consider the multiplicatively closed subset $S = \{3^m : m \in \mathbb{N}\}$ of $\mathbb{Z}$ and the $\mathbb{Z}$-module $M = \mathbb{Z}_{10} \times \mathbb{Z}_{10}$. Then $N = \overline{0} \times \overline{0}$ is a weakly S-primary submodule of M but $(N :_{\mathbb{Z}} M) = \langle 10 \rangle$ is not a weakly S-primary ideal of $\mathbb{Z}$.*

In view of Theorem 2.7, we have the following equivalent statements.

Corollary 2.9. *Let M be a faithful multiplication R-module and N be a submodule of M. The following are equivalent.*

1. N is a weakly S-primary submodule of M.
2. $(N :_R M)$ is a weakly S-primary ideal of R.
3. $N = IM$ for some weakly S-primary ideal I of R.

Let N be a proper submodule of an R-module M. Then N is said to be a maximal weakly S-primary submodule if there is no weakly S-primary submodule which contains N properly. In the following corollary, by $Z(M)$, we denote the set $\{r \in R : rm = 0 \text{ for some } m \in M\backslash\{0_M\}\}$.

Corollary 2.10. *Let N be a submodule of an R-module M such that $Z_{(N:_RM)}(R) \cup Z(M) \subseteq \sqrt{(N :_R M)}$. If N is a maximal weakly S-primary submodule of M, then N is an S-primary submodule of M.*

Proof. Let $s \in S$ be a weakly S-element of N and let $a \in R$, $m \in M$ such that $am \in N$ and $sa \notin \sqrt{(N :_R M)}$. Since $a \notin \sqrt{(N :_R M)}$, then by assumption, $a \notin Z_{(N:_RM)}(R)$ and $(0 :_M a) = 0$. Therefore, by Theorem 2.7(4), $(N :_M a)$ is a weakly S-primary submodule of M. Since N is a maximal weakly S-primary submodule, we conclude $sm \in (N :_M a) = N$ as needed. □

Proposition 2.11. *Let S be a multiplicatively closed subset of a ring R and N be a submodule of an R-module M such that $(N :_R M) \cap S = \emptyset$. Then*

1. If $(N :_M s)$ is a weakly primary submodule of M for some $s \in S$, then N is a weakly S-primary submodule of M.
2. If N is a non-zero weakly S-primary submodule of M and $S \cap Z(M) = \emptyset$, then $(N :_M s)$ is a weakly primary submodule of M for some $s \in S$.

Proof. (1) Choose $s \in S$ such that $(N :_M s)$ is a weakly primary submodule of M. Suppose that $0 \neq am \in N \subseteq (N :_M s)$ for some $a \in R$, $m \in M$. Since $(N :_M s)$ is weakly primary, we have either $a \in \sqrt{((N :_M s) :_R M)} = \sqrt{((N :_R M) :_R s)}$ or $m \in (N :_M s)$. Thus, $sa \in \sqrt{(N :_R M)}$ or $sm \in N$ and so N is a weakly S-primary submodule of M.

(2) Let s be a weakly S-element of N and $a \in R$, $m \in M$ with $0 \neq am \in (N :_M s)$. Since $S \cap Z(M) = \emptyset$, clearly we have $0 \neq sam \in N$ which implies either $s^2a \in \sqrt{(N :_R M)}$ or $sm \in N$. Thus, $sa \in \sqrt{(N :_R M)} \subseteq \sqrt{((N :_M s) :_R M)}$ or $m \in (N :_M s)$, as needed. □

If $S \cap Z(M) \neq \emptyset$, then the assertion (2) of Proposition 2.11 need not be true as we can see in the following example.

Example 2.12. *Consider $M = \mathbb{Z} \times \mathbb{Z}_{pq}$ as a $\mathbb{Z}$-module and let $S = \{p^n : n \in \mathbb{N}\}$. Here, observe that $S \cap Z(M) = S \neq \emptyset$. Now, $N = \langle 0 \rangle \times \langle \bar{0} \rangle$ is a weakly S-primary submodule of M. On the other hand, for each positive integer n, $(N :_M p^n) = \langle 0 \rangle \times \langle \bar{q} \rangle$ is not a weakly primary submodule of M. Indeed, $q.(0, \bar{1}) \in (N :_M p^n)$ but neither $q \in \sqrt{((N :_M p^n) :_R M)} = \langle 0 \rangle$ nor $(0, \bar{1}) \in (N :_M p^n)$.*

We recall from [24, Corollary 2.6], that a faithful multiplication module is finitely generated.

Proposition 2.13. *Let S be a multiplicatively closed subset of a ring R and N be a weakly S-primary submodule of a faithful multiplication R-module M.*

1. If $\{0_R\}$ is an S-primary ideal of R, then M-$rad(N)$ is an S-prime submodule of M.
2. If N is not S-primary, then $N^2 = 0_M$ and $M - rad(N) = M - rad(0_M)$. Moreover, nonzero weakly S-primary submodules and S-primary submodules coincide if R is a reduced ring.

Proof. (1) Suppose that N is a weakly S-primary submodule of M. Then $(N :_R M)$ is a weakly S-primary ideal of R by Corollary 2.9. Therefore, $\sqrt{(N :_R M)}$ is an S-prime ideal of R, [25]. Thus, M-$rad(N) = \sqrt{(N :_R M)}M$ is an S-prime submodule of M by [22, Proposition 2.9 (ii)].

(2) Suppose N is not S-primary. Then $(N :_R M)$ is a weakly S-primary ideal of R that is not S-primary by Corollary 2.9. It follows by [25, Proposition 2(1)] that $(N :_R M)^2 = 0_R$. Since M is multiplication, we have $N^2 = (N :_R M)^2 M = 0_M$ and $M - rad(N) = \sqrt{(N :_R M)}M = \sqrt{0_R}M \subseteq \sqrt{0_M : M}M = M - rad(0_M)$. The rest of the proof is straightforward. □

Proposition 2.14. *Let N be a submodule of an R-module M and S be a multiplicatively closed subset of R.*

1. *If N is a weakly S-primary submodule of M, then $S^{-1}N$ is a weakly primary submodule of $S^{-1}M$. Moreover, if $Z(M) \cap S = \emptyset$, then there exists an $s \in S$ such that $(N :_M t) \subseteq (N :_M s)$ for all $t \in S$.*
2. *If M is finitely generated and $Z(M) \cap S = \emptyset$, then the converse of (1) holds.*

