# The Josefson–Nissenzweig property for locally convex spaces

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**Abstract.** We define a locally convex space $E$ to have the *Josefson–Nissenzweig property* (JNP) if the identity map $(E', \sigma(E', E)) \to (E', \beta^*(E', E))$ is not sequentially continuous. By the classical Josefson–Nissenzweig theorem, every infinite-dimensional Banach space has the JNP. A characterization of locally convex spaces with the JNP is given. We thoroughly study the JNP in various function spaces. Among other results we show that for a Tychonoff space $X$, the function space $C_p(X)$ has the JNP iff there is a weak* null-sequence $(\mu_n)_{n\in\omega}$ of finitely supported sign-measures on $X$ with unit norm. However, for every Tychonoff space $X$, neither the space $B_1(X)$ of Baire-1 functions on $X$ nor the free locally convex space $L(X)$ over $X$ has the JNP.

## 1. Introduction

All locally convex spaces (lcs for short) are assumed to be Hausdorff and infinite-dimensional, and all topological spaces are assumed to be infinite and Tychonoff. We denote by $E'$ the topological dual of an lcs $E$. The dual space $E'$ of $E$ endowed with the weak* topology $\sigma(E', E)$ and the strong topology $\beta(E', E)$ is denoted by $E'_{w^*}$ and $E'_\beta$, respectively. For a bounded subset $B \subseteq E$ and a functional $\chi \in E'$, we put $\|\chi\|_B := \sup\{|\chi(x)| : x \in B \cup \{0\}\}$.

Josefson [17] and Nissenzweig [20] proved independently the following theorem (other proofs of this beautiful result were given by Hagler and Johnson [15] and Bourgain and Diestel [6]).

**Theorem 1.1.** [Josefson–Nissenzweig] $E$ be a Banach space. Then there is a null sequence $\{\chi_n\}_{n\in\omega}$ in $E'_{w^*}$ such that $\|\chi_n\| = 1$ for every $n \in \omega$.

Therefore the identity map $E'_{w^*} \to E'_\beta$ is *not* sequentially continuous for every Banach space. Recall that a function $f : X \to Y$ between topological spaces $X$ and $Y$ is called *sequentially continuous* if for any convergent sequence $\{x_n\}_{n\in\omega} \subseteq X$, the sequence $\{f(x_n)\}_{n\in\omega}$ converges in $Y$ and $\lim_n f(x_n) = f(\lim_n x_n)$.

The Josefson–Nissenzweig theorem was extended to Fréchet spaces in [5].

**Theorem 1.2.** [Bonet–Lindström–Valdivia] *For a Fréchet space $E$, the identity map $E'_{w^*} \to E'_\beta$ is sequentially continuous if and only if $E$ is a Montel space.*

Another extension of the Josefson–Nissenzweig theorem was provided by Bonet [4] and Lindström and Schlumprecht [18] who proved that a Fréchet space $E$ is a Schwartz space if and only if every null sequence in $E'_{w^*}$ converges uniformly to zero on some zero-neighborhood in $E$.

Studying the separable quotient problem for $C_p$-spaces and being motivated by the Josefson–Nissenzweig theorem, Banakh, Kąkol and Śliwa introduced in [3] the Josefson–Nissenzweig property for the space $C_p(X)$ of continuous real-valued functions on a Tychonoff space $X$, endowed with the topology of pointwise convergence. Namely, they defined $C_p(X)$ to have the Josefson–Nissenzweig property (JNP) if the dual space $C_p(X)'$ of $C_p(X)$ contains a weak$^*$ null sequence $\{\mu_n\}_{n\in\omega}$ of finitely supported sign-measures on $X$ such that $\|\mu_n\| := |\mu_n|(X) = 1$ for every $n \in \omega$. This definition implies that for any Tychonoff space $X$ containing a non-trivial convergent sequence, the function space $C_p(X)$ has the JNP. Yet, there exists a compact space $K$ without non-trivial convergent sequences such that the function space $C_p(K)$ has the Josefson–Nissenzweig property, see Plebanek's example in [3]. On the other hand, Banakh, Kąkol and Śliwa observed in [3] that the function space $C_p(\beta\omega)$ does not have the Josefson–Nissenzweig property.

Denote by $C_p^0(\omega)$ the subspace of the product $\mathbb{R}^\omega$ consisting of all real-valued functions on the discrete space $\omega$ that tend to zero at infinity. The following characterization of $C_p$-spaces with the Josefson–Nissenzweig property is the main result of [3].

**Theorem 1.3.** [Banakh–Kąkol–Śliwa] *For a Tychonoff space $X$, the following conditions are equivalent:*

(i) *$C_p(X)$ has the Josefson–Nissenzweig property;*
(ii) *$C_p(X)$ contains a complemented subspace isomorphic to $C_p^0(\omega)$;*
(iii) *$C_p(X)$ has a quotient isomorphic to $C_p^0(\omega)$;*
(iv) *$C_p(X)$ admits a linear continuous map onto $C_p^0(\omega)$.*

This characterization shows that the Josefson–Nissenzweig property depends only on the locally convex structure of the space $C_p(X)$ in spite of the fact that its definition involves the norm in the dual space, which is a kind of an external structure for $C_p(X)$.

The aforementioned results motivate us to define and study the Jossefson–Nissenzweig property in the class of all locally convex spaces, this is the main goal of the article.

Let $E$ be a locally convex space. Taking into account Theorem 1.2, it is natural to say that $E$ has the Jossefson–Nissenzweig property if the identity map $E'_{w^*} \to E'_\beta$ is not sequentially continuous. However, such definition is not fully consistent with the Josefson–Nissenzweig property for $C_p$-spaces: the identity map $C_p(X)'_{w^*} \to C_p(X)'_\beta$ is not sequentially continuous for any infinite Tychonoff space $X$ containing a non-trivial convergent sequence $\{x_n\}_{n\in\omega}$ because the strong dual of $C_p(X)$ is feral (see for example [13, Proposition 2.6], recall that a locally convex space $E$ is called *feral* if any bounded subset of $E$ is finite-dimensional). Indeed, the sequence $\{\frac{1}{n}\delta_{x_n}\}_{n\in\omega}$, where $\delta_x$ is the Dirac measure at the point $x$, is trivially $w^*$-null but it is not $\beta$-null because of ferality. Therefore to give a "right" definition of the Josefson–Nissenzweig property, we should consider a weaker topology on $E'$ than the strong topology $\beta(E',E)$. A locally convex topology on the dual $E'$ which covers both cases described in Theorem 1.2 and Theorem 1.3 is the topology $\beta^*(E',E)$ of uniform convergence on $\beta(E,E')$-bounded subsets of $E$, defined in [16, 8.4.3.C]. Put $E'_{\beta^*} := \left(E', \beta^*(E',E)\right)$.

**Definition 1.4.** A locally convex space $E$ is said to have the *Jossefson–Nissenzweig property* (briefly, the JNP) if the identity map $E'_{w^*} \to E'_{\beta^*}$ is not sequentially continuous. $\square$

Now, if $E$ is a Fréchet space, then $\beta^*(E',E) = \beta(E',E)$ by Corollary 10.2.2 of [16]. Therefore Theorem 1.2 states that a Fréchet space $E$ has the JNP if and only if $E$ is not a Montel space. In Section 2 we give an independent proof of Theorem 1.2 in an extended form, see Theorem 2.3. We also consider other important classes of locally convex spaces. In particular, we prove that for any Tychonoff space $X$, the space $B_1(X)$ of all Baire one functions on $X$ and the free locally convex space $L(X)$ over $X$ fail to have the JNP. In Theorem 2.5 we give an operator characterization of locally convex spaces with the JNP.

In Section 3 we study function spaces endowed with various natural topologies which have the JNP. In particular, in Corollary 3.10 we show that a function space $C_p(X)$ has the JNP if and only if it has that property in the sense of Banakh, Kąkol and Śliwa [3] mentioned above.

## 2. The Josefson–Nissenzweig property in some classes of locally convex spaces

All locally convex spaces considered in this paper are over the field $\mathbb{F}$ of real or complex numbers.

Let $E$ be a locally convex space. A closed absorbent absolutely convex subset of $E$ is called a *barrel*. The *polar* of a subset $A$ of $E$ is denoted by

$$A^\circ := \{\chi \in E' : \|\chi\|_A \leq 1\}, \quad \text{where} \quad \|\chi\|_A = \sup\left\{|\chi(x)| : x \in A \cup \{0\}\right\}.$$

Now we give a more clear and useful description of the topology $\beta^*(E', E)$. Recall that we defined a subset $B \subseteq E$ to be *barrel-bounded* if for any barrel $U \subseteq E$ there is an $n \in \omega$ such that $B \subseteq nU$. It is easy to see that each finite subset of $E$ is barrel-bounded and each barrel-bounded set in $E$ is bounded. Observe that a subset of a barrelled space is bounded if and only if it is barrel-bounded. We recall that a locally convex space $E$ is *barrelled* if each barrel in $E$ is a neighborhood of zero. A neighborhood base at zero of the topology $\beta^*(E', E)$ on $E'$ consists of the polars $B^\circ$ of barrel-bounded subsets $B \subseteq E$.