Proof. (1) Let $s \in S$ be a weakly S-element of N and $\frac{a}{s_1} \in S^{-1}R$, $\frac{m}{s_2} \in S^{-1}M$ such that $0_{S^{-1}M} \neq \frac{a}{s_1}\frac{m}{s_2} \in S^{-1}N$. Then $uam \in N$ for some $u \in S$. If $uam = 0$, then $\frac{am}{s_1 s_2} = \frac{uam}{us_1 s_2} = 0_{S^{-1}M}$, a contradiction. Thus, $0 \neq uam \in N$ which implies that either $sua \in \sqrt{(N :_R M)}$ or $sm \in N$. Thus, $\frac{a}{s_1} = \frac{sua}{sus_1} \in S^{-1}\sqrt{(N :_R M)} \subseteq \sqrt{(S^{-1}N :_{S^{-1}R} S^{-1}M)}$ or $\frac{m}{s_2} = \frac{sm}{ss_2} \in S^{-1}N$, as needed. For the rest of the proof, let $t \in S$ and $0 \neq m \in (N :_M t)$. Then $0 \neq tm \in N$ as $Z(M) \cap S = \emptyset$ and so $sm \in N$ as $st \in (N :_M M) \cap S$ gives a contradiction. Therefore, $m \in (N :_M s)$ and so $(N :_M t) \subseteq (N :_M s)$ for all $t \in S$.

(2) Suppose that M is finitely generated, $S^{-1}N$ is a weakly primary submodule of $S^{-1}M$ and there is a fixed $s \in S$ such that $(N :_M t) \subseteq (N :_M s)$ for all $t \in S$. Since $S^{-1}N$ is proper, we have $(N :_R M) \cap S = \emptyset$. Let $0 \neq am \in N$ for some $a \in R$ and $m \in M$. Then $0 \neq \frac{a}{1}\frac{m}{1} \in S^{-1}N$ as $Z(M) \cap S = \emptyset$. Since $S^{-1}N$ is weakly primary, we have either $\frac{a}{1} \in \sqrt{(S^{-1}N :_{S^{-1}R} S^{-1}M)} = S^{-1}\sqrt{(N :_R M)}$ as M is finitely generated or $\frac{m}{1} \in S^{-1}N$. Hence, $s_1 a \in \sqrt{(N :_R M)}$ for some $s_1 \in S$ or $s_2 m \in N$ for some $s_2 \in S$. If $s_1 a \in \sqrt{(N :_R M)}$, then $s_1^n a^n \in (N :_R M)$ for some positive integer n and $a^n \in ((N :_R M) : s_1^n) = ((N :_M s_1^n) :_R M) \subseteq ((N :_M s) :_R M)$ by our assumption. Thus $sa^n \in (N :_R M)$ and $sa \in \sqrt{(N :_R M)}$. If $s_2 m \in N$, then we conclude $m \in (N :_M s_2) \subseteq (N :_M s)$ and so $sm \in N$. Consequently, N is a weakly S-primary submodule of M. □

Let M be an R-module and $S \subseteq S'$ be two multiplicatively closed subsets of R. If N is a weakly S-primary submodule of M and $(N :_R M) \cap S' = \emptyset$, then it is clear that N is a weakly S'-primary submodule of M. In [15], the saturation of S is defined as the multiplicatively closed subset $S^* = \{x \in R : xy \in S \text{ for some } y \in R\}$ which contains S.

Proposition 2.15. *Let S be a multiplicatively closed subset of a ring R and N be a submodule of an R-module M. Then N is weakly S-primary if and only if N is weakly S^*-primary.*

Proof. Suppose N is weakly S^*-primary in M associated to $s^* \in S^*$. Note that $(N :_R M) \cap S = \emptyset$ as $S \subseteq S^*$. Choose $s = s^*y \in S$ for some $y \in R$ and let $0 \neq am \in N$ for some $a \in R$ and $m \in M$. Then either $s^*a \in \sqrt{(N :_R M)}$ or $s^*m \in N$. Hence, $sa \in \sqrt{(N :_R M)}$ or $sm \in N$ and N is a weakly S-primary submodule of M. Conversely, suppose that N is weakly S-primary. We need to prove that $(N :_R M) \cap S^* = \emptyset$. If $s^* \in (N :_R M) \cap S^*$, then there is $y \in R$ such that $s = s^*y \in (N :_R M) \cap S$ which is a contradiction. Thus, N is weakly S^*-primary as $S \subseteq S^*$. □

Proposition 2.16. *Let M and M' be two R-modules and $f : M \to M'$ be a homomorphism. For a multiplicatively closed subset S of R, we have:*

1. If f is an epimorphism and N is a weakly S-primary submodule of M containing $Ker(f)$, then $f(N)$ is a weakly S-primary submodule of M'.
2. If f is a monomorphism and N' is a weakly S-primary submodule of M', then $f^{-1}(N')$ is a weakly S-primary submodule of M.

Proof. (1) Let $s \in S$ be a weakly S-element of N. First, as $Ker(f) \subseteq N$, it follows that $(f(N) :_R M') \cap S = \emptyset$. Suppose that $0 \neq am' \in f(N)$ for some $a \in R$ and $m' \in M'$. Choose $m \in M$ with $m' = f(m)$. Then $0 \neq af(m) = f(am) \in f(N)$ and since $Ker(f) \subseteq N$, we have $0 \neq am \in N$. It follows that either $sa \in \sqrt{(N :_R M)}$ or $sm \in N$. Thus, clearly we have either $sa \in \sqrt{(f(N) :_R M')}$ or $sm' = f(sm) \in f(N)$ and $f(N)$ is a weakly S-primary submodule of M'.

(2) Let $s \in S$ be a weakly S-element of N' and note that clearly $(f^{-1}(N') :_R M) \cap S = \emptyset$. Let $a \in R$ and $m \in M$ such that $0 \neq am \in f^{-1}(N')$. Then $0 \neq f(am) = af(m) \in N'$ as f is a monomorphism. It follows either $sa \in \sqrt{(N' :_R M')}$ or $sf(m) \in N'$. Thus, we conclude either $sa \in \sqrt{(f^{-1}(N') :_R M)}$ or $sm \in f^{-1}(N')$ and we are done. □

Corollary 2.17. *Let S be a multiplicatively closed subset of a ring R and $K \subseteq N$ be two submodules of an R-module M.*

1. If N is a weakly S-primary submodule of M, then N/K is a weakly S-primary submodule of M/K.
2. If K' is a weakly S-primary submodule of M, then $K' \cap N$ is a weakly S-primary submodule of N.
3. If N/K is a weakly S-primary submodule of M/K and K is a weakly S-primary submodule of M, then N is a weakly S-primary submodule of M.