Let $E$ be a locally convex space. The space $E$ with the weak topology $\sigma(E, E')$ is denoted by $E_w$. The strong second dual space $(E'_\beta)'_\beta$ of $E$ will be denoted by $E''$. Denote by $\psi_E : E \to E''$ the canonical evaluation map defined by $\psi_E(x)(\chi) := \chi(x)$ for all $x \in E$ and $\chi \in E'$. Recall that $E$ is called *reflexive* if $\psi_E$ is a topological isomorphism, and $E$ is *Montel* if it is reflexive and every closed bounded subset of $E$ is compact. We recall that $E$ has the *Shur property* if the identity map $E_w \to E$ is sequentially continuous. Recall also that $E$ is called *quasi-complete* if every closed bounded subset of $E$ is complete.

**Theorem 2.1.** *Let $E$ be a quasi-complete reflexive space whose every separable bounded subset is metrizable. Then $E$ has the JNP if and only if $E$ is not Montel.*

*Proof.* Recall that each reflexive space is barrelled, see [16, Proposition 11.4.2]. Now we note that $\beta^*(E', E) = \beta(E', E)$ by Corollary 10.2.2 of [16].

To prove the "only if" part, suppose for a contradiction that $E$ is Montel. Since the strong dual of a Montel space is also Montel ([16, Proposition 11.5.4]), the strong dual space $E'_\beta$ is Montel and, by Proposition 2.3 of [10], $E'_\beta$ has the Schur property. As every Montel space is also reflexive, the Schur property exactly means that the identity map $E'_{w^*} \to E'_\beta = E'_{\beta^*}$ is sequentially continuous and hence $E$ does not have the JNP. This contradiction shows that $E$ is not Montel. To prove the "if" part, assume that $E$ is not Montel. Then, by Proposition 3.7 of [10], $E'$ contains a $\sigma(E', E)$-convergent sequence which does not converge in $\beta(E', E) = \beta^*(E', E)$. Thus $E$ has the JNP. $\square$

**Proposition 2.2.** *Let $E$ be a barrelled space, and let $\mathcal{T}$ be a locally convex topology on $E$ compatible with the duality $(E, E')$. Then the space $E$ has the JNP if and only if $(E, \mathcal{T})$ has the JNP.*

*Proof.* Note that $(E, \mathcal{T})' = E'$ and hence $(E, \mathcal{T})'_{w^*} = E'_{w^*}$. So to prove the proposition it suffices to show that also $(E, \mathcal{T})'_{\beta^*} = E'_{\beta^*}$. To this end, we shall show that $\beta^*\big(E', (E, \mathcal{T})\big) = \beta(E', E) = \beta^*(E', E)$. It is clear that these equalities hold true if the spaces $E$ and $(E, \mathcal{T})$ have the same barrel-bounded sets. Since $E$ is barrelled, a subset of $E$ is barrel-bounded if and only if it is bounded. As $E$ and $(E, \mathcal{T})$ have the same bounded sets, these spaces have the same barrel-bounded sets if we shall show that every bounded subset $B$ of $(E, \mathcal{T})$ is barrel-bounded. If $U$ is a barrel in $(E, \mathcal{T})$, it is also a barrel in $E$ and hence $U$ is a neighborhood of zero in $E$. Hence there is $n \in \omega$ such that $B \subseteq nU$. Thus $B$ is barrel-bounded. $\square$

Below we give an independent proof of Theorem 1.2. Our proof, as well as the proof of Theorem 1.2, also essentially uses the following result from [18]: *a Fréchet space $E$ is reflexive if the identity map $E'_{w^*} \to E'_\beta$ is sequentially continuous.*

**Theorem 2.3.** *For a Fréchet space $E$, the following assertions are equivalent:*

(i) *$E$ is not Montel.*
(ii) *$E$ has the JNP.*

(iii) $E_w$ *has the JNP.*

*Proof.* (i) $\Rightarrow$ (ii) Let $E$ be a non-Montel space. Since the Fréchet space $E$ is barrelled, $E'_\beta = E'_{\beta^*}$. If the identity map $E'_{w^*} \to E'_\beta = E'_{\beta^*}$ is not sequentially continuous, then $E$ has the JNP by definition. If this map is sequentially continuous, then, by [18], $E$ is reflexive. Finally, Theorem 2.1 implies that $E$ has the JNP.

(ii) $\Rightarrow$ (i) Assume that $E$ has the JNP. Then $E$ is not Montel by Theorem 2.1 (recall that every Montel space is reflexive).

(iii) $\Leftrightarrow$ (ii) follows from Proposition 2.2. $\square$

Now we present a characterization of the JNP in the terms of $C_p^0(\omega)$-valued operators. An operator $T : X \to Y$ between locally convex spaces is *β-to-β precompact* if for any barrel-bounded set $B \subseteq X$ the image $T(B)$ is barrel-precompact in $Y$. Recall that $C_p^0(\omega)$ denotes the subspace of $\mathbb{F}^\omega$ consisting of functions $\omega \to \mathbb{F}$ that tend to zero at infinity. We shall use also the following well known description of precompact subsets of the Banach space $c_0$, where $e'_n$ is the $n$th coordinate functional of $c_0$.

**Proposition 2.4.** *A subset $A$ of $c_0$ is precompact if and only if $\lim_{n\to\infty} \|e'_n\|_A = 0$.*

Now we are ready to prove an operator characterization of the JNP.

**Theorem 2.5.** *For a locally convex space $E$ over the field $\mathbb{F}$ the following conditions are equivalent:*

(i) *$E$ has the Josefson-Nissenzweig property;*
(ii) *there exists a continuous operator $T : E \to C_p^0(\omega)$, which is not β-to-β precompact.*

*Proof.* (i) $\Rightarrow$ (ii) Assume that $E$ has the JNP, and take any null sequence $\{\mu_n\}_{n\in\omega} \subseteq E'_{w^*}$ that does not converge to zero in the topology $\beta^*(E', E)$. Then there exist a barrel-bounded set $B \subseteq E$ and $\varepsilon > 0$ such that the set $\{n \in \omega : \|\mu_n\|_B > \varepsilon\}$ is infinite. Replacing $\{\mu_n\}_{n\in\omega}$ by a suitable subsequence and multiplying it by $\frac{1}{\varepsilon}$, we can assume that $\|\mu_n\|_B > 1$ for every $n \in \omega$. The $\sigma(E', E)$-null sequence $\{\mu_n\}_{n\in\omega}$ determines the continuous operator $T : X \to C_p^0(\omega)$ defined by $T(x) := (\mu_n(x))_{n\in\omega}$. For simplicity of notation, set $Y := C_p^0(\omega)$. By Lemma 2.4 of [2], the strong topology $\beta(Y, Y')$ on $Y$ is generated by the norm $\|(x_n)_{n\in\omega}\| = \sup_{n\in\omega} |x_n|$ (so $(Y, \beta(Y, Y')) = c_0$). To see that the set $T(B)$ is not precompact in the norm topology, for every $k \in \omega$ we can inductively choose a number $n_k \in \omega$ and a point $b_k \in B$ such that the following conditions are satisfied:

- $|\mu_n(b_i)| < \frac{1}{2}$ for any $i < k$ and any $n \geq n_k$;

- $|\mu_{n_k}(b_k)| > 1$;

(this is possible because $\lim_n \mu_n(b) = 0$ and $\|\mu_n\|_B > 1$). Then for any $i < k$, we have

$$\|T(b_i) - T(b_k)\| \geq |\mu_{n_k}(b_k) - \mu_{n_k}(b_i)| > 1 - \tfrac{1}{2} > \tfrac{1}{2},$$

which implies that the sequence $\{T(b_k)\}_{k\in\omega}$ has no accumulation point in the Banach space $c_0$, and hence it cannot be precompact in $c_0$.

(ii) $\Rightarrow$ (i) Assume that $E$ admits a continuous operator $T : E \to C_p^0(\omega)$ such that $T$ is not β-to-β precompact. Then for some barrel-bounded set $B \subseteq X$, the image $T(B)$ is not barrel-precompact in $C_p^0(\omega)$.