Proof. Observe that $(N/K :_R M/K) \cap S = \emptyset$ if and only if $(N :_R M) \cap S = \emptyset$.

(1). The claim follows by Proposition 2.16(1) considering the canonical epimorphism $\pi : M \to M/K$ defined by $\pi(m) = m + K$.

(2). This follows by Proposition 2.16(2) considering the natural injection $i : N \to M$ defined by $i(m) = m$ for all $m \in N$.

(3). Let $s \in S$ be a weakly S-element of N/K and $s\prime \in S$ be a weakly S-element of K. Let $a \in R$ and $m \in M$ such that $am \in N$. If $am \in K$, then either $s\prime a \in \sqrt{(K :_R M)} \subseteq \sqrt{(N :_R M)}$ or $s'm \in K \subseteq N$. If $am \notin K$, then $K \neq a(m + K) \in N/K$ which implies that either $sa \in \sqrt{(N/K :_R M/K)}$ or $s(m + K) \in N/K$. Thus, $sa \in \sqrt{(N :_R M)}$ or $sm \in N$. It follows that N is an S-primary submodule of M associated with $s = ss' \in S$. □

The converse of Corollary 2.17(1) does not hold in general. For instance, consider the submodules $N = K = \langle p_1p_2 \rangle$ of the $\mathbb{Z}$-module $\mathbb{Z}$ and the multiplicatively closed subset $S = \left\{p_3^n : n \in \mathbb{N} \cup \{0\}\right\}$ of $\mathbb{Z}$ where p_1, p_2 and p_3 are distinct prime numbers. Then clearly $N/K = 0$ is a weakly S-primary submodule of $\mathbb{Z}/K$ but N is not a weakly S-primary submodule of $\mathbb{Z}$ as $0 \neq p_1 \cdot p_2 \in N$ but neither $sp_1 \in \sqrt{(N :_{\mathbb{Z}} \mathbb{Z})} = \langle p_1p_2 \rangle$ nor $sp_2 \in N$ for all $s \in S$.

Proposition 2.18. *Let S be a multiplicatively closed subset of a ring R and N be a weakly S-primary submodule of an R-module M. For any submodule K of M with $(K :_R M)M = K$ and $(K :_R M) \cap S \neq \emptyset$, $N \cap K$ is a weakly S-primary submodule of M.*

Proof. It is clear that $(N \cap K :_R M) \cap S = \emptyset$. Suppose that $0 \neq am \in N \cap K \subseteq N$ for some $a \in R$ and $m \in M$. Then there exists a $s \in S$ with either $sa \in \sqrt{(N :_R M)}$ or $sm \in N$. Take $t \in (K :_R M) \cap S$. Then $sta \in \sqrt{(N :_R M)} \cap (K :_R M) \subseteq \sqrt{(N \cap K :_R M)}$ or $stm \in N \cap (K :_R M)M = N \cap K$. Thus, $N \cap K$ is a weakly S-primary submodule of M associated with $st \in S$. □

We note that the condition $(K :_R M) \cap S \neq \emptyset$ in Proposition 2.18 can not be omitted. Indeed, if N is weakly primary and K is as above, then $N \cap K$ need not be weakly primary. For example, consider the $\mathbb{Z}$-module $\mathbb{Z}_{72}$, $S = \{3^n : n \in \mathbb{N}\}$, $N = \langle \overline{4} \rangle$ and $K = \langle \overline{9} \rangle$. Then $N \cap K = \langle \overline{36} \rangle$ is not a weakly primary submodule of $\mathbb{Z}_{72}$ but observe from Example 2.3(2) that it is a weakly S-primary submodule. Next, we characterize weakly S-primary submodules of cartesian product of modules.

Theorem 2.19. *Let S, S' be multiplicatively closed subsets of rings R, R' respectively and N, N' be non-zero submodules of an R-module M and an R'-module M', respectively. Consider $M \times M'$ as an $(R \times R')$-module. Then the following are equivalent.*

1. $N \times N'$ is a weakly $S \times S'$-primary submodule of $M \times M'$.
2. N is an S-primary submodule of M and $(N' :_{R'} M') \cap S' \neq \emptyset$ or N' is an S'-primary submodule of M' and $(N :_R M) \cap S \neq \emptyset$
3. $N \times N'$ is an $S \times S'$-primary submodule of $M \times M'$.

Proof. (1)⇒(2). Choose a weakly $S \times S'$-element (s, s') of $N \times N'$ and $0 \neq m \in N$.

Case I: Suppose $(N :_R M) \cap S = \emptyset = (N' :_{R'} M') \cap S'$. Then for each $m' \in M'$, $(0, 0) \neq (1, 0)(m, m') \in N \times N'$ and so either $(s, s')(1, 0) \in \sqrt{(N \times N' :_{R \times R'} M \times M')} = \sqrt{(N :_R M)} \times \sqrt{(N' :_{R'} M')}$ or $(s, s')(m, m') \in N \times N'$. Hence, we have either $s^n \in (N :_R M) \cap S$ for some positive integer n or $s' \in N' \cap S' \subseteq (N' :_{R'} M') \cap S'$, a contradiction.

Case II. Assume that $(N :_R M) \cap S \neq \emptyset$, say, $s \in (N :_R M) \cap S$ and note that $(N' :_R M') \cap S' = \emptyset$. Indeed, if $s' \in (N' :_R M') \cap S'$, then $(s, s') \in (N \times N' :_{R \times R'} M \times M') \cap (S \times S')$, a contradiction. Suppose $am' \in N'$ for some $a \in R'$ and $m' \in M'$. Then $(0, 0) \neq (1, a)(m, m') \in N \times N'$ implies either $(s, s')(1, a) \in \sqrt{(N :_R M)} \times \sqrt{(N' :_{R'} M')}$ or $(s, s')(m, m') \in N \times N'$. Thus, $s'a \in \sqrt{(N' :_{R'} M')}$ or $s'm' \in N'$ and N' is an S'-primary submodule of M'.

Case III. Assume that $(N' :_{R'} M') \cap S' \neq \emptyset$. We can prove in a similar way that N is an S-primary submodule of M.

(2)⇒(3). see [7, Theorem 2.20].

(3)⇒(1). is immediate. □

In view of the above theorem, we conclude the following generalization.