Observe that any barrel $B$ in $C_p^0(\omega)$ is also a barrel in the Banach space $c_0$, and hence $B$ is a $c_0$-neighborhood of zero. This implies that a subset $A$ of $C_p^0(\omega)$ is barrel-bounded (resp. barrel-precompact) if and only if it is a bounded (resp. precompact) subset in $c_0$. Therefore $T(B)$ is not precompact in $c_0$. By Proposition 2.4 this means that $\|e'_n\|_{T(B)} \not\to 0$. For every $n \in \omega$, set $\chi_n := e'_n \circ T$. Since $e'_n \to 0$ in the weak* topology of $C_p^0(\omega)'$, we obtain that $\chi_n \to 0$ in the weak* topology of $E'$. Since

$$\|\chi_n\|_B = \|e'_n\|_{T(B)} \not\to 0 \quad \text{as } n \to \infty,$$

the sequence $\{\chi_n\}_{n\in\omega}$ does not converge to zero in the topology $\beta^*(E', E)$. Thus $E$ has the JNP. $\square$

Recall that a locally convex space $E$ is $c_0$-*barrelled* if every $\sigma(E', E)$-null sequence is equicontinuous. It is well known that each Fréchet space is barrelled and each barrelled locally convex space is $c_0$-barrelled.

**Theorem 2.6.** *Let $E$ be a $c_0$-barrelled space such that $E = E_w$. Then $E$ does not have the JNP.*

*Proof.* It is well known that the space $E$ is a dense linear subspace of $\mathbb{F}^X$ for some set $X$ (for example, $X$ can be chosen to be a Hamel basis of $E'$). Hence we can identify the dual space $E'$ with the linear space of all functions $\mu : X \to \mathbb{F}$ whose support $\mathrm{supp}(\mu) := \mu^{-1}(\mathbb{F} \setminus \{0\})$ is finite.

To see that $E$ does not have the JNP, we have to check that the identity map $E'_{w^*} \to E'_{\beta^*}$ is sequentially continuous. Fix any sequence $\mathcal{M} = \{\mu_n\}_{n \in \omega} \subseteq E'$ that converges to zero in the topology $\sigma(E', E)$. Since $E$ is $c_0$-barrelled, the sequence $\mathcal{M}$ is equicontinuous, which means that $\mathcal{M} \subseteq U^\circ$ for some open neighborhood $U \subseteq E$ of zero. We can assume that $U$ is of the basic form:

$$U = \{f \in E : |f(x)| < \varepsilon \text{ for all } x \in F\}$$

for some finite set $F \subseteq X$ and some $\varepsilon > 0$. The inclusion $\mathcal{M} \subseteq U^\circ$ implies that $\bigcup_{\mu \in \mathcal{M}} \mathrm{supp}(\mu) \subseteq F$, and hence $\mathcal{M}$ is a subset of the finite-dimensional subspace $E'_F := \{\mu \in E' : \mathrm{supp}(\mu) \subseteq F\}$. Since any finite-dimensional locally convex space carries a unique locally convex topology, the sequence $\mathcal{M} \subseteq E'_F$ converges to zero also in the topology $\beta^*(E', E)$. $\square$

A subset $B$ of a topological space $X$ is *functionally bounded* if for every continuous function $f : X \to \mathbb{R}$ the image $f(B)$ is a bounded set in the real line. By [13], for a Tychonoff space $X$, the function space $C_p(X)$ is $c_0$-barrelled if and only if every functionally bounded subset of $X$ is finite. Combining this characterization with Theorem 2.6, we obtain the following assertion.

**Proposition 2.7.** *Let $X$ be a Tychonoff space such that every functionally bounded subset of $X$ is finite. Then the function space $C_p(X)$ does not have the JNP.*

Let $X$ be a Tychonoff space. Let $B_0(X) := C_p(X)$, and for every countable ordinal $\alpha \geq 1$, let $B_\alpha(X)$ be the family of all functions $f : X \to \mathbb{F}$ that are pointwise limits of sequences $\{f_n\}_{n \in \omega} \subseteq \bigcup_{\beta < \alpha} B_\beta(X)$ in the Tychonoff product $\mathbb{F}^X$. All the spaces $B_\alpha(X)$ are endowed with the topology of pointwise convergence, inherited from the Tychonoff product $\mathbb{F}^X$. The classes $B_\alpha(X)$ of Baire-$\alpha$ functions play an essential role in Functional Analysis, see for example [7, 21].

**Proposition 2.8.** *For every Tychonoff space $X$ and each nonzero countable ordinal $\alpha$, the function space $B_\alpha(X)$ does not have the JNP.*

*Proof.* In [1], we proved that the space $B_\alpha(X)$ is barrelled. Since $B_\alpha(X)$ carries its weak topology, Theorem 2.6 applies. $\square$

A locally convex space $E$ is *quasibarrelled* if and only if the canonical map $\psi_E : E \to E''$ is a topological embedding. The next assertion complements dually Proposition 2.2.

**Proposition 2.9.** *Let $(E, \tau)$ be a quasibarrelled space, $\mathcal{T}$ be a locally convex topology on $E'$ compatible with the duality $(E, E')$, and let $H := (E', \mathcal{T})$. Then:*

(i) *$\beta^*(H', H)$ coincides with the topology $\tau$ on $E = H'$;*
(ii) *the space $H$ has the JNP if and only if $E$ does not have the Schur property.*

*Proof.* (i) Since $H' = E$ and $\sigma(H', H) = \sigma(E, E')$, a subset $S$ of $H' = E$ is $\sigma(H', H)$-bounded if and only if $S$ is bounded in $E$. Therefore a subset $U$ of $H$ is a barrel if and only if $U = S^\circ$ for some bounded subset $S \subseteq H'$. Hence a subset $B$ of $H$ is barrel-bounded if and only if $B$ is a $\beta(E', E)$-bounded subset of $E' = H$. Therefore the topology $\beta^*(H', H)$ on $H' = E$ is the topology induced on $E$ from $E''$. Since $E$ is quasibarrelled, $E$ is a subspace of $E''$. Thus the topology $\beta^*(H', H)$ on $H' = E$ coincides with the original topology $\tau$ of the space $E$.

(ii) The JNP of $H$ means that the identity map

$$\left(H', \sigma(H', H)\right) = \left(E, \sigma(E, E')\right) \to \left(H', \beta^*(H', H)\right) \overset{(i)}{=} (E, \tau)$$

is not sequentially continuous that, by definition, means that $E$ does not have the Schur property. $\quad\square$

Proposition 2.9 can be applied to one of the most important classes of locally convex spaces, namely, the class of free locally convex spaces introduced by Markov in [19]. Recall that the *free locally convex space $L(X)$* on a Tychonoff space $X$ is a pair consisting of a locally convex space $L(X)$ and a continuous map $i : X \to L(X)$ such that every continuous map $f$ from $X$ to a locally convex space $E$ gives rise to a unique continuous operator $\bar{f} : L(X) \to E$ such that $f = \bar{f} \circ i$. The free locally convex space $L(X)$ always exists and is essentially unique. We recall also that the set $X$ forms a Hamel basis of $L(X)$ and the map $i$ is a topological embedding. Various locally convex properties of free locally convex spaces are studied in [11, 12].

From the definition of $L(X)$ it easily follows the well known fact that the dual space $L(X)'$ of $L(X)$ is linearly isomorphic to the space $C(X)$. Indeed, the uniqueness of the operator $\bar{f}$ in the definition of the free locally convex space $(L(X), i)$ ensures that the operator $i' : L(X)' \to C(X)$, $i' : \mu \mapsto \mu \circ i$, is bijective and hence is a required linear isomorphism. Via the pairing $(L(X)', L(X)) = (C(X), L(X))$ we note that $C_p(X)'_{w^*} = L(X)_w$. Usually the space $L(X)_w$ is denoted by $L_p(X)$.

**Proposition 2.10.** *For every Tychonoff space $X$, the space $L(X)$ does not have the JNP.*

*Proof.* By Corollary 11.7.3 of [16], the space $E = C_p(X)$ is quasibarrelled for every Tychonoff space $X$. Since $C_p(X)$ carries its weak topology it is trivially has the Schur property. As we explained above the topology $\mathcal{T}$ of $L(X)$ is compatible with the duality $(E, E')$. Now (ii) of Proposition 2.9 applies. $\quad\square$

If $E$ is a locally convex space, we denote by $E'_\mu = \left(E', \mu(E', E)\right)$ the dual space $E'$ of $E$ endowed with the Mackey topology $\mu(E', E)$ of the dual pair $(E, E')$. The following statement is dual to Theorem 2.3.

**Theorem 2.11.** *For a Fréchet space $E$, the following assertions are equivalent:*

(i) *$E$ does not have the Schur property.*

(ii) *$\left(E', \mu(E', E)\right)$ has the JNP.*

(iii) *$\left(E', \sigma(E', E)\right)$ has the JNP.*

(iv) *$E$ has a bounded non-precompact sequence which does not have a subsequence equivalent to the unit basis of $\ell_1$.*

*Proof.* The clauses (i)–(iii) are equivalent by Proposition 2.9, and Theorem 1.2 of [10] exactly states that (i) and (iv) are equivalent. $\quad\square$

We finish this section with the following "hereditary" result.

**Proposition 2.12.** *Let an lcs $E$ have the JNP. Then:*

(i) *for every lcs $L$, the product $E \times L$ has the JNP;*

(ii) *closed subspaces and Hausdorff quotients of an lcs with the JNP may fail to have the JNP;*

(iii) *every lcs $H$ is topologically isomorphic to a closed subspace of an lcs with the JNP.*

*Proof.* (i) It is easy to check that $(E \times L)'_{\beta^*} = E'_{\beta^*} \times L'_{\beta^*}$. Since also $(E \times L)'_{w^*} = E'_{w^*} \times L'_{w^*}$ the JNP of $E$ implies that the identity map $(E \times L)'_{w^*} \to (E \times L)'_{\beta^*}$ is not continuous. Thus $E \times L$ has the JNP.

(ii) Let $E$ be a Banach space and $L$ be a Fréchet–Montel space. Then $L$ is topologically isomorphic to a closed subspace and to a Hausdorff quotient of $E \times L$, and the assertion follows from (i) and Theorem 2.3.

(iii) If $Z$ is a Banach space, then $H$ embeds into $Z \times H$. It remains to note that, by (i) and the Josefson–Nissenzweig theorem 1.1, $Z \times H$ has the JNP. $\quad\square$

## 3. The Josefson–Nissenzweig property in function spaces

Let $X$ be a set, and let $f : X \to \mathbb{F}$ be a function to the field $\mathbb{F}$ of real or complex numbers. For a subset $A \subseteq X$ and $\varepsilon > 0$, let

$$\|f\|_A := \sup(\{|f(x)| : x \in A\} \cup \{0\}) \in [0, \infty].$$

Observe that a subset $A \subseteq X$ is *functionally bounded* iff $\|f\|_A < \infty$ for any continuous function $f : X \to \mathbb{R}$. A Tychonoff space $X$ is *pseudocompact* if $X$ is functionally bounded in $X$.

For a subfamily $\mathcal{F} \subseteq \mathbb{F}^X$, we put

$$[A; \varepsilon]_{\mathcal{F}} := \{f \in \mathcal{F} : \|f\|_A \leq \varepsilon\}.$$

If the family $\mathcal{F}$ is clear from the context, then we shall omit the subscript $\mathcal{F}$ and write $[A; \varepsilon]$ instead of $[A; \varepsilon]_{\mathcal{F}}$. A family $\mathcal{S}$ of subsets of $X$ is *directed* if for any sets $A, B \in \mathcal{S}$ the union $A \cup B$ is contained in some set $C \in \mathcal{S}$.

For a Tychonoff space $X$, we denote by $C(X)$ the space of all continuous functions $f : X \to \mathbb{F}$ on $X$ and let $C^b(X)$ be the subspace of $C(X)$ consisting of all bounded functions.

A Tychonoff space $X$ is defined to be a *$\mu$-space* if every functionally bounded subset of $X$ has compact closure in $X$. We denote by $\upsilon X$, $\mu X$ and $\beta X$ the Hewitt completion (=realcompactification), the Diedonné completion and the Stone-Čech compactification of $X$, respectively. It is known ([9, 8.5.8]) that $X \subseteq \mu X \subseteq \upsilon X \subseteq \beta X$. Also it is known that all paracompact spaces and all realcompact spaces are Diedonné complete and each Diedonné complete space is a $\mu$-space, see [9, 8.5.13]. On the other hand, each pseudocompact $\mu$-space is compact. By a *compactification* of a Tychonoff space $X$ we understand any compact Hausdorff space $\gamma X$ containing $X$ as a dense subspace.

For a Tychonoff space $X$, the space $C(X)$ carries many important *locally convex topologies*, i.e., topologies turning $C(X)$ into a locally convex space. For a locally convex topology $\mathcal{T}$ on $C(X)$, we denote by $C_{\mathcal{T}}(X)$ the space $C(X)$ endowed with the topology $\mathcal{T}$. The subspace $C^b(X)$ of $C_{\mathcal{T}}(X)$ with the induced topology is denoted by $C_{\mathcal{T}}^b(X)$.

Each directed family $\mathcal{S}$ of functionally bounded sets in a Tychonoff space $X$ induces a locally convex topology $\mathcal{T}_{\mathcal{S}}$ on $C(X)$ whose neighborhood base at zero consists of the sets $[S; \varepsilon]$ where $S \in \mathcal{S}$ and $\varepsilon > 0$. The topology $\mathcal{T}_{\mathcal{S}}$ is called *the topology of uniform convergence on sets of the family $\mathcal{S}$*. The topology $\mathcal{T}_{\mathcal{S}}$ is Hausdorff if and only if the union $\bigcup \mathcal{S}$ is dense in $X$.

If $\mathcal{S}$ is the family of all finite, compact or functionally bounded subsets of $X$, respectively, then the topology $\mathcal{T}_{\mathcal{S}}$ will be denoted by $\mathcal{T}_p$, $\mathcal{T}_k$ or $\mathcal{T}_b$, and the function space $C_{\mathcal{T}_{\mathcal{S}}}(X)$ will be denoted by $C_p(X)$, $C_k(X)$ or $C_b(X)$, respectively.

Although the topologies $\mathcal{T}_p$, $\mathcal{T}_k$ and $\mathcal{T}_b$ are the most famous and well-studied there are other natural and important topologies on $C(X)$, for example, the topology $\mathcal{T}_s$ defined by the family of all finite unions of convergent sequences in $X$ and the topology $\mathcal{T}_c$ on $C(X)$ defined by the family of all *countable* functionally-bounded subsets of $X$. It is clear that

$$\mathcal{T}_p \subseteq \mathcal{T}_s \subseteq \mathcal{T}_c \subseteq \mathcal{T}_b \quad \text{and} \quad \mathcal{T}_p \subseteq \mathcal{T}_s \subseteq \mathcal{T}_k \subseteq \mathcal{T}_b.$$

In [2] we consider the following locally convex topologies on function spaces. Let $X$ be a dense subspace of a Tychonoff space $M$ (for example, $M = \mu X$, $\upsilon X$ or $\beta X$). Then the union $\bigcup \mathcal{S}$ of the directed family $\mathcal{S}$ of all finite (resp. compact, functionally bounded) subsets of $X$ is dense in $M$. Therefore $\mathcal{S}$ defines the Hausdorff locally convex vector topology $\mathcal{T}_{\mathcal{S}}$ on the space $C(M)$ denoted by $\mathcal{T}_{p \restriction X}$ (resp. $\mathcal{T}_{k \restriction X}$, $\mathcal{T}_{b \restriction X}$).

Numerous results concerning the JNP obtained below show that if a function space $C_{\mathcal{T}}(X)$ has the JNP, then so do the spaces $C_{\tau}(X)$, $C_{\mathcal{T}}(\mu X)$, $C_{\mathcal{T}}^b(X)$, $C_{\tau}^b(X)$, $C_{\mathcal{T}}^b(\mu X)$, and $C(\beta X)$, where $\tau$ is a locally convex topology on $C(X)$, stronger than $\mathcal{T}$. These facts and the aforementioned discussion motivate the following definition which is very useful to unify all proofs.

**Definition 3.1.** Let $I : E \to L$ be an injective continuous operator between locally convex spaces $E$ and $L$. We shall say that a locally convex space $H$ is *between* the spaces $E$ and $L$ if there exist injective continuous operators $T_E : E \to H$ and $T_L : H \to L$ such that $T_L \circ T_E = I$. $\quad \square$

For example, for any Tychonoff space $X$ the function space $C_k(X)$ is between $C_b(X)$ and $C_p(X)$. Also for any compactification $\gamma X$ of $X$, the space $C_k^b(X)$ is between the spaces $C(\gamma X)$ and $C_k(X)$ linked by the restriction operator $I : C(\gamma X) \to C_k(X)$, $I : f \mapsto f\!\restriction_X$.

The *support* $\mathrm{supp}(\mu)$ of a linear functional $\mu : C(X) \to \mathbb{F}$ is the set of all points $x \in X$ such that for every neighborhood $O_x \subseteq X$ of $x$ there exists a function $f \in C(X)$ such that $\mu(f) \neq 0$ and $\mathrm{supp}(f) \subseteq O_x$ where $\mathrm{supp}(f) = \overline{\{x \in X : f(x) \neq 0\}}$. The definition of $\mathrm{supp}(\mu)$ implies that it is a closed subset of $X$. We shall use the following assertions.