Theorem 2.20. *Let $M = M_1 \times M_2 \times \cdots \times M_n$ be an $R = R_1 \times R_2 \times \cdots \times R_n$-module and $S = S_1 \times S_2 \times \cdots \times S_n$ where R_i is a ring, S_i is a multiplicatively closed subset of R_i and N_i is a non-zero submodule of M_i for each $i = 1, 2, ..., n$. Then the following assertions are equivalent.*

1. $N = N_1 \times N_2 \times \cdots \times N_n$ is a weakly S-primary submodule of M.
2. There exists $i \in \{1, 2, ..., n\}$ such that N_i is an S_i-primary submodule of M_i and $(N_j :_{R_j} M_j) \cap S_j \neq \emptyset$ for all $j \neq i$.

Proof. To prove the claim, we use the mathematical induction on n. For $n = 2$, see Theorem 2.19. Assume that the claim holds for all $k < n$. Suppose $N = N_1 \times N_2 \times \cdots \times N_n$ is a weakly S-primary submodule of M. Let $R' = R_1 \times R_2 \times \cdots \times R_{n-1}$, $N' = N_1 \times N_2 \times \cdots \times N_{n-1}$ and $S' = S_1 \times S_2 \times \cdots \times S_{n-1}$. By Theorem 2.19, we have either N_n is weakly S-primary in M_n and $(N' :_{R'} M') \cap S' \neq \emptyset$ or N' is a weakly S'-primary submodule of M' and $S_n \cap (N_n :_{R_n} M_n) \neq \emptyset$. In the first case, we are done as clearly $(N_j :_{R_j} M_j) \cap S_j \neq \emptyset$ for all $j \neq n$. In the second case, we conclude the result by the induction hypothesis. □

Let R be a ring, M be an R-module and consider the idealization ring $R \ltimes M$ of M in R. It is proved in [5, Theorem 3.2] that if $I \ltimes N$ is a homogenous ideal in $R \ltimes M$, then $\sqrt{I \ltimes N} = \sqrt{I} \ltimes M$. For a multiplicatively closed subset S of R, clearly $S \ltimes N = \{(s,n) : s \in S,\ n \in N\}$ is a multiplicatively closed subset of $R \ltimes M$. We conclude this section with the following theorem discussing the weakly $S \ltimes N$-primary ideals of the idealization ring $R \ltimes M$.

Theorem 2.21. *Let S be a multiplicatively closed subset of a ring R, I be an ideal of R and $K \subseteq N$ be submodules of an R-module M with $IM \subseteq N$. If $I \ltimes N$ is a weakly $S \ltimes K$-primary ideal of $R \ltimes M$, then I is a weakly S-primary ideal of R and N is a weakly S-primary submodule of M whenever $(N :_R M) \cap S = \emptyset$. Furthermore, there exists $s \in S$ such that the following hold*

1. For all $a, b \in R$, $ab = 0$, $sa \notin \sqrt{I}$, $sb \notin I$ implies $a, b \in ann(N)$.
2. For all $c \in R$, $m \in M$, $cm = 0$, $sc \notin \sqrt{(N :_R M)}$, $sm \notin N$ implies $c \in ann(I)$ and $m \in (0 :_M I)$.

Proof. It is clear that $(S \ltimes K) \cap (I \ltimes N) = \emptyset$ if and only if $I \cap S = \emptyset$. Let (s,k) be a weakly $S \ltimes K$-primary element of $I \ltimes N$ and let $a, b \in R$ with $0 \neq ab \in I$. Then $(0,0) \neq (a,0)(b,0) \in I \ltimes N$ and so either $(s,k)(a,0) \in \sqrt{I \ltimes N} = \sqrt{I} \ltimes M$ or $(s,k)(b,0) \in I \ltimes N$. Hence, we have either $sa \in \sqrt{I}$ or $sb \in I$ and I is a weakly S-primary ideal of R. To show that N is weakly S-primary, let $0 \neq am \in N$ for $a \in R$, $m \in M$. Then $(0,0) \neq (a,0)(0,m) \in I \ltimes N$ and so $(sa, ak) = (s,k)(a,0) \in \sqrt{I \ltimes N} = \sqrt{I} \ltimes M$ or $(0, sm) = (s,k)(0,m) \in I \ltimes N$. Thus, we conclude either $sa \in \sqrt{I} \subseteq \sqrt{(N :_R M)}$ or $sm \in N$ and so N is a weakly S-primary submodule of M.

(1) Let $a, b \in R$ such that $ab = 0$, $sa \notin \sqrt{I}$ and $sb \notin I$. Suppose $a \notin ann(N)$ so that there exists $n \in N$ such that $an \neq 0$. Then $(0,0) \neq (a,0)(b,n) = (0,an) \in I \ltimes N$ and so either $(s,k)(a,0) \in \sqrt{I \ltimes N}$ or $(s,k)(b,n) \in I \ltimes N$. Hence, $sa \in \sqrt{I}$ or $sb \in I$, a contradiction. Similarly, if $b \notin ann(N)$, then we get a contradiction. Therefore, $a, b \in ann(N)$ as needed.

(2) We assume for $c \in R$, $m \in M$ that $cm = 0$, $sc \notin \sqrt{(N :_R M)}$ and $sm \notin N$. Assume on the contrary that $c \notin ann(I)$. Then there exists $a \in I$ such that $ca \neq 0$. Hence, $(0,0) \neq (c,0)(a,m) = (ca,0) \in I \ltimes N$ and so $(s,k)(c,0) \in \sqrt{I \ltimes N}$ or $(s,k)(a,m) \in I \ltimes N$. Therefore, $sc \in \sqrt{I} \subseteq \sqrt{(N :_R M)}$ or $sm + ka \in N$ which gives $sm \in N$ as $K \subseteq N$, a contradiction. Thus, $c \in ann(I)$. Secondly, assume that $m \notin (0 :_M I)$. Then there exists $a \in I$ such that $am \neq 0$ and this yields $(0,0) \neq (a,m)(c,m) = (ac, am) \in I \ltimes N$. Thus, we conclude either $(s,k)(a,m) \in \sqrt{I \ltimes N}$ or $(s,k)(c,m) \in I \ltimes N$ which implies either $sc \in \sqrt{I} \subseteq \sqrt{(N :_R M)}$ or $sm \in N$, so we get a required contradiction. □

3. (Weakly) S-primary Submodules of Amalgamation Modules

Let R be a ring, J an ideal of R and M an R-module. As a subring of $R \times R$, in [12], the amalgamated duplication of R along J is defined as

$$R \bowtie J = \{(r, r+j) : r \in R\ ,\ j \in J\}$$

Recently, in [11], the duplication of the R-module M along the ideal J denoted by $M \bowtie J$ is defined as

$$M \bowtie J = \{(m, m') \in M \times M : m - m' \in JM\}$$

which is an $(R \bowtie J)$-module with scaler multiplication defined by $(r, r+j).(m, m') = (rm, (r+j)m')$ for $r \in R$, $j \in J$ and $(m, m') \in M \bowtie J$. For various properties and results concerning this kind of modules, one may see [11].

Let J be an ideal of a ring R and N be a submodule of an R-module M. Then

$$N \bowtie J = \{(n, m) \in N \times M : n - m \in JM\}$$

and

$$\bar{N} = \{(m, n) \in M \times N : m - n \in JM\}$$

are clearly submodules of $M \bowtie J$. If S is a multiplicatively closed subset of R, then the sets $S \bowtie J = \{(s, s+j) : s \in S, j \in J\}$ and $\bar{S} = \{(r, r+j) : r+j \in S\}$ are obviously multiplicatively closed subsets of $R \bowtie J$.

In general, let $f : R_1 \to R_2$ be a ring homomorphism, J be an ideal of R_2, M_1 be an R_1-module, M_2 be an R_2-module (which is an R_1-module induced naturally by f) and $\varphi : M_1 \to M_2$ be an R_1-module homomorphism. The subring

$$R_1 \bowtie^f J = \{(r, f(r) + j) : r \in R_1, j \in J\}$$

of $R_1 \times R_2$ is called the amalgamation of R_1 and R_2 along J with respect to f. In [13], the amalgamation of M_1 and M_2 along J with respect to φ is defined as

$$M_1 \bowtie^{\varphi} JM_2 = \{(m_1, \varphi(m_1) + m_2) : m_1 \in M_1 \text{ and } m_2 \in JM_2\}$$

which is an $(R_1 \bowtie^f J)$-module with the scaler product defined as

$$(r, f(r) + j)(m_1, \varphi(m_1) + m_2) = (rm_1, \varphi(rm_1) + f(r)m_2 + j\varphi(m_1) + jm_2)$$

For submodules N_1 and N_2 of M_1 and M_2, respectively, one can easily justify that the sets

$$N_1 \bowtie^{\varphi} JM_2 = \{(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie^{\varphi} JM_2 : m_1 \in N_1\}$$

and

$$\overline{N_2}^{\varphi} = \{(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie^{\varphi} JM_2 : \ \varphi(m_1) + m_2 \in N_2\}$$

are submodules of $M_1 \bowtie^{\varphi} JM_2$. Moreover if S_1 and S_2 are multiplicatively closed subsets of R_1 and R_2, respectively, then

$$S_1 \bowtie^f J = \{(s_1, f(s_1) + j) : s \in S_1, \ \ j \in J\}$$

and

$$\overline{S_2}^f = \{(r, f(r) + j) : r \in R_1, \ \ f(r) + j \in S_2\}$$

are clearly multiplicatively closed subsets of $R_1 \bowtie^f J$.

Note that if $R = R_1 = R_2$, $M = M_1 = M_2$, $f = Id_R$ and $\varphi = Id_M$, then the amalgamation of M_1 and M_2 along J with respect to φ is exactly the duplication of the R-module M along the ideal J. Moreover, in this case, we have $N_1 \bowtie^{\varphi} JM_2 = N \bowtie J$, $\overline{N_2}^{\varphi} = \bar{N}$, $S_1 \bowtie^f J = S \bowtie J$ and $\overline{S_2}^f = \bar{S}$.

The proof of the following lemma is straightforward.

Lemma 3.1. *Let $M_1 \bowtie^{\varphi} JM_2$, $N_1 \bowtie^{\varphi} JM_2$ and $\overline{N_2}^{\varphi}$ be as above. Then*

1. $(r_1, f(r_1) + j) \in (N_1 \bowtie^{\varphi} JM_2 :_{R_1 \bowtie^f J} M_1 \bowtie^{\varphi} JM_2)$ *if and only if* $r_1 \in (N_1 :_{R_1} M_1)$.
2. *If f and φ are epimorphisms, then* $(r_1, f(r_1)+j) \in (\overline{N_2}^{\varphi} :_{R_1 \bowtie^f J} M_1 \bowtie^{\varphi} JM_2)$ *if and only if* $f(r_1)+j \in (N_2 :_{R_2} M_2)$.

Theorem 3.2. *Consider the $(R_1 \bowtie^f J)$-module $M_1 \bowtie^{\varphi} JM_2$ defined as above. Let S be a multiplicatively closed subsets of R_1 and N_1 be submodule of M_1. Then*

1. $N_1 \bowtie^{\varphi} JM_2$ is an $S \bowtie^f J$-primary submodule of $M_1 \bowtie^{\varphi} JM_2$ if and only if N_1 is an S-primary submodule of M_1.

2. $N_1 \bowtie^{\varphi} JM_2$ is a weakly $S \bowtie^f J$-primary submodule of $M_1 \bowtie^{\varphi} JM_2$ if and only if the following hold
   (a) N_1 is a weakly S-primary submodule of M_1.
   (b) For $r_1 \in R_1$, $m_1 \in M_1$ with $r_1m_1 = 0$ but $s_1r_1 \notin \sqrt{(N_1 :_{R_1} M_1)}$ and $s_1m_1 \notin N_1$ for all $s_1 \in S$, then $f(r_1)m_2 + j\phi(m_1) + jm_2 = 0$ for every $j \in J$ and $m_2 \in JM_2$.

Proof. It is easy to verify that $(N_1 \bowtie^{\varphi} JM_2 :_{R_1 \bowtie^f J} M_1 \bowtie^{\varphi} JM_2) \cap (S \bowtie^f J) = \emptyset$ if and only if $(N_1 :_{R_1} M_1) \cap S = \emptyset$.

(1) ($\Rightarrow$) Suppose $(s, f(s) + j)$ is a weakly $S \bowtie^f J$-element of $N_1 \bowtie^{\varphi} JM_2$. Let $r_1 \in R_1$ and $m_1 \in M_1$ such that $r_1m_1 \in N_1$. Then $(r_1, f(r_1)) \in R_1 \bowtie^f J$, $(m_1, \varphi(m_1)) \in M_1 \bowtie^{\varphi} JM_2$ and $(r_1, f(r_1))(m_1, \varphi(m_1)) = (r_1m_1, \varphi(r_1m_1)) \in N_1 \bowtie^{\varphi} JM_2$. By assumption, we have either

$$(s, f(s) + j)(r_1, f(r_1)) \in \sqrt{(N_1 \bowtie^{\varphi} JM_2 :_{R_1 \bowtie^f J} M_1 \bowtie^{\varphi} JM_2)}$$

or

$$(s, f(s) + j)(m_1, \varphi(m_1)) \in N_1 \bowtie^{\varphi} JM_2$$

In the first case, we conclude by Lemma 3.1 that $sr_1 \in \sqrt{(N_1 :_{R_1} M_1)}$. In the second case, we get $sm_1 \in N_1$ and so N_1 is an S-primary submodule of M_1.