**Lemma 3.2.** *Let $X$ be a Tychonoff space, and let $\mathcal{S}$ be a directed family of functionally bounded sets in $X$. If a functional $\mu \in C(X)'$ is continuous in the topology $\mathcal{T}_\mathcal{S}$, then $\mathrm{supp}(\mu) \subseteq \overline{S}$ for some set $S \in \mathcal{S}$ such that $[S; 0] \subseteq \mu^{-1}(0)$.*

*Proof.* By the continuity of $\mu$ in the topology $\mathcal{T}_\mathcal{S}$, there exist a set $S \subseteq \mathcal{S}$ and $\varepsilon > 0$ such that $\mu([S; \varepsilon]) \subseteq (-1, 1)$. Then

$$[S; 0] = \bigcap_{n \in \mathbb{N}} [S; \tfrac{\varepsilon}{n}] \subseteq \bigcap_{n \in \omega} \mu^{-1}\!\left(\left(-\tfrac{1}{n}, \tfrac{1}{n}\right)\right) = \mu^{-1}(0).$$

It remains to prove that $\mathrm{supp}(\mu) \subseteq \overline{S}$. In the opposite case we can find a function $f \in C(X)$ such that $\mu(f) \neq 0$ and $\mathrm{supp}(f) \cap \overline{S} = \emptyset$. On the other hand, $f \in [S; 0] \subseteq \mu^{-1}(0)$ and hence $\mu(f) = 0$. This contradiction shows that $\mathrm{supp}(\mu) \subseteq \overline{S}$. $\square$

**Lemma 3.3.** *Let $X$ be a Tychonoff space. If a linear functional $\mu \in C(X)'$ is continuous in the topology $\mathcal{T}_k$, then $\mathrm{supp}(\mu)$ is a compact subset of $X$ and $[\mathrm{supp}(\mu); 0] \subseteq \mu^{-1}(0)$.*

*Proof.* By the continuity of $\mu$ in the topology $\mathcal{T}_k$, there exist a compact subset $K \subseteq X$ and $\varepsilon > 0$ such that $\mu([K; \varepsilon]) \subseteq (-1, 1)$. By (the proof of) Lemma 3.2, $\mathrm{supp}(\mu) \subseteq K$ and $[K; 0] \subseteq \mu^{-1}(0)$. Since $\mathrm{supp}(\mu)$ is a closed subset of $X$, $\mathrm{supp}(\mu)$ is closed in $K$ and hence $\mathrm{supp}(\mu)$ is a compact subset of $X$.

It remains to prove that $[\mathrm{supp}(\mu); 0] \subseteq \mu^{-1}(0)$. To derive a contradiction, assume that $[\mathrm{supp}(\mu); 0] \nsubseteq \mu^{-1}(0)$ and hence there exists a continuous function $f \in C(X)$ such that $\mu(f) \neq 0$ but $f\!\restriction_{\mathrm{supp}(\mu)} = 0$. Multiplying $f$ by a suitable constant, we can assume that $\mu(f) = 2$. Embed the space $X$ into its Stone–Čech compactification $\beta X$. By the Tietze–Urysohn Theorem, there exists a continuous function $\bar{f} \in C(\beta X)$ such that $\bar{f}\!\restriction_K = f\!\restriction_K$. It follows from $[K; 0] \subseteq \mu^{-1}(0)$ that $\mu(f) = \mu(\bar{f}\!\restriction_X)$.

Consider the open neighborhood $U = \{x \in \beta X : |\bar{f}(x)| < \varepsilon\}$ of $\mathrm{supp}(\mu)$ in $\beta X$. By the definition of support $\mathrm{supp}(\mu)$, every point $x \in K \setminus U$ has an open neighborhood $O_x \subseteq \beta X$ such that $\mu(g) = 0$ for any function $g \in C(X)$ with $\mathrm{supp}(g) \subseteq O_x \cap X$. Observe that $U \cup \bigcup_{x \in K \setminus U} O_x$ is an open neighborhood of the compact set $K$ in $\beta X$. So there is a finite family $F \subseteq K \setminus U$ such that $K \subseteq U \cup \bigcup_{x \in F} O_x$. Let $1_{\beta X}$ denote the constant function $\beta X \to \{1\}$. By the paracompactness of the compact space $\beta X$, there is a finite family $\{\lambda_0, \ldots, \lambda_n\}$ of continuous functions $\lambda_i : \beta X \to [0, 1]$ such that $\sum_{i=0}^n \lambda_i = 1_{\beta X}$ and for every $i \in \{0, \ldots, n\}$, the support $\mathrm{supp}(\lambda_i)$ is contained in some set $V \in \{\beta X \setminus K, U\} \cup \{O_x : x \in F\}$. We lose no generality assuming that

$$\bigcup_{i=0}^j \mathrm{supp}(\lambda_i) \subseteq \beta X \setminus K, \qquad \bigcup_{i=j+1}^s \mathrm{supp}(\lambda_i) \subseteq U,$$

and for every $i \in \{s+1, \ldots, n\}$ there exists $x_i \in F$ such that $\mathrm{supp}(\lambda_i) \subseteq O_{x_i}$.

Replacing the functions, $\lambda_0, \ldots, \lambda_j$ by the single function $\sum_{i=0}^j \lambda_i$ and the functions $\lambda_{j+1}, \ldots, \lambda_s$ by the single function $\sum_{j+1}^s \lambda_i$, we can assume that $j = 0$ and $s = 1$. In this simplified case we have $\mathrm{supp}(\lambda_0) \subseteq \beta X \setminus K$, $\mathrm{supp}(\lambda_1) \subseteq U$ and $\mathrm{supp}(\lambda_i) \subseteq O_{x_i}$ for all $i \in \{2, \ldots, n\}$.

For every $i \in n$, consider the function $f_i \in C(\beta X)$, defined by $f_i(x) := \lambda_i(x) \cdot \bar{f}(x)$ for $x \in \beta X$. Then $\bar{f} = \sum_{i \in n} f_i$.

It follows from $\mathrm{supp}(f_0) \subseteq \mathrm{supp}(\lambda_0) \subseteq \beta X \setminus K$ and $[K; 0] \subseteq \mu^{-1}(0)$ that $f_0\!\restriction_K = 0$ and $\mu(f_0\!\restriction_X) = 0$.

Since $K \subseteq U$, $\bar{f}(U) \subseteq (-\varepsilon, \varepsilon)$ and $\mathrm{supp}(f_1) \subseteq \mathrm{supp}(\lambda_1) \subseteq U$, the function $f_1{\restriction}_X = (\bar{f} \cdot \lambda_1){\restriction}_X$ belongs to the set $[K; \varepsilon] \subseteq \mu^{-1}\big((-1, 1)\big)$ and hence

$$\big|\mu\big(f_1{\restriction}_X\big)\big| \le 1. \tag{1}$$

For every $i \in \{2, \dots, n\}$, we have $\mathrm{supp}(f_i) \subseteq \mathrm{supp}(\lambda_i) \subseteq O_{x_i}$ and hence $\mu(f_i{\restriction}_X) = 0$ by the choice of $O_{x_i}$. Now we see that

$$2 = \mu(f) = \mu(\bar{f}{\restriction}_X) = \mu(f_0{\restriction}_X) + \mu(f_1{\restriction}_X) + \mu\Big(\sum_{i=2}^{n} f_i{\restriction}_X\Big) = \mu(f_1{\restriction}_X),$$

which contradicts (1). $\square$

**Lemma 3.4.** *Let X be a Tychonoff space. For any bounded subset* $\mathcal{M} \subseteq (C_b(X))'_{w^*}$ *the set* $\mathrm{supp}(\mathcal{M}) = \bigcup_{\mu \in \mathcal{M}} \mathrm{supp}(\mu)$ *is functionally bounded in X.*

*Proof.* To derive a contradiction, assume that the set $\mathrm{supp}(\mathcal{M})$ is not functionally bounded in $X$. Then there exists a continuous function $\varphi : X \to [0, \infty)$ such that the set $\varphi(\mathrm{supp}(\mathcal{M}))$ is not bounded in the real line. Inductively we shall choose sequences of functionals $\{\mu_n\}_{n \in \omega} \subseteq \mathcal{M}$, points $\{x_n\}_{n \in \omega} \subseteq \mathrm{supp}(\mathcal{M})$ and functionally bounded sets $\{S_n\}_{n \in \omega}$ in $X$ such that for every $n \in \omega$ the following conditions are satisfied:

1. $\varphi(x_n) > 3 + \sup \varphi(\bigcup_{i<n} S_i)$;
2. $x_n \in \mathrm{supp}(\mu_n) \subseteq \overline{S_n}$ and $[S_n; 0] \subseteq \mu^{-1}(0)$.