($\Leftarrow$) Let s be a weakly S-element of N_1. Let $(r_1, f(r_1) + j) \in R_1 \bowtie^f J$ and $(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie^{\varphi} JM_2$ such that $(r_1, f(r_1) + j)(m_1, \varphi(m_1) + m_2) \in N_1 \bowtie^{\varphi} JM_2$. Then $r_1m_1 \in N_1$ and so either $sr_1 \in \sqrt{(N_1 :_{R_1} M_1)}$ or $sm_1 \in N_1$. If $sr_1 \in \sqrt{(N_1 :_{R_1} M_1)}$, then by Lemma 3.1, $(s, f(s))(r_1, f(r_1) + j) \in \sqrt{(N_1 \bowtie^{\varphi} JM_2 :_{R_1 \bowtie^f J} M_1 \bowtie^{\varphi} JM_2)}$ and if $sm_1 \in N_1$, then $(s, f(s))(m_1, \varphi(m_1) + m_2) \in N_1 \bowtie^{\varphi} JM_2$. Thus, $N_1 \bowtie^{\varphi} JM_2$ is an $S \bowtie^f J$-primary submodule of $M_1 \bowtie^{\varphi} JM_2$ associated to $(s, f(s)) \in S \bowtie^f J$.

(2) ($\Rightarrow$) Suppose $(s, f(s) + j)$ is a weakly $S \bowtie^f J$-element of $N_1 \bowtie^{\varphi} JM_2$. Let $r_1 \in R_1$ and $m_1 \in M_1$ such that $0 \neq r_1m_1 \in N_1$. Then $(0, 0) \neq (r_1, f(r_1))(m_1, \varphi(m_1)) = (r_1m_1, \varphi(r_1m_1)) \in N_1 \bowtie^{\varphi} JM_2$. By assumption, either

$$(s, f(s) + j)(r_1, f(r_1)) \in \sqrt{(N_1 \bowtie^{\varphi} JM_2 :_{R_1 \bowtie^f J} M_1 \bowtie^{\varphi} JM_2)}$$

or $(s, f(s) + j)(m_1, \varphi(m_1)) \in N_1 \bowtie^{\varphi} JM_2$. Thus, $sr_1 \in \sqrt{(N_1 :_{R_1} M_1)}$ by Lemma 3.1 or $sm_1 \in N_1$ and so N_1 is weakly S-primary in M_1. We use the contrapositive to prove the other part. Let $r_1 \in R_1$, $m_1 \in M_1$ with $r_1m_1 = 0$ and $f(r_1)m_2 + j\phi(m_1) + jm_2 \neq 0$ for some $j \in J$ and some $m_2 \in JM_2$. Then

$$\begin{aligned}(0, 0) &\neq (r_1, f(r_1) + j)(m_1, \varphi(m_1) + m_2) \\ &= (0, f(r_1)m_2 + j\varphi(m_1) + jm_2) \in N_1 \bowtie^{\varphi} JM_2\end{aligned}$$

By assumption, either

$$(s, f(s) + j)(r_1, f(r_1) + j) \in \sqrt{(N_1 \bowtie^{\varphi} JM_2 :_{R_1 \bowtie^f J} M_1 \bowtie^{\varphi} JM_2)}$$

or $(s, f(s) + j)(m_1, \varphi(m_1) + m_2) \in N_1 \bowtie^{\varphi} JM_2$ and so again $sr_1 \in \sqrt{(N_1 :_{R_1} M_1)}$ or $sm_1 \in N_1$ as needed.

($\Leftarrow$) Let s be a weakly S-element of N_1, $(r_1, f(r_1) + j) \in R_1 \bowtie^f J$ and $(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie^{\varphi} JM_2$ such that

$$\begin{aligned}(0, 0) &\neq (r_1m_1, \varphi(r_1m_1) + f(r_1)m_2 + j\varphi(m_1) + jm_2) \\ &= (r_1, f(r_1) + j)(m_1, \varphi(m_1) + m_2) \in N_1 \bowtie^{\varphi} JM_2\end{aligned}$$

If $0 \neq r_1m_1$, then the proof is similar to that of (1). Suppose $r_1m_1 = 0$. Then $f(r_1)m_2 + j\varphi(m_1) + jm_2 \neq 0$ and so by assumption there exists $s' \in S$ such that either $s'r_1 \in \sqrt{(N_1 :_{R_1} M_1)}$ or $s'm_1 \in N_1$. Thus, $(s', f(s'))(r_1, f(r_1) + j) \in \sqrt{(N_1 \bowtie^{\varphi} JM_2 :_{R_1 \bowtie^f J} M_1 \bowtie^{\varphi} JM_2)}$ or $(s', f(s'))(m_1, \varphi(m_1) + m_2) \in N_1 \bowtie^{\varphi} JM_2$. Therefore, $N_1 \bowtie^{\varphi} JM_2$ is a weakly $S \bowtie^f J$-primary submodule of $M_1 \bowtie^{\varphi} JM_2$ associated to $(ss', f(ss')) \in S \bowtie^f J$. $\square$

In particular, if we take $S = \{1_{R_1}\}$ and consider $S \bowtie^f 0 = \{(1_{R_1}, 1_{R_2})\})$ in Theorem 3.2, then we get the following corollary.

Corollary 3.3. *Consider the $(R_1 \bowtie^f J)$-module $M_1 \bowtie^\varphi JM_2$ defined as in Theorem 3.2 and let N_1 be a submodule of M_1. Then*

1. $N_1 \bowtie^\varphi JM_2$ is a primary submodule of $M_1 \bowtie^\varphi JM_2$ if and only if N_1 is a primary submodule of M_1.
2. $N_1 \bowtie^\varphi JM_2$ is a weakly primary submodule of $M_1 \bowtie^\varphi JM_2$ if and only if the following hold
   (a) N_1 is a weakly primary submodule of M_1.
   (b) For $r_1 \in R_1, m_1 \in M_1$ with $r_1 m_1 = 0$ but $r_1 \notin \sqrt{(N_1 :_{R_1} M_1)}$ and $m_1 \notin N_1$, then $f(r_1)m_2 + j\phi(m_1) + jm_2 = 0$ for every $j \in J$ and $m_2 \in JM_2$.