To start the inductive construction, choose any point $x_0 \in \mathrm{supp}(\mathcal{M})$ with $\varphi(x_0) > 3$ and find a functional $\mu_0 \in \mathcal{M}$ such that $x_0 \in \mathrm{supp}(\mu_0)$. By Lemma 3.2, there exists a functionally bounded set $S_0 \subseteq X$ such that $\mathrm{supp}(\mu_0) \subseteq \overline{S_0}$ and $[S_0, 0] \subseteq \mu_0^{-1}(0)$. Assume that for some $n \in \mathbb{N}$ we have chosen functionally bounded sets $S_0, \dots, S_{n-1}$ in $X$ satisfying the inductive conditions (1) and (2). As the set $\varphi(\mathrm{supp}(\mathcal{M}))$ is unbounded and the set $\varphi(\bigcup_{i<n} S_i)$ is bounded in the real line, we can find a point $x_n \in \mathrm{supp}(\mathcal{M})$ satisfying the inductive condition (1). Since $x_n \in \mathrm{supp}(\mathcal{M}) = \bigcup_{\mu \in \mathcal{M}} \mathrm{supp}(\mu)$ we can find a functional $\mu_n \in \mathcal{M}$ such that $x_n \in \mathrm{supp}(\mu_n)$. By Lemma 3.2, for the functional $\mu_n \in C_b(X)'$ there exists a functionally bounded set $S_n \subseteq X$ satisfying the inductive condition (2). This completes the inductive step.

Now, for every $n \in \omega$, consider the open neighborhood $O_n = \{x \in X : |\varphi(x) - \varphi(x_n)| < 1\}$ of the point $x_n$. The inductive condition (1) ensures that $\varphi(x_n) - \varphi(x_i) > 3$ for any $i < n$, which implies that the family $(O_n)_{n \in \omega}$ is discrete in $X$. Moreover, $O_m \cap S_n = \emptyset$ for any numbers $n < m$.

For every $n \in \omega$, the definition of the support $\mathrm{supp}(\mu_n) \ni x_n$ implies the existence of a function $f_n \in C(X)$ such that $\mu_n(f_n) \neq 0$ and $\mathrm{supp}(f_n) \subseteq O_n$. Multiplying $f_n$ by a suitable constant, we can assume that

$$\mu_n(f_n) > n + \sum_{i<n} |\mu_n(f_i)|. \tag{2}$$

Since the family $(O_n)_{n \in \omega}$ is discrete, so is the family $(\mathrm{supp}(f_n))_{n \in \omega}$. Consequently, the function $f = \sum_{n \in \omega} f_n : X \to \mathbb{F}$ is well-defined and continuous.

For every numbers $n < m$ we have $\mathrm{supp}(f_m) \cap S_n \subseteq O_m \cap S_n = \emptyset$ and hence $f{\restriction}_{S_n} = \sum_{i \le n} f_i{\restriction}_{S_n}$. Since $[S_n; 0] \subseteq \mu_n^{-1}(0)$, we have

$$\mu_n(f) = \sum_{i \le n} \mu_n(f_i) \ge \mu_n(f_n) - \sum_{i<n} |\mu_n(f_i)| > n$$

according to (2). Consequently, $\sup\{|\mu(f)| : \mu \in \mathcal{M}\} \ge \sup_{n \in \omega} |\mu_n(f)| = \infty$, which contradicts the boundedness of the set $\mathcal{M}$ in $C(X)'_{w^*}$. $\square$

The next theorem shows that the JNP has some "hereditary" type property with respect to finer locally convex topologies.

**Theorem 3.5.** *Let $\gamma X$ be a compactification of a Tychonoff space $X$. Let $Y$ be an lcs between $C_k(X)$ and $C_p(X)$, and let $Z$ be an lcs between $C(\gamma X)$ and $Y$. If the lcs $Y$ has the JNP, then $Z$ has the JNP, too.*

*Proof.* We can identify the lcs $Y$ with the function space $C(X)$ endowed with a suitable locally convex topology $\mathcal{T}$ such that $\mathcal{T}_p \subseteq \mathcal{T} \subseteq \mathcal{T}_k$. Since $Z$ is between $C(\gamma X)$ and $Y$, there exist injective operators $T_\gamma : C(\gamma X) \to Z$ and $T : Z \to Y$ such that $T \circ T_\gamma(f) = f\restriction_X$ for every $f \in C(\gamma X)$.

Assuming that the lcs $Y$ has the JNP, find a null sequence $\{\mu_n\}_{n \in \omega} \subseteq Y'_{w^*}$ such that $\inf_{n \in \omega} \|\mu_n\|_B > \varepsilon$ for some barrel-bounded set $B \subseteq Y$ and some $\varepsilon > 0$. For every $n \in \omega$, choose an element $f_n \in B$ such that $|\mu_n(f_n)| > \varepsilon$. Since $\mathcal{T} \subseteq \mathcal{T}_k \subseteq \mathcal{T}_b$, the linear functionals $\mu_n \in Y'$ are continuous in the topology $\mathcal{T}_b$. The continuity of the identity operator $C_b(X) \to Y$ implies that the sequence $(\mu_n)_{n \in \omega}$ is weak* null in $C_b(X)'$. By Lemma 3.4, the set $S = \bigcup_{n \in \omega} \mathrm{supp}(\mu_n)$ is functionally bounded in $X$. It follows that the set $[S; 1]$ is a barrel in $C_p(X)$ and hence in $Y$. Since $B$ is barrel-bounded in $Y$, there exists $r \in \mathbb{N}$ such that $B \subseteq r \cdot [S; 1]$.

For every $n \in \omega$, the functional $\mu_n$ is continuous in the topology $\mathcal{T} \subseteq \mathcal{T}_k$ and hence, by Lemma 3.3, $\mu_n$ has compact support $\mathrm{supp}(\mu_n)$. By the Tietze–Urysohn Theorem, there exists a continuous function $g_n \in C(\gamma X)$ such that $g_n \restriction_{\mathrm{supp}(\mu_n)} = f_n \restriction_{\mathrm{supp}(\mu_n)}$ and $\|g_n\|_{\gamma X} = \|f_n\|_{\mathrm{supp}(\mu_n)}$ and hence, by Lemma 3.3, $\mu_n\big(g_n\restriction_X\big) = \mu_n(f_n)$.

The definition of the number $r$ guarantees that

$$\sup_{n \in \omega} \|g_n\|_{\gamma X} = \sup_{n \in \omega} \|f_n\|_{\mathrm{supp}(\mu_n)} \leq \sup_{f \in B} \|f\|_S \leq r.$$

This means that the set $\{g_n\}_{n \in \omega}$ is bounded in the Banach space $C(\gamma X)$. Since Banach spaces are barrelled, $\{g_n\}_{n \in \omega}$ is barrel-bounded in $C(\gamma X)$. Now the continuity of the operator $T_\gamma : C(\gamma X) \to Z$ ensures that the set $D = \{T_\gamma(g_n)\}_{n \in \omega}$ is barrel-bounded in $Z$. For every $n \in \omega$, consider the linear continuous functional $\lambda_n = \mu_n \circ T \in Z'$. The convergence $\mu_n \to 0$ in $Y'_{w^*}$ implies the convergence $\lambda_n \to 0$ in $Z'_{w^*}$.

Finally, observe that for every $n \in \omega$ we have

$$\lambda_n(T_\gamma(g_n)) = \mu_n(T \circ T_\gamma(g_n)) = \mu_n(g_n\restriction X) = \mu_n(f_n)$$

and hence $\|\lambda_n\|_D \geq |\mu_n(f_n)| > \varepsilon$ and $\|\lambda_n\|_D \not\to 0$, witnessing that the lcs $Z$ has the JNP. $\square$

Applying Theorem 3.5 to $\gamma X = \beta X$, $Y = C_p(X)$ and $Z = C_k(X)$ we obtain

**Corollary 3.6.** *Let $X$ be a Tychonoff space such that the space $C_p(X)$ has the JNP. Then also the space $C_k(X)$ has the JNP. Moreover, the weak\* null sequence $\{\mu_n\}_{n \in \omega}$ in the dual space $C_k(X)'$ witnessing the JNP of $C_k(X)$ can be chosen such that all $\mu_n$ have finite support.*

**Corollary 3.7.** *Let $X$ be a Tychonoff space such that the space $C_k(X)$ has the JNP. Then:*

(i) *the spaces $C_k(\mu X)$, $C_k(\nu X)$, $C(\beta X)$ and $C_b(X)$ have the JNP;*
(ii) *the spaces $C_k^b(\mu X)$, $C_k^b(\nu X)$, and $C_b^b(X)$ have the JNP.*

*Moreover, the weak\* null sequence $\{\mu_n\}_{n \in \omega}$ in their dual spaces witnessing the JNP can be chosen such that all $\mu_n$ have compact support contained in $X$.*

*Proof.* The corollary follows from Theorem 3.5 applied to $\gamma X = \beta X$, $Y = C_k(X)$ and $Z$ is one of the spaces from (i) and (ii). $\square$

If $E$ is a locally convex space, we denote by $E_\beta$ the space $E$ endowed with the locally convex topology $\beta(E, E')$ whose neighborhood base at zero consists of barrels. To characterize function spaces $C_p(X)$ and $C_k(X)$ having the JNP we shall use the following proposition.