Theorem 3.4. *Consider the $(R_1 \bowtie^f J)$-module $M_1 \bowtie^\varphi JM_2$ defined as in Theorem 3.2 where f and φ are epimorphisms. Let S be a multiplicatively closed subsets of R_2 and N_2 be a submodule of M_2. Then*

1. $\overline{N_2}^{\varphi}$ is an $\overline{S}^f$-primary submodule of $M_1 \bowtie^\varphi JM_2$ if and only if N_2 is an S-primary submodule of M_2.
2. $\overline{N_2}^{\varphi}$ is a weakly $\overline{S}^f$-primary submodule of $M_1 \bowtie^\varphi JM_2$ if and only if the following hold
   (a) N_2 is a weakly S-primary submodule of M_2.
   (b) For $r_1 \in R_1, m_1 \in M_1, m_2 \in JM_2, j \in J$ with $(f(r_1) + j)(\varphi(m_1) + m_2) = 0$ but $s(f(r_1) + j) \notin \sqrt{(N_2 :_{R_2} M_2)}$ and $s(\varphi(m_1) + m_2) \notin N_2$ for all $s \in S$, then $r_1 m_1 = 0$.

Proof. In view of Lemma 3.1, we can easily prove that $(\overline{N_2}^{\varphi} :_{R_1 \bowtie^f J} M_1 \bowtie^\varphi JM_2) \cap \overline{S}^f = \emptyset$ if and only if $(N_2 :_{R_2} M_2) \cap S = \emptyset$.

(1). ($\Rightarrow$) Suppose N_2 is an S-primary submodule of M_2 associated to $s = f(t) \in S$. Let $(r_1, f(r_1)+j) \in R_1 \bowtie^f J$ and $(m_1, \varphi(m_1)+m_2) \in M_1 \bowtie JM_2$ such that $(r_1, f(r_1)+j)(m_1, \varphi(m_1)+m_2) \in \overline{N_2}^{\varphi}$. Then $(f(r_1)+j)(\varphi(m_1)+m_2) \in N_2$ and so $s(f(r_1)+j) \in \sqrt{(N_2 :_{R_2} M_2)}$ or $s(\varphi(m_1)+m_2) \in N_2$. If $s(f(r_1)+j) \in \sqrt{(N_2 :_{R_2} M_2)}$, then by Lemma 3.1, we have $(t, s)(r_1, f(r_1) + j) \in \sqrt{(\overline{N_2}^{\varphi} :_{R_1 \bowtie^f J} M_1 \bowtie^\varphi JM_2)}$. If $s(\varphi(m_1) + m_2) \in N_2$, then $(t, s)(m_1, \varphi(m_1) + m_2) \in \overline{N_2}^{\varphi}$. Therefore, $\overline{N_2}^{\varphi}$ is $\overline{S}^f$-primary in $M_1 \bowtie^\varphi JM_2$.

($\Leftarrow$) Suppose $\overline{N_2}^{\varphi}$ is an $\overline{S}^f$-primary submodule of $M_1 \bowtie^\varphi JM_2$ associated to $(t, f(t) + j) = (t, s) \in \overline{S}^f$. Let $r_2 = f(r_1) \in R_2$ and $m_2 = \varphi(m_1) \in M_2$ such that $r_2 m_2 \in N_2$. Then $(r_1, r_2)(m_1, m_2) \in \overline{N_2}^{\varphi}$ where $(r_1, r_2) \in R_1 \bowtie^f J$ and $(m_1, m_2) \in M_1 \bowtie^\varphi JM_2$. By assumption,

$$(t, s)(r_1, r_2) \in \sqrt{(\overline{N_2}^{\varphi} :_{R_1 \bowtie^f J} M_1 \bowtie^\varphi JM_2)} \text{ or } (t, s)(m_1, m_2) \in \overline{N_2}^{\varphi}$$

In the first case, Lemma 3.1 implies $sr_2 \in \sqrt{(N_2 :_{R_2} M_2)}$. In the second case, we conclude $sm_2 \in N_2$ and the result follows.

(2). ($\Rightarrow$) Let $(t, f(t) + j) = (t, s)$ be a weakly $\overline{S}^f$-element of $\overline{N_2}^{\varphi}$. Let $r_2 = f(r_1) \in R_2$ and $m_2 = f(m_1) \in M_2$ such that $0 \neq r_2 m_2 \in N_2$. Then $(0.0) \neq (r_1, r_2)(m_1, m_2) \in \overline{N_2}^{\varphi}$ where $(r_1, r_2) \in R_1 \bowtie^f J$ and $(m_1, m_2) \in M_1 \bowtie^\varphi JM_2$. Thus, either $(t, s)(r_1, r_2) \in \sqrt{(\overline{N_2}^{\varphi} :_{R_1 \bowtie^f J} M_1 \bowtie^\varphi JM_2)}$ or $(t, s)(m_1, m_2) \in \overline{N_2}^{\varphi}$. Hence, either $sr_2 \in \sqrt{(N_2 :_{R_2} M_2)}$ by Lemma 3.1 or $sm_2 \in N_2$ and s is a weakly S-element of N_2. For the other part, we use contrapositive. Let $r_1 \in R_1$, $m_1 \in M_1$, $m_2 \in JM_2$, $j \in J$ with $(f(r_1) + j)(\varphi(m_1) + m_2) = 0$ and suppose $r_1 m_1 \neq 0$. Then $(0, 0) \neq (r_1, f(r_1) + j)(m_1, \varphi(m_1) + m_2) \in \overline{N_2}^{\varphi}$ and so $(t, s)(r_1, f(r_1) + j) \in \sqrt{(\overline{N_2}^{\varphi} :_{R_1 \bowtie^f J} M_1 \bowtie JM_2)}$ or $(t, s)(m_1, \varphi(m_1) + m_2) \in \overline{N_2}^{\varphi}$. Hence, either $s(f(r_1) + j) \in \sqrt{(N_2 :_{R_2} M_2)}$ or $s(\varphi(m_1) + m_2) \in N_1$ and the result follows.