**Proposition 3.8.** *Let $X$ be a Tychonoff space, and let $\mathcal{T}$ be a locally convex topology on $C(X)$ such that $\mathcal{T}_p \subseteq \mathcal{T} \subseteq \mathcal{T}_k$. Then:*

(i) *for every barrel $D$ in $C_\mathcal{T}(X)$, there are a functionally bounded subset $A$ of $X$ and $\varepsilon > 0$ such that $[A; \varepsilon] \subseteq D$.*

(ii) *a subset $\mathcal{F} \subseteq C_{\mathcal{T}}(X)$ is barrel-bounded if and only if for any functionally bounded set $A \subseteq X$, the set $\mathcal{F}(A) := \bigcup_{f \in \mathcal{F}} f(A)$ is bounded in $\mathbb{F}$;*

(iii) $\left(C_{\mathcal{T}}(X)\right)_\beta = C_b(X)$.

*Proof.* (i) Let $\upsilon X$ be the realcompactification of $X$, and let $R : C_k(\upsilon X) \to C_k(X)$, $R : f \mapsto f\restriction_X$, be the restriction operator. Since every continuous function $f : X \to \mathbb{F}$ admits a unique continuous extension to $\upsilon X$, the operator $R$ is bijective. As $\upsilon X$ is a $\mu$-space, the Nachbin–Shirota theorem [16, 11.7.5] implies that $C_k(\upsilon X)$ is barrelled. The continuity of the operators $C_k(\upsilon X) \to C_k(X) \to C_{\mathcal{T}}(X)$ implies that for every barrel $D$ in $C_{\mathcal{T}}(X)$, the preimage $R^{-1}(D)$ is a barrel in $C_k(\upsilon X)$. Since the space $C_k(\upsilon X)$ is barrelled, $R^{-1}(D)$ is a neighborhood of zero. So, there exists a compact subset $K \subseteq \upsilon X$ and $\varepsilon > 0$ such that $[K; \varepsilon] \subseteq R^{-1}(D)$. It follows that the set $A = K \cap X$ is functionally bounded and closed in $X$. It is clear that the set $[A; \varepsilon]$ is a barrel in $C_p(X)$ and hence also in $C_{\mathcal{T}}(X)$. We claim that $[A; \varepsilon] \subseteq D$. Indeed, suppose for a contradiction that $[A; \varepsilon] \setminus D$ contains some function $f$. Since the barrel $D$ is closed in $\mathcal{T}$ and the identity operator $C_k(X) \to C_{\mathcal{T}}(X)$ is continuous, there exist $\delta > 0$ and a compact set $C \subseteq X$ such that $(f + [C; \delta]) \cap D = \emptyset$. If follows from $f \in [A; \varepsilon]$ that $f[C \cap K] = f[C \cap A] \subseteq [-\varepsilon, \varepsilon]$. By the Tietze–Urysohn Theorem, there exists a continuous function $g : K \to [-\varepsilon, \varepsilon]$ such that $g(x) = f(x)$ for every $x \in K \cap C$. Define the function $\varphi : C \cup K \to \mathbb{F}$ by the formula

$$\varphi(x) = \begin{cases} f(x) & \text{if } x \in C; \\ g(x) & \text{if } x \in K; \end{cases}$$

and observe that it is well-defined and continuous. By the Tietze–Urysohn Theorem, the function $\varphi$ can be extended to a bounded continuous function $\psi : \upsilon X \to \mathbb{F}$. Then $\psi\restriction_X \in (f + [C; \delta]) \cap R([K; \varepsilon]) \subseteq (f + [C; \delta]) \cap D$, which contradicts the choice of $C$ and $\delta$. This contradiction shows that $B \subseteq D$.

(ii) To prove the "only if" part, take a functionally bounded set $A \subseteq X$ and observe that the set $[A; 1]$ is a barrel in $C_p(X)$, and hence $B$ is a barrel in $C_{\mathcal{T}}(X)$. If $\mathcal{F}$ is barrel-bounded, then $\mathcal{F} \subseteq n \cdot [A; 1]$ for some $n \in \omega$, which implies $\mathcal{F}(A) \subseteq \{z \in \mathbb{F} : |z| \le n\}$.

To prove the "if" part, assume that for any functionally bounded set $A \subseteq X$, the set $\mathcal{F}(A)$ is bounded in $\mathbb{F}$. To see that $\mathcal{F}$ is barrel-bounded in $C_{\mathcal{T}}(X)$, fix any barrel $D \subseteq C_{\mathcal{T}}(X)$. By (i), there are a functionally bounded set $Z$ in $X$ and $\varepsilon > 0$ such that $[Z; \varepsilon] \subseteq D$. It is clear that $[Z; \varepsilon]$ is a barrel in $C_p(X)$ and hence in $C_{\mathcal{T}}(X)$. By our assumption, the set $\mathcal{F}(Z)$ is bounded, and hence the real number $r = \sup\{|f(x)| : f \in \mathcal{F}, x \in Z\}$ is well-defined. It is clear that $\mathcal{F} \subseteq [Z; r] = \frac{r}{\varepsilon}[Z; \varepsilon] \subseteq \frac{r}{\varepsilon}D$. Thus $\mathcal{F}$ is barrel-bounded.

(iii) Set $E = C_{\mathcal{T}}(X)$. By (i), the family $\mathcal{B}$ of sets of the form $[A; \varepsilon]$, where $A$ is functionally bounded in $X$ and $\varepsilon > 0$, is a neighborhood base at zero of the topology $\beta(E, E')$. On the other hand, by the definition of $\mathcal{T}_b$, the family $\mathcal{B}$ is also a neighborhood base at zero of the topology $\mathcal{T}_b$. Thus $\beta(E, E') = \mathcal{T}_b$, as desired. $\square$

In the following theorem for a Tychonoff space $X$ and a functional $\mu \in C(X)'$ we put

$$\|\mu\| := \|\mu\|_{[X; 1]}.$$

**Theorem 3.9.** *$X$ be a Tychonoff space, and let $\mathcal{T}$ be a locally convex topology on $C(X)$ such that $\mathcal{T}_p \subseteq \mathcal{T} \subseteq \mathcal{T}_k$. The function space $C_{\mathcal{T}}(X)$ has the JNP if and only if there is a null sequence $\{\mu_n\}_{n \in \omega} \subseteq C_{\mathcal{T}}(X)'_{w^*}$ such that $\|\mu_n\| = 1$ for every $n \in \omega$.*

*Proof.* To prove the "if" part, assume that the weak dual $E'_{w^*}$ of the locally convex space $E = C_{\mathcal{T}}(X)$ contains a null sequence $\{\mu_n\}_{n \in \omega} \subseteq E'_{w^*}$ such that $\|\mu_n\| = 1$ for all $n \in \omega$. To see that $E$ has the Josefson–Nissenzweig property, it suffices to show that the sequence $\{\mu_n\}_{n \in \omega}$ diverges in the topology $\beta^*(E', E)$.

Since the null sequence $\mathcal{M} = \{\mu_n\}_{n \in \omega} \subseteq E'_{w^*}$ is bounded, we can apply Lemma 3.4 and conclude that the set $\mathrm{supp}(\mathcal{M})$ is functionally bounded in $X$. By (ii) of Proposition 3.8, the set $[X; 1]$ is barrel-bounded in $E$. Since $\|\mu_n\|_{[X; 1]} = \|\mu_n\| = 1 \not\to 0$, the sequence $\{\mu_n\}_{n \in \omega}$ does not converge to zero in the topology $\beta^*(E', E)$. Thus the identity map $E'_{w^*} \to E'_{\beta^*}$ is not sequentially continuous and $E = C_{\mathcal{T}}(X)$ has the JNP.

To prove the "only if" part, assume that the space $E = C_{\mathcal{T}}(X)$ has the Josefson–Nissenzweig property and hence there is a sequence $\mathcal{M} = \{\mu_n\} \subseteq E'_{w^*}$ that converges to zero in the topology $\sigma(E', E)$ but not in the topology $\beta^*(E', E)$. By Lemma 3.4, the set $\mathrm{supp}(\mathcal{M})$ is functionally bounded in $X$.

We claim that $\|\mu_n\| \not\to 0$. Indeed, suppose for a contradiction that $\lim_n \|\mu_n\| = 0$. In this case we shall prove that the sequence $\{\mu_n\}_{n \in \omega}$ converges to zero in the topology $\beta^*(E', E)$. Given any barrel-bounded set $B \subseteq E$, we need to find an $n \in \omega$ such that $\|\mu_k\|_B \leq 1$ for every $k \geq n$. By (ii) of Proposition 3.8, the number

$$r := \sup\{|f(x)| : f \in B, \ x \in \mathrm{supp}(\mathcal{M})\}$$

is finite. Since $\|\mu_n\| \to 0$, there exists a number $m \in \omega$ such that $r \cdot \|\mu_k\| \leq 1$ for all $k \geq m$. Fix $k \geq m$ and choose an arbitrary $f \in B$. Since $\mathcal{T} \subseteq \mathcal{T}_k$, the functional $\mu_k$ has compact support $\mathrm{supp}(\mu_n)$, according to Lemma 3.3. By the Tietze–Urysohn Theorem, there is an extension $\hat{f} \in C(X)$ of the function $f\!\upharpoonright_{\mathrm{supp}(\mu_n)}$ such that $\|\hat{f}\|_X = \|f\|_{\mathrm{supp}(\mu_n)} \leq r$. By Lemma 3.3, we have $\mu_k(\hat{f}) = \mu_k(f)$. Therefore

$$|\mu_k(f)| = \left|\mu_k(\hat{f})\right| \leq \|\mu_k\| \cdot \|\hat{f}\|_X \leq \|\mu_k\| \cdot r \leq 1,$$

which implies that $\|\mu_k\|_B \leq 1$. Therefore, the sequence $\{\mu_k\}_{k \in \omega}$ converges to zero in the topology $\beta^*(E', E)$, which is a desired contradiction.

By the claim, there is $\varepsilon > 0$ such that the set $\Omega = \{n \in \omega : \|\mu_n\| \geq \varepsilon\}$ is infinite. Write $\Omega$ as $\Omega = \{n_k\}_{k \in \omega}$, where $n_0 < n_1 < \cdots$, and observe that the sequence $\{\eta_k\}_{k \in \omega}$ of functionals

$$\eta_k := \frac{\mu_{n_k}}{\|\mu_{n_k}\|}$$

converges to zero in the topology $\sigma(E', E)$ and consists of functionals of norm 1. $\quad\square$

The next corollary of Theorem 3.9 and Lemma 3.2 shows that for $C_p$-spaces the Josefson–Nissenzweig property introduced in Definition 1.4 is equivalent to the JNP introduced in [3].

**Corollary 3.10.** *For a Tychonoff space $X$, the function space $C_p(X)$ has the JNP if and only if there is a null sequence $\{\mu_n\}_{n \in \omega} \subseteq C_p(X)'_{w^*}$ that consists of finitely supported sign-measures of norm 1.*

Applying Theorem 3.9 and Lemma 3.2 to the compact-open topology $\mathcal{T} = \mathcal{T}_k$ on $C(X)$, we obtain

**Corollary 3.11.** *For a Tychonoff space $X$, the function space $C_k(X)$ has the JNP if and only if there is a null sequence $\{\mu_n\}_{n \in \omega} \subseteq C_k(X)'_{w^*}$ that consists of compactly supported sign-measures of norm 1.*

**Corollary 3.12.** *If a Tychonoff space $X$ contains a non-trivial convergent sequence, then the function space $C_p(X)$ has the JNP.*

*Proof.* Let $\{x_n\}_{n \in \omega} \subseteq X$ be a non-trivial sequence that converges to some point $x \in X \setminus \{x_n\}_{n \in \omega}$. For every $n \in \omega$, consider the functional $\chi_n \in C_p(X)'$ defined by $\chi_n(f) = \frac{1}{2}\big(f(x_n) - f(x)\big)$ for any $f \in C_p(X)$. It follows that $\{\chi_n\}_{n \in \omega}$ is a null sequence in $C_p(X)'_{w^*}$ with $\|\chi_n\| = 1$ for all $n \in \omega$. By Corollary 3.10, the function space $C_p(X)$ has the JNP. $\quad\square$

For function spaces over pseudocompact spaces, the JNP has even better "hereditary" properties.

**Theorem 3.13.** *Let $X$ be a pseudocompact space, $D$ be a dense subset in $X$, and let $\tau$ and $\mathcal{T}$ be two locally convex topologies on $C(X)$ such that $\mathcal{T}_{p \upharpoonright D} \subseteq \tau \subseteq \mathcal{T} \subseteq \mathcal{T}_b$. If the space $C_\tau(X)$ has JNP, then also the space $C_{\mathcal{T}}(X)$ has the JNP.*

*Proof.* If the lcs $C_\tau(X)$ has the JNP, then there exists a null sequence $\{\mu_n\}_{n \in \omega} \subseteq C_\tau(X)'_{w^*}$ such that $\|\mu_n\|_B \not\to 0$ for some barrel-bounded set $B \subseteq C_\tau(X)$. Since $\tau \subseteq \mathcal{T}$, the functionals $\mu_n$ are $\mathcal{T}$-continuous and hence $\{\mu_n\}_{n \in \omega}$ is a null sequence in $C_{\mathcal{T}}(X)'_{w^*}$. We claim that the set $B$ remains barrel-bounded in $C_{\mathcal{T}}(X)$. Indeed, since $\mathcal{T} \subseteq \mathcal{T}_b$, any barrel $A$ in $C_{\mathcal{T}}(X)$ is a barrel in $C_b(X)$. Since $X$ is pseudocompact, $C_b(X)$ is a Banach space whose topology is generated by the norm $\|\cdot\|_X$. Since Banach spaces are barrelled, the barrel $A$ is a neighborhood

of zero and hence $[X; \varepsilon] \subseteq A$ for some $\varepsilon > 0$. As $D$ is dense in $X$, the barrel $[X; \varepsilon]$ is a barrel in $C_{p \upharpoonright D}(X)$ and hence a barrel in $C_\tau(X)$. Since the set $B$ is barrel-bounded in $C_\tau(X)$, there exists a positive real number $r$ such that $B \subseteq r \cdot [X; \varepsilon] \subseteq r \cdot A$, which means that $B$ is barrel-bounded in $C_{\mathcal{T}}(X)$. Since $\|\mu_n\|_B \not\to 0$, the locally convex space $C_{\mathcal{T}}(X)$ has the JNP. $\square$

Recall that a Tychonoff space $X$ is called an *F-space* if every functionally open set $A$ in $X$ is $C^*$-*embedded* in the sense that every bounded continuous function $f : A \to \mathbb{R}$ has a continuous extension $\bar{f} : X \to \mathbb{R}$. For numerous equivalent conditions for a Tychonoff space $X$ being an *F*-space, see [14, 14.25]. In particular, the Stone–Čech compactification $\beta\Gamma$ of a discrete space $\Gamma$ is a compact *F*-space. The following example generalizes the example given in (2) after Definition 1 of [3] with a more detailed proof.

**Example 3.14.** For any infinite compact *F*-space $K$, the function space $C_p(K)$ does not have the JNP.

*Proof.* Let $S = \{\mu_n\}_{n \in \omega} \subseteq C_p(K)'_{w^*}$ be a null sequence. Then $S$ is a weak$^*$ null sequence in the dual space $M(K)$ of all regular Borel sign-measures on $K$ of the Banach space $C(K)$. By Corollary 4.5.9 of [8], the Banach space $C(K)$ has the Grothendieck property, which implies that $\mu_n \to 0$ in the weak topology of the Banach space $M(K)$. But since $S$ contains only sign-measures with finite support, it is contained in the Banach subspace $\ell_1(K)$ of $M(K)$. Now the Schur property of $\ell_1(K)$ implies that $S$ converges to zero in $\ell_1(K)$, i.e. $\|\mu_n\| \to 0$. By Corollary 3.10, the function space $C_p(K)$ does not have the JNP. $\square$

By Corollary 3.6, if the space $C_p(X)$ has the JNP, then also the space $C_k(X)$ has the JNP. However, the converse is not true in general as Example 3.14 shows. So the JNP is not equivalent for $C_p(X)$ and $C_k(X)$.

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