($\Leftarrow$) Suppose $s = f(t) \in S$ is a weakly S-element of N_2. Let $(r_1, f(r_1) + j) \in R_1 \bowtie^f J$ and $(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie^\varphi JM_2$ such that $(0, 0) \neq (r_1, f(r_1) + j)(m_1, \varphi(m_1) + m_2) \in \overline{N_2}^{\varphi}$. Then $(f(r_1) + j)(\varphi(m_1) + m_2) \in N_2$. If

$(f(r_1)+j)(\varphi(m_1)+m_2)=0$, then $r_1m_1\neq 0$. So by assumption, there exists $s'=f(t')\in S$ such that $s'(f(r_1)+j)\in\sqrt{(N_2:_{R_2}M_2)}$ or $s'(\varphi(m_1)+m_2)\in N_2$. It follows that $(t',s')(r_1,f(r_1)+j)\in\sqrt{(\overline{N_2}^{\varphi}:_{R_1\bowtie^f J}M_1\bowtie^{\varphi}JM_2)}$ or $(t',s')(m_1,\varphi(m_1)+m_2)\in\overline{N_2}^{\varphi}$. Hence, $\overline{N_2}^{\varphi}$ is a weakly $\overline{S}^f$-primary submodule of $M_1\bowtie^{\varphi}JM_2$ associated to (tt',ss'). If $(f(r_1)+j)(\varphi(m_1)+m_2)\neq 0$, then the result follows as in the proof of (1). $\square$

In particular, if we take $S=\{1_{R_2}\}$ and consider the multiplicatively closed subset $\overline{S}^f=\{(1_{R_1},1_{R_2})\}$ of $R_1\bowtie^f J$ in Theorem 3.4, then we get the following results for primary and weakly primary submodules of amalgamation modules.

Corollary 3.5. *Let $M_1\bowtie^{\varphi}JM_2$ and N_2 be defined as in Theorem 3.4. Then*

1. $\overline{N_2}^{\varphi}$ is a primary submodule of $M_1\bowtie^{\varphi}JM_2$ if and only if N_2 is a primary submodule of M_2.
2. $\overline{N_2}^{\varphi}$ is a weakly primary submodule of $M_1\bowtie^{\varphi}JM_2$ if and only if the following hold
   (a) N_2 is a weakly primary submodule of M_2.
   (b) For $r_1\in R_1$, $m_1\in M_1$, $m_2\in JM_2$, $j\in J$ with $(f(r_1)+j)(\varphi(m_1)+m_2)=0$ but $(f(r_1)+j)\notin\sqrt{(N_2:_{R_2}M_2)}$ and $(\varphi(m_1)+m_2)\notin N_2$, then $r_1m_1=0$.

Corollary 3.6. *Let N be a submodule of an R-module M, J an ideal of R and S a multiplicatively closed subset of R. The following are equivalent:*

1. N is an S-primary submodule of M.
2. $N\bowtie J$ is an $(S\bowtie J)$-primary submodule of $M\bowtie J$.
3. $\overline{N}$ is an $\overline{S}$-primary submodule of $M\bowtie J$.

Corollary 3.7. *Let N be a submodule of an R-module M, J an ideal of R and S a multiplicatively closed subset of R. The following are equivalent:*

1. $N\bowtie J$ is a weakly $(S\bowtie J)$-primary submodule of $M\bowtie J$
2. N is a weakly S-primary submodule of M and for $r\in R$, $m\in M$ with $rm=0$ but $sr\notin\sqrt{(N:_{R_1}M)}$ and $sm\notin N$ for all $s\in S$, then $(r+j)m'=0$ for every $j\in J$ and $m'\in JM_2$.

Corollary 3.8. *Let N be a submodule of an R-module M, J an ideal of R and S a multiplicatively closed subset of R.*

1. $\overline{N}$ is a weakly $\overline{S}$-prime submodule of $M\bowtie J$.
2. N is a weakly S-prime submodule of M and for $r\in R$, $m\in M$, $m'\in JM$, $j\in J$ with $(r+j)(m+m')=0$ but $s(r+j)\notin(N:_R M)$ and $s(m+m')\notin N$ for all $s\in S$, then $rm=0$.

Next, we justify that the second condition of Corollary 3.7 (2) can not be ignored.

Example 3.9. *Consider the ideal $J=2\mathbb{Z}$ of $\mathbb{Z}$ and the submodule $N=0\times\langle\bar{0}\rangle$ of the $\mathbb{Z}$-module $M=\mathbb{Z}\times\mathbb{Z}_6$. Then*

$$M\bowtie J=\{(m,m')\in M\times M: m-m'\in JM=2\mathbb{Z}\times\langle\bar{2}\rangle\}$$

and

$$N\bowtie J=\{(n,m)\in N\times M: n-m\in 2\mathbb{Z}\times\langle\bar{2}\rangle\}$$

Obviously, N is a weakly primary submodule of M. On the other hand, $N\bowtie J$ is not a weakly primary submodule of $M\bowtie J$. Indeed, $(2,4)\in\mathbb{Z}\bowtie J$ and $((0,\bar{3}),(0,\bar{1}))\in M\bowtie J$ with $(2,4).((0,\bar{3}),(0,\bar{1}))=((0,\bar{0}),(0,\bar{4}))\in N\bowtie J$. But $(2,4)\notin\sqrt{((N\bowtie J):_{\mathbb{Z}\bowtie I}(M\bowtie J))}$ as $2\notin\sqrt{(N:_{\mathbb{Z}}M)}=\langle 0\rangle$ and $((0,\bar{3}),(0,\bar{1}))\notin N\bowtie J$. We note that if we take $r=2$ and $m=(0,\bar{3})\in M$, then clearly, $rm=0$, $r\notin\sqrt{(N:_R M)}=0$ and $m\notin N$ but for $m'=(0,\bar{2})\in JM=2\mathbb{Z}\times\langle\bar{2}\rangle$, we have $(r+0)m'\neq 0$.

Also, if the second condition of (2) in Corollary 3.8 does not hold, then we may find a weakly S-primary submodule N of M such that $\overline{N}$ is not a weakly $\overline{S}$-primary submodule of $M\bowtie J$.

Example 3.10. *Let N, M and J be as in Example 3.9. Choose $(2,4)\in\mathbb{Z}\bowtie J$ and $((0,\bar{1}),(0,\bar{3}))\in M\bowtie J$. Then we have $(2,4).((0,\bar{1}),(0,\bar{3}))\in\overline{N}$ but clearly $(2,4)\notin\sqrt{(\bar{N}:_{\mathbb{Z}\bowtie I}(M\bowtie J))}$ and $((0,\bar{1}),(0,\bar{3}))\notin\overline{N}$. Therefore, $\bar{N}$ is not a weakly primary submodule of $M\bowtie J$.*

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